Chapter 3

From Graphs to Matroids

3.1 Overview

In this chapter, we construct matroids from graphs and matrices. In Section 3.2, we start with graphs. We encode them by certain binary matrices that lead to matroids we call graphic. For these matroids, we adapt a number of the graph concepts and operations of Chapter 2, for example the operation of taking minors. In Section 3.3, we generalize the construction of Section 3.2 by starting with arbitrary binary matrices, not just those arising from graphs. From the binary matrices, we deduce the binary matroids. In Section 3.4, we carry the generalization one step further. This time, we begin with abstract matrices, which represent a proper generalization of matrices over fields. From the abstract matrices, we define all matroids.

It is easy to determine matroids that cannot be produced from any graph, or from any binary matrix, or even from any matrix over any field. On the other hand, compact characterizations of the matroids that cannot be obtained from the matrices over some given field are usually difficult to find. We meet an exception in Section 3.5. There we characterize the matroids producible from the binary matrices by excluding a certain 4-element matroid, called U_4^2, as a minor. Sections 3.4 and 3.5 may be skipped by the reader who is only interested in binary matroids. Later, we occasionally refer to the material on general matroids to point out extensions. The final section, 3.6, lists references.

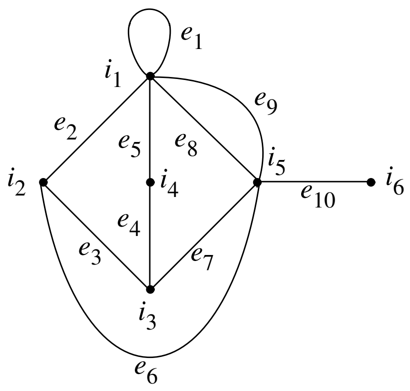
The chapter requires knowledge of the definitions of Chapter 2. To assist the reader, we will repeat certain definitions.

3.2 Graphs Produce Graphic Matroids

In this section, we deduce the graphic matroids from graphs by the following two-step process. In the first step, we encode graphs G by certain binary matrices called node/edge incidence matrices F. In the second step, we derive from the matrices F the graphic matroids M. For insight into the structure of the matroids M so created, we transform the node/edge incidence matrices F by elementary row operations into certain binary matrices B. The latter matrices contain in compact form all facts about M. Thus, we say that the matrices B represent M. We translate a number of graph definitions and concepts for G into statements about the matrices B and the graphic matroids M. In particular, we link trees, cycles, cotrees, and cocycles of G to features of B and M. We also describe the effect of taking minors in the graphs G on B and M, and characterize k-connectivity of G in terms of partitions of B and M. Finally, we establish the relationship between graphs that give rise to the same graphic matroid, and conclude with a handy procedure for deciding whether a certain 1-element binary extension of a graphic matroid is graphic.

Throughout this section, we assume G to be a connected graph with edge set E. The following graph will serve as an example.

(3.2.1)



Graph G

Observe the node symbols i_1, $i_2, \ldots,$ i_6. Recall that each one of these symbols stands for the subset of edges incident at the respective node.

Node/Edge Incidence Matrix

We may represent G by a binary matrix F called the *node/edge incidence matrix.* Each node of G corresponds to a row of that matrix, and each edge to a column. Suppose an edge e connects nodes i and j in G. Into column e of the matrix, we place one 1 into row i, a second 1 into row j, and 0s into the remaining rows. This rule accommodates all edges of G except for loops. There are several ways to treat loops. Here we decide to place only 0s into the columns of loops. For the example graph G of (3.2.1), the resulting matrix F is

(3.2.2)
$$F = \begin{array}{c|cccccccccc|} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \\ \hline i_1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ i_2 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ i_3 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ i_4 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ i_5 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ i_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

Node/edge incidence matrix F of G of (3.2.1)

Except for the endpoints of loops, we can reconstruct G from F. Thus, modulo that small defect, F represents G.

Recall from Chapter 2 the following definitions. A *cycle* of G is the edge set of a walk where all nodes are distinct and where one returns to the departure node. A *tree* of G is a maximal edge subset of E without cycles. A *cocycle* of G is a minimal set of edges that intersects every tree of G. A *cotree* of G is the set $E - X$ for some tree X of G. The *rank* of G is the cardinality of any one of its trees. Thus, it is the number of nodes of G minus 1.

In the discussion to follow, we always assume that the graph G has a cycle and a cocyle. Toward the end of this section, we address the special (actually elementary) situation where G has no cycle or no cocycle.

Over GF(2), the linear dependence of $n \geq 1$ vectors, say of f^1, $f^2, \ldots,$ f^n, is characterized as follows: That set is GF(2)-dependent if there exists a nonempty subset $J \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{j \in J} f^j = 0$ (summation in GF(2), of course). Declare the vectors $f^1, \ldots, f^n$ to be *minimally* GF(2)-*dependent*, for short GF(2)-*mindependent*, if they are GF(2)-dependent, and if every proper subset of these vectors is GF(2)-independent.

For example, column e_1 of F of (3.2.2) is GF(2)-mindependent. So are the columns e_2, e_3, e_4, e_5. The first case corresponds to the loop e_1 of G, the second one to the cycle $\{e_2, e_3, e_4, e_5\}$. Indeed, by the just-given definition, a set of GF(2)-mindependent columns of the node/edge incidence matrix must correspond to a subgraph $\overline{G}$ of G with the following two properties. First, each node of $\overline{G}$ has even degree. Second, there is no subgraph of $\overline{G}$ but the empty one where each node has even degree. Evidently, the cycles of G are the only subgraphs of G with these two properties. This implies that the subgraphs of G without cycles correspond to the GF(2)-independent column subsets of F. In particular, the trees of G correspond to the bases of F, and the rank of G is the GF(2)-rank of F.

So far, we have interpreted cycles and trees of G in terms of F. How do cocycles and cotrees manifest themselves in F? We will answer that question in a moment. In the meantime, the reader may want to try obtaining an answer.

Graphic Matroid

Suppose we are just interested in the collection, say $\mathcal{I}$, of trees of G and their subsets. That interest may be surprising. But the set $\mathcal{I}$ contains a significant amount of information about G. Exactly how much, we will see later in this section. For example, with $\mathcal{I}$ we can decide whether or not an edge subset C of E is a cycle. The answer is "yes" if and only if C is not in $\mathcal{I}$ and every proper subset of C is in $\mathcal{I}$. On the other hand, we cannot decide with $\mathcal{I}$ which nodes are the endpoints of loops.

For the graph G of (3.2.1), $\mathcal{I}$ includes the sets $\{e_2, e_3, e_4, e_7, e_{10}\}$ and $\{e_3, e_5, e_6\}$. We know that each set in $\mathcal{I}$ is the index set of a column submatrix F' of F with GF(2)-independent columns. Conversely, every such index set is recorded in $\mathcal{I}$.

Still assume that we are just interested in $\mathcal{I}$. We are tempted to combine the edge set E of G and the set $\mathcal{I}$ to the ordered pair $M = (E, \mathcal{I})$. The set E is the *groundset* of M, and $\mathcal{I}$ is the collection of *independent sets* of M. Sometimes, we want to emphasize that M is deduced from G and denote it by $M(G)$. In subsequent sections of this chapter, we will see that $M(G)$ is a special case of what then will be called *binary matroids* or even just *matroids*. In the spirit of those definitions, we call $M(G)$ the *graphic matroid* of G.

Representation Matrix

We have established the collection $\mathcal{I}$ from the node/edge incidence matrix F of G. Elementary row operations performed on F do not affect GF(2)-independence of columns. Thus, we may determine $\mathcal{I}$ from any matrix derived from F by such operations. We discuss a special case of such row operations next.

First, we delete one row from F getting a matrix F'. Since each column of F has an even number of 1s, the sum of the rows of F is the zero vector. Hence, F and F' have same GF(2)-rank.

Second, we perform binary row operations to convert the column submatrix of F' corresponding to some tree of G to an identity matrix.

For the example matrix F of (3.2.2), we select $\{e_2, e_3, e_4, e_7, e_{10}\}$ as tree of G. When we apply the preceding two-step procedure to F, we get the matrix of (3.2.3) below. Note the row indices of the matrix. They are in agreement with the notation introduced in Section 2.3.

$$
(3.2.3) \qquad
\begin{array}{c|cccccccccc|}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \\
\hline
e_2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
e_3 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
e_4 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
e_7 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
e_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
$$

columns of identity matrix

Matrix obtained from F of (3.2.2)

The matrix of (3.2.3) is a bit difficult to read. We thus permute its columns to collect the identity submatrix at the left end. This change results in the matrix A below.

$$
(3.2.4) \qquad
A = \begin{array}{c|ccccc|ccccc|}
 & e_2 & e_3 & e_4 & e_7 & e_{10} & e_1 & e_5 & e_6 & e_8 & e_9 \\
\hline
e_2 & 1 & & & & & 0 & 1 & 0 & 1 & 1 \\
e_3 & & 1 & & & & 0 & 1 & 1 & 1 & 1 \\
e_4 & & & 1 & & & 0 & 1 & 0 & 0 & 0 \\
e_7 & & & & 1 & & 0 & 0 & 1 & 1 & 1 \\
e_{10} & & & & & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Matrix A deduced from the matrix of (3.2.3)

The unspecified entries in the identity submatrix of A are to be taken as zeros, in agreement with the convention introduced in Section 2.3 about unspecified matrix entries.

In the case of a general graph G, let X be some tree of G, and Y be the corresponding cotree $E - X$. Then for some binary matrix B, the matrix A is of the form

$$
(3.2.5) \qquad
A = \begin{array}{c|c|c|}
 & X & Y \\
\hline
X & I & B \\
\hline
\end{array}
$$

Matrix A for general graph G with tree X

As explained in Section 2.3, the same information is conveyed by B with row index set X and column index set Y; that is, by

$$
(3.2.6) \qquad
\begin{array}{c|c|}
 & Y \\
\hline
X & B \\
\hline
\end{array}
$$

Matrix B for general graph G with tree X

For our example case,

(3.2.7)
$$B = \begin{array}{c|ccccc} & e_1 & e_5 & e_6 & e_8 & e_9 \\ \hline e_2 & 0 & 1 & 0 & 1 & 1 \\ e_3 & 0 & 1 & 1 & 1 & 1 \\ e_4 & 0 & 1 & 0 & 0 & 0 \\ e_7 & 0 & 0 & 1 & 1 & 1 \\ e_{10} & 0 & 0 & 0 & 0 & 0 \end{array}$$
(rows indexed by X, columns indexed by Y)

Matrix B for graph G of (3.2.1)

The matrix B may be viewed as a binary encoding of the matroid $M = (E, \mathcal{I})$. Since M was defined to be a graphic matroid, we call B a *graphic matrix*. We also say that B *represents* M over GF(2), or that B is a *representation matrix* of M. In the literature, the term *standard representation matrix* is sometimes used. The node/edge incidence matrix is then a *nonstandard representation matrix*. We omit "standard" since we almost always employ matrices like B to represent M.

Tree, Subgraph Rank

We show how trees, cycles, cotrees, cocycles, and the rank of subgraphs of G manifest themselves in B of (3.2.6). We repeatedly make use of some partition (X_1, X_2) of X, and of some partition (Y_1, Y_2) of Y. Typically, we just specify one set of X_1, X_2, and one set of Y_1, Y_2. For any such partitions, we assume B to be partitioned as

(3.2.8)
$$B = \begin{array}{c|c|c} & Y_1 & Y_2 \\ \hline X_1 & B^1 & D^2 \\ \hline X_2 & D^1 & B^2 \end{array}$$

Partitioned version of B

Let Z be a tree of G. Define $X_2 = Z \cap X$ and $Y_1 = Z \cap Y$. In A of (3.2.5), the submatrix $\overline{A}$ indexed by $Z = X_2 \cup Y_1$ is thus a GF(2)-basis of A of the form

(3.2.9)
$$\overline{A} = \begin{array}{c|c|c} & X_2 & Y_1 \\ \hline X_1 & 0 & B^1 \\ \hline X_2 & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} & D^1 \end{array}$$

Submatrix of A indexed by $Z = X_2 \cup Y_1$

The submatrix B^1 of $\overline{A}$ is square, and, by cofactor expansion, GF(2)-nonsingular. Conversely, any square and GF(2)-nonsingular submatrix B^1 of B defined by (X_1, X_2) and (Y_1, Y_2) corresponds to a tree $Z = X_2 \cup Y_1$ of G. More generally, let the submatrix B^1 of B of (3.2.8) be of any size and with GF(2)-rank $B^1 = k$. Then the subgraph of G containing precisely the edges of $X_2 \cup Y_1$ has rank equal to $|X_2| + k$.

Cycle

We know that a cycle C of G corresponds to a column submatrix $\overline{A}$ of A with GF(2)-mindependent columns. Analogously to the tree case, let such a column submatrix be indexed by $X_2 = C \cap X$ and $Y_1 = C \cap Y$. Once more, (3.2.9) displays the general case. GF(2)-mindependence of the columns of $\overline{A}$ implies that they must add up to 0 in GF(2). Thus, each row of B^1 (resp. D^1) must contain an even (resp. odd) number of 1s. Note that this parity condition on the row sums of B^1 and D^1 uniquely determines X_1 and X_2 when Y_1 is specified. Furthermore, by GF(2)-mindependence, no nonempty proper column submatrix of $\overline{A}$, say indexed by $X_2' \subseteq X_2$ and $Y_1' \subseteq Y_1$, satisfies the analogous parity condition.

Fundamental Cycle

The special case of a cycle with $|Y_1| = 1$, say $Y_1 = \{y\}$, is of particular interest. If column y contains no 1s, then y is a loop. Otherwise, X_2 is the index set of the rows with a 1 in column y. The cycle is $X_2 \cup \{y\}$. It is the *fundamental cycle* that y forms with a subset of the tree X. By reversing these arguments, we obtain a quick way of constructing B. Let the tree X be given. We take each edge $y \in Y(= E - X)$ in turn and find the fundamental cycle C that y forms with a subset of X. Then column y of B is the characteristic vector of $C - \{y\}$. That is, the entry in column y and row $x \in X$ is 1 if $x \in (C - \{y\})$, and is 0 otherwise.

Parallel Elements

A cycle of cardinality 2, say $\{y, z\}$ with $y \in Y$, manifests itself in B by two parallel columns indexed by y and z, or by a unit vector column indexed by y and with the 1 in row z. We say that y and z are *in parallel* in M. We define any element to be parallel to itself. Then "is parallel to" is easily checked to be an equivalence relation. The equivalence classes are the *parallel classes* of M. They are precisely the parallel classes of G.

Cotree

Recall that a cotree of G is the set $E-Z$ for some tree Z of G. Consider the transpose of B of (3.2.8), i.e.,

(3.2.10)
$$B^t = \begin{array}{c|c|c} & X_1 & X_2 \\ \hline Y_1 & (B^1)^t & (D^1)^t \\ \hline Y_2 & (D^2)^t & (B^2)^t \end{array}$$

Transpose of B of (3.2.8)

Suppose $Z = X_2 \cup Y_1$. The cotree $X_1 \cup Y_2$ corresponds to the GF(2)-nonsingular submatrix $(B^1)^t$ of B in the same way in which the tree Z is related to the GF(2)-nonsingular B^1 of B.

Append an identity to B^t, getting

(3.2.11)
$$A^* = \begin{array}{c|c|c} & Y & X \\ \hline Y & I & B^t \end{array}$$

B^t with additional identity matrix

Evidently, every cotree indexes a GF(2)-basis of A^*, and vice versa.

Cocycle

We know that a cocycle is a minimal set that intersects every tree. Put differently, a cocycle is a minimal set that is not contained in any cotree. The latter definition shows that cocycles are related to cotrees in the same way that cycles are related to trees. Thus, for the interpretation of cocycles in terms of B, the previous discussion for cycles becomes applicable once we make suitable substitutions. The special cycle case with $|Y_1| = 1$ becomes the special cocycle case with $|X_1| = 1$, say with $X_1 = \{x\}$. If row x of B contains no 1s, then x is a coloop. Otherwise, let Y_1 be the index set of the columns with a 1 in row x. Then $Y_1 \cup \{x\}$ is a cocycle. It is the *fundamental cocycle* that x forms with a subset of the cotree Y. We may use fundamental cocycles to construct the rows of B. The process is analogous to the construction of the columns of B via fundamental cycles.

Coparallel or Series Elements

A cocycle of cardinality 2, say $\{x, z\}$ with $x \in X$, manifests itself in B by two parallel rows indexed by x and z, or by a unit vector row indexed by

x and with the 1 in column z. We also say that x and z are *coparallel*, or *in series*. Declare any element to be in series with itself. Then "is in series with" is an equivalence relation. The equivalence classes are the *coparallel* or *series classes* of M. They are precisely the series classes of G, according to the nonstandard definition of "series" for graphs in Section 2.2.

Dual of Graphic Matroid

For a given graph G, let $\mathcal{I}^*$ be the set of cotrees and their subsets. The fact that cycles are related to $\mathcal{I}$ in the same way in which cocycles are related to $\mathcal{I}^*$ suggests that we tie $\mathcal{I}$ and $\mathcal{I}^*$ together by duality. Specifically, we define the pair $M^* = (E, \mathcal{I}^*)$ to be the *dual matroid* of M. The prefix "co-" dualizes a term. For example, each set in $\mathcal{I}^*$ is *co-independent*. Consistent with the definition of *graphic matroid*, we call M^* the *cographic matroid* of G, and denote it by $M(G)^*$. By the above discussion, B^t represents M^*. For this reason, we call B^t *cographic*.

Let G be a connected plane graph, and G^* be its dual. In Section 2.2, the following is shown. The cotrees of G are the trees of G^*, and the cocycles of G are the cycles of G^*. Thus, in this special case, M^* is the graphic matroid of G^*. Consistent with graph terminology, we call M, as well as all of its representation matrices B, *planar*. This definition and the preceding observation imply that planarity of M implies graphicness of M and M^*, or equivalently, graphicness and cographicness of M.

Is it possible that the dual of a graphic matroid is not graphic? The answer is "yes." Toward the end of this section, we include two examples of a graphic M where M^* is not graphic. In Chapter 10, it is proved that M^* is graphic if and only if G is planar.

Pivot

According to the pivot rule (2.3.3) of Section 2.3, a GF(2)-pivot in B of (3.2.6), say with pivot element B_{xy}, is carried out as follows.

(3.2.12)

(3.2.12.1) We replace for every $u \in (X - \{x\})$ and every $w \in (Y - \{y\})$, B_{uw} by $B_{uw} + (B_{uy} \cdot B_{xw})$ (operations in GF(2)).

(3.2.12.2) We exchange the indices x and y.

Let B' be the matrix produced from B by the pivot. From Section 2.3, we know that this change corresponds to elementary row operations and one column exchange that transform the matrix A composed of B and an identity matrix, i.e.,

(3.2.13)

$$A = \begin{array}{c|c|c|} & X & Y \\ \hline X & I & B \\ \hline \end{array}$$

Matrix A derived from B

to

(3.2.14)

$$A' = \begin{array}{c|c|c|} & X' & Y' \\ \hline X' & I & B' \\ \hline \end{array}$$

Matrix A' derived from B'

where $X' = (X - \{x\}) \cup \{y\}$ and $Y' = (Y - \{y\}) \cup \{x\}$. Hence, by pivots we can deduce from B any matrix B'' that corresponds to a specified tree X'' of G. This fact is very useful. It permits us always to select a B'' that is particularly convenient for our purposes. One such case we see next, when we discuss the effects of edge deletions, additions, contractions, and expansions. We briefly review these graph operations.

A *deletion* of a noncoloop edge is the removal of that edge. A *deletion* of a coloop is accomplished by a contraction, which is defined next. A *contraction* of a nonloop edge is the contraction of the edge so that its endpoints become one vertex. A *contraction* of a loop is a deletion. An *addition* of an edge is the addition of an edge that does not become a coloop. An *expansion* by an edge involves splitting of a node into two nodes, which are joined by the new edge. The new edge cannot be a loop. A *reduction* is a deletion or a contraction. An *extension* is an addition or an expansion. For disjoint $U, W \subseteq E$, the *minor* $G/U\backslash W$ is the graph produced by contraction of the edges of U and deletion of the edges of W. The process is well defined because of an implicit ordering of the edges of G. No such ordering needs to be assumed when U contains no cycle and W no cocycle. Any graph producible from G by some sequence of reductions is obtainable as $G/U\backslash W$, where U and W obey the cycle/cocycle condition just stated.

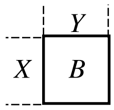
Matroid Deletion, Addition, Contraction, Expansion, Minor

We translate the above graph operations into matroid language. We start with the taking of minors. So let $G/U\backslash W$ be one such minor. As just stated, we may assume U and W to obey the cycle/cocycle condition. We claim that G has a tree X and a cotree $Y = E - X$ such that $U \subseteq X$ and

$W \subseteq Y$. The proof is as follows. Since W contains no cocycle of G, the minor $G \backslash W$ has the same rank as G. Evidently, $G \backslash W$ contains U. Since U does not contain a cycle of G, U also does not contain a cycle of $G \backslash W$. Thus, in $G \backslash W$ we may extend U to a tree X of $G \backslash W$. Since $G \backslash W$ and G have same rank, X is also a tree of G. By the construction, the tree X of G contains U, and the cotree $Y = E - X$ of G contains W.

Let B be the matrix of (3.2.6) corresponding to X and Y, i.e., B is

(3.2.15)



Matrix B for general graph G with tree X

Derive B' from B by deleting the rows indexed by U and the columns indexed by W. We claim that B' represents the graphic matroid of $G/U \backslash W$. We denote that matroid by $M/U \backslash W$, and we call it the *minor* of M obtained by *contraction* of U and *deletion* of W. The proof is as follows.

Each tree Z' of $G/U \backslash W$ is contained in $E - (U \cup W)$. Indeed, $Z' \cup U$ must be a tree $Z \subseteq (E - W)$ of G. Conversely, for every tree $Z \subseteq (E - W)$ of G with $U \subseteq Z$, the set $Z' = Z - U$ is a tree of $G/U \backslash W$. We know from the earlier discussion that $Z = X_2 \cup Y_1$, with $X_2 \subseteq X$ and $Y_1 \subseteq Y$, is a tree of G if and only if the square submatrix B^1 of B indexed by $X_1 = X - X_2$ and Y_1 is GF(2)-nonsingular. Thus, the trees Z' of $G/U \backslash W$ correspond precisely to the square GF(2)-nonsingular submatrices B^1 of B'. Hence, B' represents $M/U \backslash W$, the graphic matroid of $G/U \backslash W$.

Recall from Section 2.2 that the cycles and cocycles of G undergo the following changes as we go from G to $G/U \backslash W$. The cycles of $G/U \backslash W$ are the minimal nonempty sets of the collection $\{C - U \mid C \subseteq E - W;\ C = \text{ cycle of } G\}$. The cocycles of $G/U \backslash W$ are the minimal nonempty sets of the collection $\{C^* - W \mid C^* \subseteq E - U;\ C^* = \text{ cocycle of } G\}$. Correspondingly, the circuits and cocircuits of $M/U \backslash W$ may be derived from those of M.

We just proved that each deletion (resp. contraction) of an edge of G that is not a coloop (resp. loop) produces the removal of a column (resp. row) from a properly selected matrix B. Each addition (resp. expansion), the operation inverse to deletion (resp. contraction), corresponds to the adjoining of a column (resp. row) to B.

Suppose G is a plane graph, and G^* its dual. We know that B^t represents the graphic matroid of G^*. Now row vectors of B appear as column vectors in B^t, and column vectors of B appear as row vectors in B^t. A deletion in G induces a column removal in B, and thus a row removal in B^t. The latter removal corresponds to a contraction in G^*. By matroid arguments, we have proved that deletions in G correspond to contractions in G^*.

Up to this point, we have assumed that G has a cycle and a cocycle. Now suppose that this is not the case. Then G consists only of loops incident at one node, or of coloops, or is empty. In the first case, we define B to have no rows and as many columns as G has loops. In the second case, B is to have no columns and as many rows as G has coloops. In the third situation, B is to be the 0×0 matrix. By the definitions of Chapter 2, in the first two situations B is a trivial matrix, and in the third one the empty matrix. A trivial (resp. empty) B represents a *trivial* (resp. *empty*) *matroid*. The empty matroid has $E = \emptyset$, $\mathcal{I} = \mathcal{I}^* = \{\emptyset\}$, and has no circuits or cocircuits. We leave it to the reader to verify that the above reduction and extension results are valid for the special case of a trivial or empty B.

Separations and Connectivity

We turn to Tutte k-separations, for short k-separations, of G. We show how such separations manifest themselves in B of (3.2.15). Recall from Chapter 2 that a *k-separation* of G with $k \geq 1$ is a pair (E_1, E_2) of nonempty sets that partition E and that have the following properties. First, $|E_1|, |E_2| \geq k$. Second, the subgraph G_1 (resp. G_2) obtained from G by removal of the edges of E_2 (resp. E_1) must be connected. Third, identification of k nodes of G_1 with k nodes of G_2 must produce G. For $k \geq 2$, the graph G is *Tutte k-connected*, for short *k-connected*, if G has no l-separation for $1 \leq l < k$.

Given a k-separation (E_1, E_2) of G and given the matrix B of (3.2.15), define for $i = 1, 2$, $X_i = E_i \cap X$ and $Y_i = E_i \cap Y$. Thus, B can be partitioned as

(3.2.16)
$$B = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & B^1 & D^2 \\ \hline X_2 & D^1 & B^2 \\ \hline \end{array}$$

Partitioned version of B

We claim that

(3.2.17) $\qquad$ $\text{GF(2)-rank } D^1 + \text{GF(2)-rank } D^2 = k - 1.$

For a proof, append to B an identity matrix, getting

(3.2.18)
$$A = \begin{array}{c|c|c|c|c|} & X_1 & X_2 & Y_1 & Y_2 \\ \hline X_1 & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} & 0 & B^1 & D^2 \\ \hline X_2 & 0 & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} & D^1 & B^2 \\ \hline \end{array}$$

Matrix B of (3.2.16) with additional identity matrix

Derive from A the matrix A^1 (resp. A^2) by deleting the columns indexed by $X_2 \cup Y_2$ (resp. $X_1 \cup Y_1$). Then

$$\begin{aligned} \text{GF(2)-rank } A^1 &= |X_1| + \text{GF(2)-rank } D^1 \\ \text{GF(2)-rank } A^2 &= |X_2| + \text{GF(2)-rank } D^2 \end{aligned} \tag{3.2.19}$$

The matrix A may be obtained from the node/edge incidence matrix F of G by row operations. Thus, for $i = 1, 2$, the rank of G_i is equal to GF(2)-rank A^i. We combine this result with (3.2.19) and get

$$\begin{aligned} &(\text{rank of } G_1) + (\text{rank of } G_2) \\ &\quad = \text{GF(2)-rank } A^1 + \text{GF(2)-rank } A^2 \\ &\quad = |X_1| + |X_2| + \text{GF(2)-rank } D^1 + \text{GF(2)-rank } D^2 \\ &\quad = (\text{rank of } G) + \text{GF(2)-rank } D^1 + \text{GF(2)-rank } D^2 \end{aligned} \tag{3.2.20}$$

Finally, the graphs G_1 and G_2 are connected, and identification of k nodes of these graphs produces G. For $i = 1, 2$, let the graph G_i have $n_i + k$ nodes. Then G has $n_1 + n_2 + k$ nodes. With these definitions,

$$\begin{aligned} (\text{rank of } G_1) + (\text{rank of } G_2) &= (n_1 + k - 1) + (n_2 + k - 1) \\ &= (n_1 + n_2 + k - 1) + (k - 1) \\ &= (\text{rank of } G) + (k - 1) \end{aligned} \tag{3.2.21}$$

Then (3.2.17) follows directly from (3.2.20) and (3.2.21).

Definition of k-Separation and k-Connectivity

The preceding discussion motivates the following definitions. For any $k \geq 1$, the matrix B of (3.2.15) is *k-separable* if B can be partitioned as in (3.2.16) such that

$$\begin{aligned} &|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq k \\ &\text{GF(2)-rank } D^1 + \text{GF(2)-rank } D^2 \leq k - 1 \end{aligned} \tag{3.2.22}$$

The pair $(X_1 \cup Y_1, X_2 \cup Y_2)$ is a *k-separation* of B. For $k \geq 2$, the matrix B is *k-connected* if B has no l-separation for $1 \leq l < k$.

Connectivity in M is defined via that of B. That is, for $k \geq 1$, M is *k-separable* if B is k-separable. For $k \geq 2$, M is *k-connected* if M (equivalently, B) has no l-separation for $1 \leq l < k$. In particular, the empty matroid is k-connected for all $k \geq 2$.

The above definitions and observations validate the following lemma.

(3.2.23) Lemma. *Let G be a connected graph, and M be the graphic matroid of G. For $k \geq 1$, any k-separation of G induces a k-separation of M.*

Let us naïvely attempt to prove the converse. That is, we assume a k-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ of M as given by (3.2.16) and (3.2.22). We let G_1 (resp. G_2) be the subgraph created from G by removal of the edges of $E_2 = X_2 \cup Y_2$ (resp. $E_1 = X_1 \cup Y_1$). Now we try to reverse the order of the arguments made earlier about (3.2.16)–(3.2.21).

The inequality GF(2)-rank D^1 + GF(2)-rank $D^2 \leq k - 1$ of (3.2.22) creates a first difficulty. We would like equality. This problem is easily avoided. We simply restrict ourselves to a k-separation of M with minimal k. Equivalently, we may demand M, and hence B, to be k-connected and k-separable.

By (3.2.20) and (3.2.21), we have

$$(\text{rank of } G_1) + (\text{rank of } G_2) = (\text{rank of } G) + k - 1 \tag{3.2.24}$$

We also know that $|E_1|, |E_2| \geq k$ by (3.2.22). We could prove (E_1, E_2) to be a k-separation of G if we could show G_1 and G_2 to be connected. Try as we might, this we cannot do. For good reason, since (E_1, E_2) is not always a k-separation of G, as we shall see.

Link between Graph and Matroid Separations

So let us scale down our goal. Let us simply strive to obtain a detailed description of the structure of G with k-separable M, k being minimal. To this end, we temporarily abandon our notion of nodes as edge subsets. Instead, we adopt the customary notion of nodes as points. Thus, a node of G may occur in several subgraphs of G. We retain the assumption that G is a connected graph with edge set E.

We need a few definitions to describe and prove the structure of G. Let $H_1, H_2, \ldots, H_p$, $p \geq 2$, be subgraphs of G whose edge sets partition E. Each H_i contains at least one edge. The *connecting vertices* of H_i are the vertices of H_i that also occur in some H_j, $j \neq i$. The remaining vertices of H_i are *internal*. The vertices that occur in both H_i and H_j, $i \neq j$, are the *common* vertices of H_i and H_j. The *sum* of $H_1, H_2, \ldots, H_r$, $r \leq p$, written as $H_1 + H_2 + \cdots + H_r$, is the not necessarily connected subgraph of G whose edge set is the union of the edge sets of $H_1, H_2, \cdots, H_r$.

Let $L(H_1, H_2, \ldots, H_p)$ be the following connected graph. Its nodes are labeled $H_1, H_2, \ldots, H_p$. As many parallel arcs connect node H_i with node H_j of $L(H_1, H_2, \ldots, H_p)$ as H_i and H_j have vertices in common. We declare the graphs $H_1, H_2, \ldots, H_p$ to be *connected in tree fashion* if $L(H_1, H_2, \ldots, H_p)$ is a tree. They are *connected in cycle fashion* if the latter graph is a cycle. To avoid confusion, we use "vertex" and "edge" in connection with G, and "node" and "arc" when $L(H_1, H_2, \ldots, H_p)$ is involved.

We are now ready to describe and prove the structure of G.

(3.2.25) Theorem. *Let M be the graphic matroid of a connected graph G. Assume (E_1, E_2) is a k-separation of M with minimal $k \geq 1$. Define G_1 (resp. G_2) from G by removing the edges of E_2 (resp. E_1) from G. Let R_1, $R_2, \ldots,$ R_g be the connected components of G_1, and S_1, $S_2, \ldots,$ S_h be those of G_2.*

(a) *If $k = 1$, then the R_i and S_j are connected in tree fashion.*
(b) *If $k = 2$, then the R_i and S_j are connected in cycle fashion.*
(c) *If $k \geq 3$, either* (c.1) *or* (c.2) *below holds.*
  (c.1) *Each of G_1 and G_2 is connected (thus $G_1 = R_1$ and $G_2 = S_1$) and contains a cycle or an internal vertex. The two graphs have exactly k vertices in common.*
  (c.2) *One of g and h is equal to 2, and the other one is equal to k. Without loss of generality assume $g = 2$ and $h = k$. Then S_1, $S_2, \ldots,$ S_h contain exactly one edge each. The union of the edge sets of the S_i (which is E_2) is a cocycle of G of cardinality k. The two connected components R_1 and R_2 of G_1 contain at least one cycle each.*

Proof. The edge sets of R_1, $R_2, \ldots,$ R_g and S_1, $S_2, \ldots,$ S_h partition E. Let m_i (resp. n_j) be the number of internal vertices of R_i (resp. S_j). Define p_{ij} to be the number of vertices R_i and S_j have in common. We accomplish the proof via a series of claims.

Claim 1. The graph $L(R_1, \ldots, R_g, S_1, \ldots, S_h)$ has $g + h$ nodes and

$$\sum_{i,j} p_{ij} = g + h + k - 2 \tag{3.2.26}$$

arcs, and thus is a tree plus $k - 1$ arcs.

Proof. The number of nodes of $L(R_1, \ldots, R_g, S_1, \ldots, S_h)$ is by definition $g+h$. Since the graphs R_i, S_j, and G are connected, we have (rank of R_i) $= m_i + \sum_j p_{ij} - 1$, (rank of S_j) $= n_j + \sum_i p_{ij} - 1$, and (rank of G) $= \sum_i m_i + \sum_j n_j + \sum_{i,j} p_{ij} - 1$. Furthermore, (rank of G_1) $= \sum_i$(rank of R_i) $= \sum_i m_i + \sum_{i,j} p_{ij} - g$, and (rank of G_2) $= \sum_j$(rank of S_j) $= \sum_j n_j + \sum_{i,j} p_{ij} - h$. By (3.2.24), (rank of G_1)+(rank of G_2) = (rank of G)$+k-1$, and thus $(\sum_i m_i + \sum_{i,j} p_{ij} - g) + (\sum_j n_j + \sum_{i,j} p_{ij} - h) = (\sum_i m_i + \sum_j n_j + \sum_{i,j} p_{ij} - 1) + k - 1$. Solving the latter equation for $\sum_{i,j} p_{ij}$, which is the number of arcs of $L(R_1, \ldots, R_g, S_1, \ldots, S_h)$, we obtain $\sum_{i,j} p_{ij} = g + h + k - 2$. Q. E. D. Claim 1

Claim 2. Parts (a) and (b) of the theorem hold.

Proof. Suppose $k = 1$. By Claim 1, $L(R_1, \ldots, R_g, S_1, \ldots, S_h)$ is a tree, and (a) follows. Let $k = 2$. Then $L(R_1, \ldots, R_g, S_1, \ldots, S_h)$ has exactly one cycle. If there are additional arcs, then $L(R_1, \ldots, R_g, S_1, \ldots, S_h)$ has a degree 1 node, and thus, G and M are 1-separable, a contradiction of the minimality of k. Thus, (b) follows. Q. E. D. Claim 2

As a result of Claim 2, we may assume from now on that $k \geq 3$.

Claim 3. At least one of the graphs $R_1, \ldots, R_g, S_1, \ldots, S_h$ has at least as many edges as it has connecting vertices.

Proof. Assume otherwise. Thus, each subgraph R_i (resp. S_j) of G is a tree on $\sum_j p_{ij}$ (resp. $\sum_i p_{ij}$) vertices. Hence, G has $\sum_{i,j} p_{ij}$ vertices and $\sum_i(\sum_j p_{ij} - 1) + \sum_j(\sum_i p_{ij} - 1) = 2\sum_{i,j} p_{ij} - g - h$ edges. Using $\sum_{i,j} p_{ij} = g + h + k - 2$ of (3.2.26), we see that G has $g + h + k - 2$ vertices and $g + h + 2(k - 2)$ edges. Since G is k-connected, the degree of each vertex of G is at least k. Thus,

$$k \cdot (\text{number of vertices}) \leq 2 \cdot (\text{number of edges}) \tag{3.2.27}$$

Accordingly, $k(g + h + k - 2) \leq 2(g + h + 2(k - 2))$, which implies $g + h \leq 4 - k \leq 1$, a contradiction. Q. E. D. Claim 3

Claim 4. For all i and j,

$$(\text{rank of } R_i) + (\text{rank of } S_j) \geq (\text{rank of } R_i + S_j) \tag{3.2.28}$$

The inequality is strict if and only if $p_{ij} \geq 2$.

Proof. Direct computation verifies the claim. Q. E. D. Claim 4

By Claim 3, we may assume from now on that R_1 has at least as many edges as it has connecting vertices. Equivalently, R_1 has an internal vertex or a cycle.

Claim 5. If $g = 1$, i.e., if $G_1 = R_1$, then case (c.1) applies.

Proof. If each S_j has no internal vertex and is a tree, then G_2 has $|E_2| = \sum_j p_{1j} - h$ edges. Using (3.2.26) with $g = 1$, we have $|E_2| = k - 1$, which contradicts $|E_2| \geq k$. Thus, without loss of generality, S_1 has at least as many edges as it has vertices. Assume $h \geq 2$. Since G does not have a 1-separation, S_2 has at least two vertices in common with G_1. Shift the edge set of S_2 from E_2 to E_1. The resulting pair (E_1^0, E_2^0) corresponds to the subgraphs $R_1 + S_2$ and $S_1 + S_3 + \cdots + S_h$ of G and, with the aid of (3.2.28), is easily checked to be an l-separation of M with $l < k$. But this contradicts the minimality of k. Thus, $h = 1$. By (3.2.26), $G_1 = R_1$ and $G_2 = S_1$ have k vertices in common, so case (c.1) holds. Q. E. D. Claim 5

From now on, we assume $g \geq 2$.

Claim 6. If in $L(R_1, \ldots, R_g, S_1, \ldots, S_h)$ the node S_j has degree 2, then the subgraph S_j of G contains exactly one edge.

Proof. By the assumption, the subgraph S_j of G has two vertices in common with $R_1, \ldots, R_g$. Suppose the edge set of S_j, say E', has at least two edges. Then $(E', E - E')$ is a 2-separation of M, which is not possible since $k \geq 3$. Q. E. D. Claim 6

Claim 7. Case (c.2) applies.

Proof. Shift the edge sets of $R_2, \ldots, R_g$ from E_1 to E_2. The resulting partition (E_1^1, E_2^1) of E corresponds to graphs R_1 and $G_2^1 = R_2 + \cdots + R_g + S_1 + \cdots + S_h$. By (3.2.28), that partition is easily seen to be an l-separation of M with $l \leq k$. By the minimality of k, we have $l = k$. Then by Claim 5, R_1 and G_2^1 have exactly k vertices in common, and G_2^1 is connected. Suppose S_1, $S_2, \ldots, S_t$ are the S_j graphs that have vertices in common with R_1. Since G_2^1 is connected, so is the graph $L(R_2, \ldots, R_g, S_1, \ldots, S_h)$. Indeed, the latter graph is obtained from $L(R_1, \ldots, R_g, S_1, \ldots, S_h)$ by the removal of node R_1, and thus, by Claim 1, has $g + h - 1$ nodes and $g + h - 2$ arcs. Hence, $L(R_2, \ldots, R_g, S_1, \ldots, S_h)$ is a tree. Any tip node of that tree must correspond to some S_j, $1 \leq j \leq t$, since otherwise G and M are 1-separable. For the same reason, $t \geq 2$. Let S_1 be a tip node of the tree. Then the subgraph S_1 of G has exactly one vertex in common with $R_2 + \cdots + R_g + S_1 + \cdots + S_h$.

Similar arguments show that the graphs $R_1 + S_1$ and $R_2 + \cdots + R_g + S_2 + \cdots + S_h$ also correspond to a k-separation of G and M. Thus, we conclude the following. R_1 and $R_2 + \cdots + R_g + S_1 + \cdots + S_h$ have exactly k vertices in common. $R_1 + S_1$ and $R_2 + \cdots + R_g + S_2 + \cdots + S_h$ also have exactly k vertices in common. S_1 and $R_2 + \cdots + R_g + S_2 + \cdots + S_h$ have exactly one vertex in common. The last three statements imply that R_1 and S_1 have exactly one vertex in common. Hence, the node S_1 in $L(R_1, \ldots, R_g, S_1, \ldots, S_h)$ has degree 2. By Claim 6, the subgraph S_1 of G contains just an edge.

Inductively, we now show that each of the subgraphs $S_2, \ldots, S_t$ contains just one edge. Specifically, we assume for some $r < t$, that $S_1, \ldots,$ S_r have just one edge each. We then prove that one of the subgraphs S_j of G, $r + 1 \leq j \leq t$, say S_{r+1}, has just one edge. For the case $r + 1 = t$, we also show that $t = h = k$ and $g = 2$. We omit the detailed arguments since they are very similar to the above ones.

It remains to be shown that each of R_1 and R_2 contains a cycle. We know that each subgraph $S_1, \ldots, S_h$ has just one edge. So if R_1 or R_2 has no cycle, i.e., is a tree, then G has a degree 2 vertex, and G and M are 2-separable, a contradiction. Thus, (c.2) holds. Q. E. D. Claim 7

The above claims clearly establish Theorem (3.2.25). □

Theorem (3.2.25) has several important results as corollaries. They are the main reason why, in this book, we prefer Tutte connectivity over vertex or cycle connectivity for graphs.

(3.2.29) Corollary. *Let M be the graphic matroid of a connected graph G. For any $k \geq 2$, M is k-connected if and only if this is so for G.*

Proof. By Lemma (3.2.23), any l-separation of G is an l-separation of M. Via Theorem (3.2.25), we argue that any l-separation of M with minimal l implies that G has an l-separation, as follows. The cases (a), (b), and (c.1) of the theorem are straightforward. For case (c.2), the graphs $R_1 + S_1 + \cdots + S_h$ and R_2 produce the desired l-separation of G. $\square$

(3.2.30) Corollary. *Let G be a connected plane graph. Then for any $k \geq 2$, G is k-connected if and only if this is so for the dual graph G^*.*

Proof. By Corollary (3.2.29) and matroid duality, the graphs G and G^* and the matroids M and M^* are all k-connected if this is so for any one of these graphs and matroids. $\square$

Recall from Chapter 2 that a matrix A is connected if the associated bipartite graph $\mathrm{BG}(A)$ is connected. It is easily seen that A is not connected if and only if A is a trivial matrix, i.e., of size $m \times 0$ or $0 \times m$, for some $m \geq 1$, or A has a row or column without 1s, or A has a block decomposition. Let us apply this result to a binary matrix B representing a graphic matroid M. It is easily checked via (3.2.16) that B is 2-connected if and only if B is connected. Correspondingly, we define the matroid M to be *connected* if it is 2-connected. Note that "G is connected" is a statement quite different from "B is connected" or "M is connected." The latter two statements are by Corollary (3.2.29) equivalent to "G is 2-connected." Admittedly, the use of "connected" has become a bit confusing by the above definitions. But that use is so well accepted in matroid theory that we employ it here, too. The next corollary summarizes the above relationship for future reference.

(3.2.31) Corollary. *Let G be a connected graph and B be a representation matrix of the graphic matroid M of G. Then G is 2-connected if and only if B (and hence M) is connected.*

Finally, we have the following characterization of a 3-connected graph in terms of any representation matrix of its graphic matroid.

(3.2.32) Corollary. *The following statements are equivalent for a connected graph G with edge set E and for any representation matrix B of the graphic matroid M of G.*

(i) *G is 3-connected.*
(ii) *If $|E| \geq 2$: G has no loops or coloops.*
If $|E| \geq 4$: G has no parallel or series edges. Furthermore, deletion of at most two stars does not produce a disconnected graph.

(iii) *M is 3-connected.*
(iv) *B is connected, has no parallel or unit vector rows or columns, and has no partition as in* (3.2.16) *with* GF(2)-rank $D^1 = 1$, $D^2 = 0$, *and* $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq 3$.
(v) *Same as* (iv), *but* $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq 5$.

Proof. Corollary (3.2.29) proves (i) $\Longleftrightarrow$ (iii). The implications (i) $\Longleftrightarrow$ (ii) and (iii) $\Longleftrightarrow$ (iv) are easily seen, and (iv) $\Longrightarrow$ (v) is trivial. The possibly surprising (v) $\Longrightarrow$ (iv) is established by a straightforward checking process as follows. In B of (3.2.16), assume GF(2)-rank $D^1 = 1$ and $D^2 = 0$. If the length of B^1 is 3 or 4, then B can be seen to have a zero column or row, or parallel or unit vector rows or columns. Any such case is already excluded by the first part of (iv). Thus, it suffices to require $|X_1 \cup Y_1| \geq 5$, and by duality, $|X_2 \cup Y_2| \geq 5$. $\square$

In the remainder of this section, we address the following questions: How are the graphs related that produce a given graphic matroid? How can one obtain a graph that generates a given graphic matroid? When does a binary matrix correspond to a graphic matroid?

Graph 2-Isomorphism

We begin with the first question. So let M be a graphic matroid. Declare any two connected graphs G and H that produce M to be *2-isomorphic*. Necessarily, G and H have the same edge set, say E. Each one of the following sets of edge subsets of G or H completely determines M, and thus must be the same for G and H: the set of trees, the set of cycles, the set of cotrees, and the set of cocycles. For the same reason, G and H have the same rank function, the same k-separations for any $k \geq 1$, and the same connectivity. Despite the numerous relationships between G and H, these graphs may be quite different. For example, in (3.2.34) below, the first graph and the last graph of the sequence are quite distinct, yet will be shown to be 2-isomorphic.

We start with the case where M is 1-separable. We claim that the 2-connected components of G, say $G_1, G_2, \ldots, G_t$, are connected in tree fashion. By Theorem (3.2.25), G consists of $p \geq 2$ connected subgraphs that are connected in tree fashion. Select a case with p maximum. If one of the subgraphs is not 2-connected, then that subgraph itself consists of $q \geq 2$ subgraphs that are connected in tree fashion. Evidently, this contradicts the maximality of p. By 2-isomorphism, the edge set of each 2-connected component G_i of G must be that of a 2-connected component, say H_i, of H. Thus, $H_1, H_2, \ldots, H_t$ are the 2-connected components of H, which are also connected in tree fashion. We emphasize that the tree structure produced by $G_1, \ldots, G_t$ may be entirely different from that of $H_1, \ldots, H_t$.

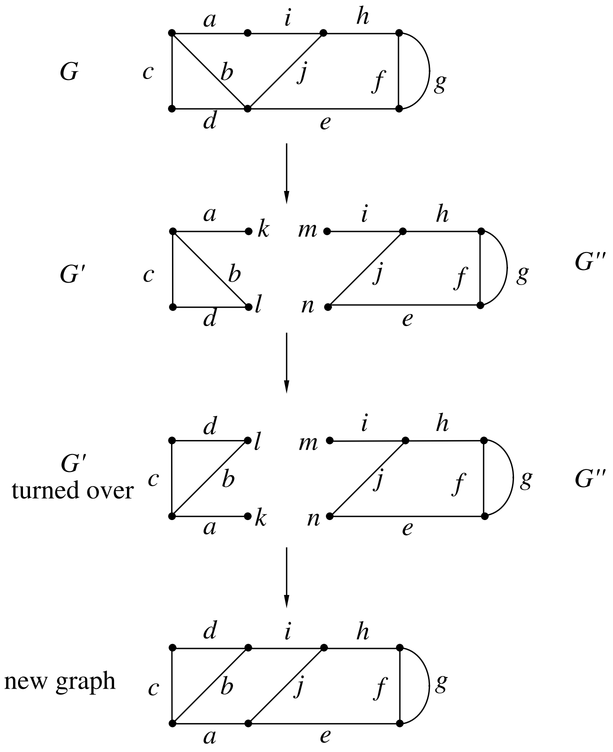
If for all i, we had $G_i = H_i$, then we would have completely explained the difference between G and H. But this may not be so. Thus, we must understand the relationships between 2-connected but different G_i and H_i. To simplify the notation, we assume M, G, and H to be themselves 2-connected. The next lemma shows that G is equal to H if we strengthen that assumption to 3-connectedness.

(3.2.33) Lemma. *Let G and H be 3-connected and 2-isomorphic graphs. Then $G = H$.*

Proof. By trivial checking, we may assume that G has at least six edges. Let Z be any star of G. By Corollary (3.2.32) (ii), Z is a cocycle of G and $G\backslash Z$ is 2-connected. Thus, Z is a cocycle of H and $H\backslash Z$ is 2-connected. This is only possible if Z is a star of H. We conclude that each star of G is one of H, and vice versa. Then $G = H$. ∎

By Lemma (3.2.33) and the earlier discussion, just one case remains, where M, G, and H are 2-connected and 2-separable. Take a 2-separation of G. It induces two subgraphs G' and G''. Assume that identification of nodes k and l of G' with nodes m and n, respectively, of G'' produces G. Instead, let us identify k of G' with n of G'', and l of G' with m of G''. Here is an example of this operation.

(3.2.34)



Example of switching operation

Roughly speaking, we have switched the nodes of attachment of G' with G''. For this reason, we say that the new graph is obtained from G by a *switching*. It is easy to see that G and the new graph have the same set of cycles. Thus, they are 2-isomorphic.

Let G' and G'' be the just-defined subgraphs of a 2-separation of G. The graph G' may not be 2-connected. By Theorem (3.2.25), the 2-connected components of G' are connected in tree fashion. That tree must be a path, since otherwise G is 1-separable. Apply the same arguments to G''. Combine the two observations. Thus, G has 2-connected subgraphs, say $G_1, G_2, \ldots, G_t$, for some $t \geq 2$, that are connected in cycle fashion. In the notation of Theorem (3.2.25), $G_1 + G_2 + \cdots + G_t = G$. By 2-isomorphism, H also has 2-connected subgraphs, say $H_1, H_2, \ldots, H_t$, where for all i, G_i and H_i have the same edge set. Clearly, $H_1 + H_2 + \cdots + H_t = H$. We now establish how the H_i are linked in H.

(3.2.35) Lemma. *$H_1, H_2, \ldots, H_t$ are connected in cycle fashion.*

Proof. Each 2-connected G_i with at least two edges constitutes one side of a 2-separation of G. By 2-isomorphism and Theorem (3.2.25), this also holds for the 2-connected H_i and H. Thus, H_i has exactly two nodes in common with the remaining subgraphs H_j of H. The same conclusion holds trivially if G_i and H_i have just one edge. Let C be any cycle of G that includes at least one edge of G_i and at least one edge of some G_j, $j \neq i$. Then C must include at least one edge each from $G_1, G_2, \ldots, G_t$. The analogous fact holds for the H_i. These observations imply that $H_1, H_2, \ldots, H_t$ are connected in cycle fashion. □

We now link 2-isomorphism and switching.

(3.2.36) Theorem. *Let G and H be 2-connected and 2-isomorphic graphs. Then H may be obtained from G by switchings.*

Proof. If G is 3-connected, Lemma (3.2.33) applies. Otherwise, as explained above, for some $t \geq 2$, let $G_1, G_2, \ldots, G_t$ be 2-connected subgraphs of G linked in cycle fashion and satisfying $G_1 + G_2 + \cdots + G_t = G$. By Lemma (3.2.35), the corresponding 2-connected subgraphs $H_1, H_2, \ldots, H_t$ of H are connected in cycle fashion as well.

Consider G_1 by itself. Join the two nodes that G_1 has in common with the remaining G_i, $i \geq 2$, by an edge u. Let G_1' be the resulting 2-connected graph. Analogously, add an edge u to H_1, getting H_1'. By the structure of G and H, the graphs G_1' and H_1' are 2-isomorphic. By induction, G_1' can by switchings be transformed to H_1'. The same switchings can be performed in G when we view the edge u of G_1' as representing $G_2 + \cdots + G_t$. Thus, by certain switchings, every G_i of G becomes H_i. Finally, the subgraphs H_i of the new G can by switchings be so positioned that H results. □

Chapter 10 includes a polynomial algorithm that for a given graphic matroid M finds a graph G producing M. The algorithm can even determine whether an arbitrary binary matroid is graphic. The algorithm essentially consists of two subroutines. One of them is called the graphicness testing subroutine. We describe and validate it next, using Theorem (3.2.36).

Graphicness Testing Subroutine

The input to the subroutine consists of a matrix B' given by

(3.2.37)

$$B' = \begin{array}{c|c|c|} & Y & z \\ \hline X & B & b \\ \hline \end{array}$$

Input matrix for graphicness testing subroutine

The submatrix B indexed by X and Y is known to be graphic. Also given is a graph G that produces B. That graph is known to be 3-connected or to be a subdivision of a 3-connected graph. In the subroutine, we want to decide whether B' is graphic. We first analyze the relationships among B', B, b, and G.

We know that the row index set X of B is a tree of G. Let Z be the set of rows $x \in X$ for which $b_x = 1$. In the tree X of G, paint each edge of Z red. We leave the remaining edges of X unpainted.

Suppose B' is graphic. Let H' be a graph for B', with the edges of Z painted red as in G. In that graph, the set X is also a tree. Moreover, according to the column vector b and the painting rule, the red edges must form a fundamental cycle with the edge z. Thus, the red edges form a path in H' as well as in $H' \backslash z = H$. The graph H generates B, as does G. Hence, the 2-connected G and H are 2-isomorphic. By Theorem (3.2.36), H is obtainable from G by switchings. Thus, graphicness of B' implies that G can by switchings be transformed to a graph where the red edges form a path.

Conversely, assume that G can by switchings be changed into a graph H where the red edges form a path. Add an edge z to H whose endpoints are those of the path. Let H' be the resulting graph. Then in H' the fundamental cycle that z forms with X is $\{z\} \cup Z$, and thus, H' generates B'. Therefore, B' is graphic.

Recall that any graph G to be processed by the graphicness testing subroutine either is 3-connected or is a subdivision of a 3-connected graph. In the first case, no switchings are possible. Thus, B' is graphic if and only if the red edges form a path in G. Assume the second case, i.e., G is a subdivision of a 3-connected graph. Then in any 2-separation of G, the edge set of one side is readily seen to be a subset of some series class of G. Evidently, any switching amounts to a resequencing of some edges of the series class. Conversely, any resequencing of the edges of a series class can clearly be accomplished by switchings. It is a trivial matter to check whether resequencing of series class edges can result in a red path. Thus, we can readily decide whether B' is graphic. We leave it to the reader to write down the rules formally.

Suppose B' is found to be graphic. The subroutine then outputs the graph H' and stops. If B' has been determined to be not graphic, then the subroutine says so and stops.

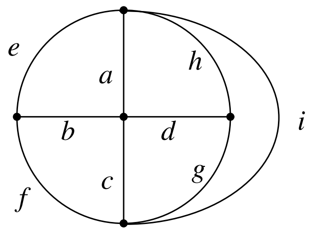
We demonstrate the subroutine with four examples. In the first case, we have

(3.2.38)
$$B' = \begin{array}{cc} & \overbrace{\begin{array}{ccccc} e & f & g & h & i \end{array}}^{Y} \; z \\ X \left\{ \begin{array}{c} a \\ b \\ c \\ d \end{array} \right. & \begin{array}{|ccccc|c|} \hline 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ \hline \end{array} \end{array}$$

Example 1 for graphicness test

The graph G

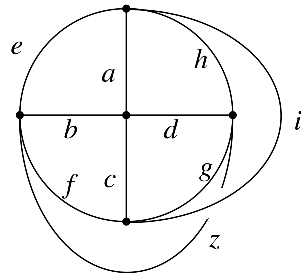
(3.2.39)



Graph G for Example 1

produces the submatrix B of B' indexed by X and Y. Evidently, G is isomorphic to the wheel W_4 plus one additional edge and is 3-connected. According to our graphicness test for B', the edges of X corresponding to the 1s in column z of B' must be painted red. Thus, we paint the edges b and d. These red edges form a path, so B' is graphic. In G, we join the two endpoints of that path by an additional edge z to obtain the following graph for B'.

(3.2.40)



Graph H' for Example 1

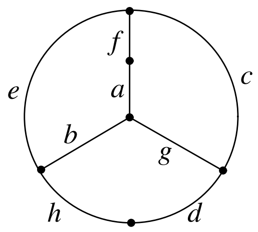
That graph is isomorphic to K_5, the complete graph on five vertices. Thus, B' of (3.2.38) represents up to index changes $M(K_5)$, the graphic matroid of K_5.

The second example involves the matrix

(3.2.41)
$$B' = \begin{array}{cc|ccc|c|} & & \multicolumn{3}{c|}{\longleftarrow Y \longrightarrow} & \\ & & f & g & h & z \\ \hline & a & 1 & 0 & 0 & 1 \\ & b & 1 & 1 & 0 & 0 \\ X & c & 0 & 1 & 1 & 0 \\ & d & 0 & 0 & 1 & 1 \\ & e & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

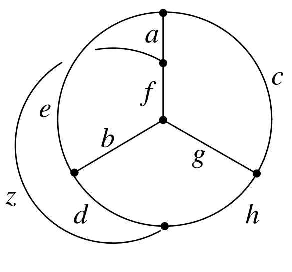
Example 2 for graphicness test

A graph G for the submatrix B indexed by X and Y is given by

(3.2.42)


Graph G for Example 2

Evidently, the graph is a subdivision of the 3-connected wheel W_3. This time, we must paint the edges a, d, and e red. The red edges do not form a path, but we can obtain a path by resequencing a and f as well as d and h. Following such sequencing, we join the endpoints of the resulting red path by a new edge z to obtain the graph

(3.2.43)


Graph H' for Example 2

Thus, B' is graphic. We leave it to the reader to verify that the graph for B' is isomorphic to $K_{3,3}$, the complete bipartite graph with three vertices on either side. We conclude that B' of (3.2.41) represents up to index changes $M(K_{3,3})$, the graphic matroid of $K_{3,3}$.

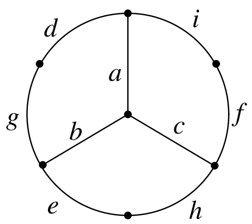
For the third example, we re-index and repartition the transpose of the matrix of (3.2.38) to get

(3.2.44)
$$B' = \begin{array}{c|ccc|c} & g & h & i & z \\ \hline a & 1 & 0 & 1 & 0 \\ b & 1 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 1 & 0 & 0 & 1 \\ e & 0 & 1 & 0 & 1 \\ f & 0 & 0 & 1 & 1 \end{array}$$

Example 3 for graphicness test

A graph G producing B is

(3.2.45)



Graph G for Example 3

The graph is clearly a subdivision of the 3-connected wheel W_3. This time we must paint the edges d, e, and f red. Obviously, no resequencing of d and g, of e and h, and of f and i can result in a red path. Thus, B' is not graphic. Since it is up to indexing the transpose of the matrix of (3.2.38), it represents up to a change of indices $M(K_5)^*$. Thus, that matroid is not graphic.

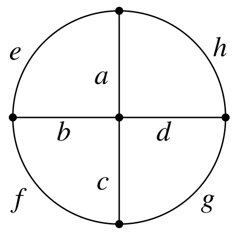
For the fourth case, we re-index and repartition the transpose of the matrix of (3.2.41) as

(3.2.46)
$$B' = \begin{array}{c|cccc|c} & e & f & g & h & z \\ \hline a & 1 & 0 & 0 & 1 & 1 \\ b & 1 & 1 & 0 & 0 & 1 \\ c & 0 & 1 & 1 & 0 & 1 \\ d & 0 & 0 & 1 & 1 & 1 \end{array}$$

Example 4 for graphicness test

The graph G of (3.2.47) below produces the submatrix B of B' indexed by X and Y. Evidently, G is isomorphic to the 3-connected wheel W_4.

(3.2.47)



Graph G for Example 4

According to column z of B', we paint in G all edges of X red. The red edges do not form a path in G. Thus, B' is not graphic. Since B' is up to a re-indexing the transpose of the matrix of (3.2.41), it represents up to a change of indices $M(K_{3,3})^*$. Thus, that matroid is not graphic.

The last two examples establish the following result.

(3.2.48) Lemma. *The matroids $M(K_5)^*$ and $M(K_{3,3})^*$ are not graphic.*

There are easier ways to prove Lemma (3.2.48). The matroid $M(K_5)^*$ has ten elements and rank 6. Thus, a graph for that matroid has ten edges and seven nodes. Since $M(K_5)^*$ is 3-connected, the degree of each node of the graph is at least 3. But then the graph must have at least eleven edges, a contradiction. The matroid $M(K_{3,3})^*$ has nine elements and rank 4. Since contraction of any edge in $K_{3,3}$ reduces that graph to the 3-connected wheel graph W_4, deletion of any element from $M(K_{3,3})^*$ must result in a 3-connected minor. We conclude that a graph for $M(K_{3,3})^*$ must have nine edges and five nodes, and deletion of any edge from that graph must produce a 3-connected minor. There is only one candidate graph. It is K_5 minus one edge. But that graph has two vertices of degree 3, and the deletion of some edge produces a 2-separable minor, a contradiction.

We conclude this section by recording the existence of the polynomial graphicness testing subroutine.

(3.2.49) Lemma. *There is a polynomial algorithm, called the graphicness testing subroutine, for the following problem. Input is a binary matrix $B' = [B \mid b]$, where B is graphic. Also given is a graph G for B. It is known that G is 3-connected or is a subdivision of a 3-connected graph. The algorithm decides whether B' is graphic. In the affirmative case, a graph H for B' is also produced.*

We mentioned earlier that the graphicness testing subroutine will in Chapter 10 be combined with a second subroutine to a polynomial test for graphicness of binary matroids. That second subroutine carries out the following task. It analyzes the connectivity of the binary matroid for which graphicness is to be decided. In doing so, the subroutine converts the given problem into a sequence of subproblems, each of which can be solved by the above graphicness testing subroutine.

In the next section, we extend the class of graphic matroids to the class of binary matroids.

3.3 Binary Matroids Generalize Graphic Matroids

So far, we have used graphs to create the graphic matroids. In the process, we have developed quite a few matroid concepts. In this section, we generalize the graphic matroids to the binary ones. Indeed, the definitions and concepts for graphic matroids introduced in the preceding section are already so rich that most of them need at most a trivial adjustment to make them suitable for binary matroids. Thus, this section is long on definitions and short on motivation of concepts and explanations.

The reader familiar with matroid theory will surely notice that many results of this section hold for general matroids, not just binary ones. In the next section, 3.4, we include a list of these results. We cover binary matroids here in such detail for two reasons. First, we want to exhibit how features and properties of matroids are motivated by elementary linear algebra arguments. Binary matrices and matroids are the perfect vehicle to display that relationship. Second, the techniques and arguments of this section are used in much more complicated settings in subsequent chapters. The discussion of this chapter thus sets the stage and prepares the reader for that material.

We proceed as follows. We define the binary matroids via binary matrices, along with fundamental concepts such as base, circuit, cobase, cocircuit, rank function, and representation matrix. We introduce matroid minors via the reduction operations of deletion and contraction and explain the effect these operations have on bases, circuits, cobases, and cocircuits.

Next, we describe (Tutte) k-separations and k-connectivity. We link these concepts to the related ones for graphs. At that time, we are ready for a census of the 3-connected binary matroids with at most eight elements.

In this book, the presentation relies heavily on what we call the matrix viewpoint of binary matroids. In the remainder of the section, we show that other viewpoints are just as important. In particular, we introduce the submodularity of the rank function and prove with that concept a basic 3-connectivity result. We now begin the detailed presentation.

Binary Matroid

Let F be a binary matrix with a column index set E. Let $\mathcal{I}$ be the collection of subsets $Z \subseteq E$ such that the column submatrix of F indexed by Z has GF(2)-independent columns. We consider $Z = \emptyset$ to be in $\mathcal{I}$. The sets Z of $\mathcal{I}$ are *independent*. Declare $M = (E, \mathcal{I})$ to be the *binary matroid* generated by F. The set E is the *groundset* of the matroid. A *base* X of M is a maximum cardinality subset of $\mathcal{I}$. Equivalently, X indexes the columns of a GF(2)-basis of F. A *circuit* C of M is a minimal subset of E that is not contained in any base of M. Equivalently, C indexes a GF(2)-mindependent column submatrix of F. A *cobase* Y of M is the set $E - X$ for some base X. A *cocircuit* C^* of M is a minimal subset of E that is not contained in any cobase of M. The *rank* of a subset $Z \subseteq E$, denoted by $r(Z)$, is the cardinality of a maximal independent subset contained in Z. The function $r(\cdot)$ is the *rank function* of M. Collect in a set $\mathcal{I}^*$ all cobases Y of M and all their subsets. The sets Z^* of $\mathcal{I}^*$ are *co-independent*. The pair $M^* = (E, \mathcal{I}^*)$ is the *dual matroid* of M.

Suppose M is the graphic matroid of a connected graph G with edge set E. Then a base (resp. a circuit, a cobase, a cocircuit, the rank function) of M is a tree (resp. a cycle, a cotree, a cocycle, the rank function) of G.

Representation Matrix

Suppose a matrix A is deduced from F by elementary row operations. Clearly, GF(2)-independence of columns is not affected by such a change. Thus, we may determine $\mathcal{I}$ from A instead of F. A special case is as follows. First, we delete GF(2)-dependent rows from F, getting, say, F'. Second, we perform binary row operations to convert the column submatrix of F' indexed by some base X to an identity. With $Y = E - X$, we thus have for some binary matrix B,

(3.3.1)
$$A = \begin{array}{c|c|c|} & X & Y \\ \hline X & I & B \\ \hline \end{array}$$

Matrix A for matroid M with base X

We allow the special cases $X = \emptyset$ or $Y = \emptyset$. B is then a trivial or empty matrix. The information contained in A is also conveyed by the submatrix B of A, which has the form

(3.3.2)
$$\begin{array}{c|c|} & Y \\ \hline X & B \\ \hline \end{array}$$

Matrix B for matroid M with base X

The binary B is a *representation matrix* of M. We also say that B *represents* M over GF(2). In the literature, the term *standard representation* *matrix* is sometimes used for A or B. The matrix F is then a *nonstandard representation matrix*. But we almost always work with B, so the abbreviated terminology suffices.

Bases, circuits, and the rank function of M manifest themselves in B as follows. For any partition of X into X_1 and X_2 and of Y into Y_1 and Y_2, we assume B to be partitioned as

(3.3.3)
$$B = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & B^1 & D^2 \\ \hline X_2 & D^1 & B^2 \\ \hline \end{array}$$

Partitioned version of B

Base, Rank Function

A set $Z \subseteq E$ is a base of M if and only if $X_2 = Z \cap X$ and $Y_1 = Z \cap Y$ induce a partition in B where B^1 is square and GF(2)-nonsingular. More generally, let Z be an arbitrary subset of E. Then B^1 defined via $X_2 = Z \cap X$ and $Y_1 = Z \cap Y$ has GF(2)-rank k if and only if Z has rank $r(Z) = |X_2| + k$ in M.

Circuit

Let $C \subseteq E$. Define $X_2 = C \cap X$ and $Y_1 = C \cap Y$. Then C is a circuit of M if and only if in B of (3.3.3), the number of 1s in the rows of B^1 (resp. D^1) is even (resp. odd), and for any proper subset $C' \subset C$, the corresponding $(B^1)'$ and $(D^1)'$, defined by $X_2' = C' \cap X$ and $Y_1' = C' \cap Y$, do not satisfy that parity condition. Note that $Y_1 = C \cap Y$ and the parity condition uniquely determine X_2, and thus B^1 and D^1.

Recall that a $\{0, 1\}$ matrix is column Eulerian if each row contains an even number of 1s, or equivalently, if the columns sum (in GF(2)) to 0. By the above discussion, any circuit of M indexes a column submatrix of $A = [I \mid B]$ that is column Eulerian. Conversely, any column submatrix of A that is column Eulerian and that does not contain a proper column submatrix with that property corresponds to a circuit of M.

We describe three special cases of circuits using the above notation.

Loop

Suppose $|C| = 1$, say $C = \{y\}$. The element y is a *loop* of M. Necessarily, $Y_1 = \{y\}$, and column y of B must be a zero vector. Conversely, any zero column vector of B corresponds to a loop.

Parallel Elements, Triangle

Suppose $|C| = 2$. The two elements of C, say y and z, are said to be *parallel*. Both elements of C cannot be in X, so we may assume $y \in Y$. Two cases are possible. We have either $X_2 = \{z\}$ and $Y_1 = \{y\}$, or $X_2 = \emptyset$ and $Y_1 = \{y, z\}$. In the first case, column y of B is a unit vector with 1 in row z. In the second case, columns y and z of B are parallel. Conversely, a column unit vector or two parallel columns of B correspond to two parallel elements of M. If $|C| = 3$, then C is a *triangle*.

Fundamental Circuit

Suppose $|C| \geq 2$ and $|Y_1| = 1$, say $Y_1 = \{y\}$. Then X_2 is the index set of the rows of B with 1s in column y, and $C = X_2 \cup \{y\}$. The circuit C is called the *fundamental circuit* the element y forms with the base X of M. The fundamental circuits C_y that the elements $y \in Y$ form with X allow a fast construction of B. Indeed, each column $y \in Y$ of B is the characteristic vector of $C_y - \{y\}$. That is, column y has 1s in the rows indexed by $C_y - \{y\}$, and 0s elsewhere.

Cobase, Cocircuit

Cobases and cocircuits of M are exhibited by B as follows. Let $Z \subseteq E$. As before, define $X_2 = Z \cap X$, $X_1 = X - X_2$, $Y_1 = Z \cap Y$, and $Y_2 = Y - Y_1$. By the earlier definition of base and cobase, the set Z is a base of M and $Z^* = E - Z = X_1 \cup Y_2$ is a cobase of M if and only if in the transpose of B,

(3.3.4)
$$B^t = \begin{array}{c|c|c|} & X_1 & X_2 \\ \hline Y_1 & (B^1)^t & (D^1)^t \\ \hline Y_2 & (D^2)^t & (B^2)^t \\ \hline \end{array}$$

Transpose of B of (3.3.3)

the submatrix $(B^1)^t$ is square and GF(2)-nonsingular. Append to B^t an identity matrix, getting

(3.3.5)
$$A^* = \begin{array}{c|c|c|} & Y & X \\ \hline Y & I & B^t \\ \hline \end{array}$$

B^t with additional identity matrix

Evidently, Z^* is a cobase of M if and only if Z^* indexes the columns of a GF(2)-basis of A^*.

Let $C^* \subseteq E$, $X_1 = C^* \cap X$, $X_2 = X - X_1$, $Y_2 = C^* \cap Y$, and $Y_1 = Y - Y_2$. Then C^* is a cocircuit of M if and only if the earlier described circuit condition holds for C^* and B^t instead of C and B. We describe three special cases in terms of B.

Coloop

Suppose $|C^*| = 1$, say $C^* = \{x\}$. The element x is a *coloop* of M. Necessarily, $X_1 = \{x\}$, and row x of B must be a zero vector. Conversely, any zero row vector of B corresponds to a coloop.

Coparallel or Series Elements, Triad

Suppose $|C^*| = 2$. The two elements of C^*, say x and z, are said to be *coparallel* or *in series*. Both elements cannot be in Y, so we may assume $x \in X$. Two cases are possible. We have either $X_1 = \{x\}$ and $Y_2 = \{z\}$, or $X_1 = \{x, z\}$ and $Y_2 = \emptyset$. In the first case, row x of B is a unit vector with 1 in column z. In the second case, rows x and z of B are parallel. Conversely, a row unit vector or two parallel rows of B correspond to two series elements of M. If $|C^*| = 3$, then C^* is a *triad*.

Fundamental Cocircuit

Suppose $|C^*| \geq 2$, and $|X_1| = 1$, say $X_1 = \{x\}$. Then Y_2 is the index set of the columns of B with 1s in row x, and $C^* = Y_2 \cup \{x\}$. The cocircuit C^* is called the *fundamental cocircuit* the element x forms with the cobase Y of M. Analogously to the circuit case, the fundamental cocircuits permit a fast construction of B.

Binary Spaces

We relate circuits and cocircuits of M to binary spaces on E, i.e., to the linear subspaces of $\mathrm{GF}(2)^E$. Specifically, consider the nullspace S of A of (3.3.1), which is given by $\{s \mid A \cdot s = 0\}$. Evidently, the vectors $s \in S$ with minimal support are exactly the characteristic vectors of the circuits of M. Any basis of these vectors generates S.

By definition of M^* from A^* of (3.3.5), the minimal support vectors of the nullspace $S^* = \{s^* \mid A^* \cdot s^* = 0\}$ of A^* are exactly the characteristic vectors of the cocircuits of M^*.

So, in a way, the circuits of M generate S. Correspondingly, the cocircuits produce S^*, which is well known (and easily proved) to be the orthogonal complement of S.

Intersection of Circuits and Cocircuits

Frequently, a judicious choice of the base X and the cobase Y of M produces a B that simplifies the proof of some result. Equivalently, we may want to proceed by GF(2)-pivots from a given representation matrix of M to some other one to exhibit a particular aspect of M. The pivots were covered in Section 2.3, and also in (3.2.12). Thus, we omit details about that operation.

Here is an example result that is easily proved with a clever choice of X and Y. That result ties circuits to cocircuits by a parity condition.

(3.3.6) Lemma. *Let C (resp. C^*) be a circuit (resp. cocircuit) of a binary matroid M. Then $|C \cap C^*|$ is even.*

Proof. Choose a cobase Y that contains all elements of C^* save one, say x. Let B be the related matrix. Thus, row x of B is the characteristic vector of $C^* - \{x\}$. If $x \in C$ (resp. $x \notin C$), then by the previously described parity condition for circuits, row x of B has an odd (resp. even) number of 1s in the columns indexed by $C \cap Y$. Either case proves $|C \cap C^*|$ to be even. □

Of course, Lemma (3.3.6) also follows trivially from the just-cited orthogonality of the binary spaces S and S^* produced by the circuits and cocircuits, respectively, of M.

Symmetric Difference of Circuits

We should mention the following result about circuits.

(3.3.7) Lemma. *Let C_1 and C_2 be two circuits of a binary matroid M. Then $(C_1 \cup C_2) - (C_1 \cap C_2)$, the symmetric difference of C_1 and C_2, is a disjoint union of circuits of M.*

Proof. Given B, define $A = [I \mid B]$. As observed earlier, the column submatrix of A indexed by C_1 (resp. C_2) is column Eulerian. The same fact holds for the column submatrix indexed by $Z = (C_1 \cup C_2) - (C_1 \cap C_2)$. Thus, the columns indexed by Z are GF(2)-dependent. Hence, Z contains a circuit C. Then $Z - C$ is also column Eulerian, and the desired conclusion follows by induction. □

Note that the proof of Lemma (3.3.7) remains valid when each of C_1 and C_2 is a disjoint union of circuits. Thus, we have the following seemingly more general result.

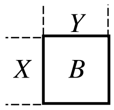
(3.3.8) Lemma. *Let each of C_1 and C_2 be a disjoint union of circuits of a binary matroid M. Then the symmetric difference of C_1 and C_2 is a disjoint union of circuits of M.*

Deletion, Contraction, Reduction

We turn to *deletions* and *contractions* of binary matroids. Any such operation is a *reduction.*

The operations are defined as follows. Let B be the matrix of (3.3.2) representing M. Thus, B is

(3.3.9)



Matrix B for matroid M with base X

A *deletion* of an element $w \in E$ from M leads to a matroid $\overline{M}$ represented as follows. If w is not a coloop of M, select any representation matrix B of M where $w \in Y$. Delete column w from B. The resulting matrix $\overline{B}$ represents $\overline{M}$. It is easy to verify that the same $\overline{M}$ results, regardless of which specific B is selected to determine $\overline{B}$. The proof consists of showing that all possible $\overline{B}$ are obtainable from each other by GF(2)-pivots. If w is a coloop of M, we declare the deletion to be a contraction, which is covered next. A *contraction* of an element $u \in E$ in M leads to a matroid $\overline{M}$ represented as follows. If u is not a loop of M, select any representation matrix B of M where $u \in X$. Delete row u from B. The resulting matrix $\overline{B}$ represents $\overline{M}$. Here, too, the outcome does not depend on the selection of the specific B. If u is a loop, declare the contraction to be a deletion.

Even after the discussion of deletions and contractions for graphs and graphic matroids in Sections 2.2 and 3.2, the reader may still be a bit puzzled that we have declared the deletion of a coloop of M to be a contraction, and the contraction of a loop of M to be a deletion. Below, we motivate these rules using the matrix $A = [I \mid B]$. Suppose we intend to delete an element w from M. Our goal is to transform A to a matrix $\overline{A} = [I \mid \overline{B}]$ so that the index sets of independent column submatrices of A are precisely the index sets of independent column submatrices of A that do not include w. Note that this goal is a generalization of the goal for edge deletions in graphs, as covered in Section 2.2. There we relied on tree subsets instead of independent column submatrices. It turns out that we must consider three cases of A to determine the desired $\overline{A}$. In the first case, w indexes a column of B. Then deletion of column w from A, and thus from B, gives the desired $\overline{A}$ and $\overline{B}$. This case is covered by the deletion rule given above. In the second case, w indexes a column of the identity I, and row w of B is nonzero. Then by row operations in A, or equivalently by a GF(2)-pivot in B, we achieve the first case, again in conformance with the deletion rule given above. The third case is like the second one, except that row w of B is zero. Note that w is then a coloop of M. Suppose we delete column w

from A. Then row w of the resulting matrix A' is zero and has no influence on the independence of column subsets of A'. Thus, we might as well drop that row from A'. But these two steps, deletion of column w from A followed by deletion of row w, correspond precisely to the deletion of row w from B, and thus to the contraction of the coloop w. The situation for the contraction of a loop may be explained in the same manner using $A^* = [I \mid B^t]$ instead of A.

Uniqueness of Reductions

Let U and W be two disjoint subsets of E. Suppose we contract the elements of U and delete the elements of W. In a moment, we outline a proof showing that the outcome is not affected by the order in which the reductions are carried out. Assuming that result, we are justified in denoting the resulting unique matroid by $M/U\backslash W$. Any such matroid is a *minor* of M. For convenience, we consider M itself to be a minor of M. Analogously to the case of $G/U\backslash W$ in Section 2.2, we may simplify the notation for $M/U\backslash W$. Thus, we may write M/U, $M\backslash W$, M/u when $U = \{u\}$, etc.

That the order of the reductions is irrelevant may be shown by induction as follows: One reverses the sequence of two successive reduction steps and proves that this change has no effect on the outcome. We leave it to the reader to carry out the elementary case analysis. Note that the preceding uniqueness result for reduction sequences is at variance with the situation for graphs. According to Section 2.2, the ordering of reduction sequences does matter when graphs are involved. Indeed, in that section we introduced a technical device to enforce a certain ordering. We have just seen that the sequence of the reductions is irrelevant in binary matroids, and thus in graphic matroids. This implies that the graphs producible by differing sequences must all correspond to the same matroid minor. Thus, all such graphs are 2-isomorphic. In this book, we almost always look at graphs from the matroid standpoint, and can afford to ignore differences between 2-isomorphic graphs. Indeed, any one graph of a collection of 2-isomorphic candidates may be used to carry out proofs or to develop ideas about matroids. These facts motivated our choice of the technical device of Section 2.2.

Circuit/Cocircuit Condition

Analogously to the case for graphs, for any minor $\overline{M}$ of M, there are disjoint $U, W \subseteq E$ such that U contains no circuit, W contains no cocircuit, and $\overline{M} = M/U\backslash W$. We say that such U and W satisfy the *circuit/cocircuit condition*. The proof of the existence of U and W is almost trivial. We know that a representation matrix for $\overline{M}$ can be deduced from one for M by a sequence of operations, each of which is a GF(2)-pivot or the deletion of a row or column. We could perform all GF(2) pivots initially, then carry out the deletion of rows and columns. Let U be the index set of the rows so deleted, and W be that of the columns. Clearly, U and W satisfy the circuit/cocircuit condition, and $\overline{M} = M/U\backslash W$ as desired.

Bases, Cobases, Circuits, and Cocircuits of Minors

Sometimes, it is convenient to assume the circuit/cocircuit condition. An instance comes up next. We want to express bases, cobases, circuits, and cocircuits of a minor $\overline{M} = M/U\backslash W$ in terms of the related sets of M. We claim that the formulation below suffices if U and W satisfy the circuit/cocircuit condition.

(3.3.10)
The set of bases of $\overline{M} = M/U\backslash W$ is
$\{X - U \mid U \subseteq X \subseteq E - W;\ \ X = \text{base of } M\}$;
the set of cobases of $\overline{M}$ is
$\{X^* - W \mid W \subseteq X^* \subseteq E - U;\ \ X^* = \text{cobase of } M\}$

(3.3.11)
The set of circuits of $\overline{M} = M/U\backslash W$ consists of
the minimal members of
$\{C - U \mid C \subseteq E - W;\ C = \text{ circuit of } M\}$;
the set of cocircuits of $\overline{M}$ consists of
the minimal members of
$\{C^* - W \mid C^* \subseteq E - U;\ C^* = \text{ cocircuit of } M\}$

Validation of (3.3.10) and (3.3.11) is not difficult. By the circuit/cocircuit condition on U and W, contraction of U and deletion of W may be translated to submatrix-taking in some matrix B of (3.3.9) representing M. Thus, one only needs to examine the effect of such submatrix-taking on bases, cobases, circuits, and cocircuits. We omit the arguments since they closely follow the presentation of Section 3.2 about minors of graphic matroids.

Display of Minor, Visible Minor

Let M with representation matrix B have $\overline{M}$ as a minor. Suppose $\overline{B}$ represents $\overline{M}$. If $\overline{B}$ is a submatrix of B, then we say that B *displays* $\overline{M}$ *via* $\overline{B}$, or more briefly, that B *displays* $\overline{M}$. Note that the submatrix of B claimed to be $\overline{B}$ must match not only the numerical entries, but also the row and column index sets of $\overline{B}$. We also say that the minor $\overline{M}$ is *visible* by the display of $\overline{B}$ in B. By the definition of minor via pivots and row/column deletions, we have the following simple but useful lemma.

(3.3.12) Lemma. *Let M be a binary matroid with a minor $\overline{M}$, and $\overline{B}$ be a representation matrix of $\overline{M}$. Then M has a representation matrix B that displays $\overline{M}$ via $\overline{B}$ and thus makes the minor $\overline{M}$ visible.*

Two special cases of Lemma (3.3.12) involve so-called contraction and deletion minors, to be discussed next.

Contraction Minor

Define a minor $\overline{M}$ of M on a set E to be a *contraction minor* of M if for some $U \subseteq E$, $\overline{M} = M/U$. If M is the graphic matroid of a graph G, then any minor of G corresponding to a contraction minor of M is called a *contraction minor* of G. The next lemma shows how a contraction minor $\overline{M}$ manifests itself in representation matrices of M displaying $\overline{M}$.

(3.3.13) Lemma. *The following statements are equivalent for any binary matroid M and any minor $\overline{M}$ of M. Let $\overline{B}$ be a representation matrix of $\overline{M}$.*

(i) *$\overline{M}$ is a contraction minor of M.*
(ii) *M has a representation matrix B displaying $\overline{M}$ via $\overline{B}$, where B is of the form*

(3.3.14)
$$B = \begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\longleftarrow Y \longrightarrow} \\ & & \overline{Y} & W \\ \hline & \overline{X} & \overline{B} & 0 \\ \cline{2-4} X & U & \multicolumn{2}{c|}{0/1} \\ \hline \end{array}$$

Matrix B displaying contraction minor $\overline{M}$

(iii) *Every representation matrix B of M displaying $\overline{M}$ via $\overline{B}$ is of the form given by (3.3.14).*

Proof. Assuming (i), we deduce (iii) as follows. Let B display $\overline{M}$ via $\overline{B}$. Thus, B is of the form given by (3.3.14), except that possibly the submatrix of B indexed by $\overline{X}$ and W may not be zero. Assume that submatrix to be nonzero. Now $\overline{M} = M/(U \cup W)$ by assumption. Then by the rule for contractions, a matrix for $\overline{M}$ is produced from B by deletion of the rows of U, followed by one or more pivots and deletion of one or more rows, and finally by deletion of some zero columns. But the resulting matrix has fewer rows than $\overline{B}$ and cannot represent $\overline{M}$, a contradiction. Trivially, (iii) implies (ii). Finally, the contraction rule proves that (ii) implies (i). □

Suppose we have a representation matrix B of a binary matroid M. Assume that B displays a minor $\overline{M}$ of M. Then parts (ii) and (iii) of Lemma (3.3.13) provide a simple way of ascertaining whether or not $\overline{M}$ is a contraction minor of M. In the notation of (3.3.14), the answer is "yes" if and only if the submatrix of B indexed by $\overline{X}$ and W is 0.

Deletion Minor

A minor $\overline{M} = M \backslash W$ is a *deletion minor* of M. Correspondingly to the contraction case, we define *deletion minors* of a graph G via those of the graphic matroid of G. By duality, Lemma (3.3.13) implies the following result.

(3.3.15) Lemma. *The following statements are equivalent for any binary matroid M and any minor $\overline{M}$ of M. Let $\overline{B}$ be a representation matrix of $\overline{M}$.*

(i) *$\overline{M}$ is a deletion minor of M.*

(ii) *M has a representation matrix B displaying $\overline{M}$ via $\overline{B}$, where B is of the form*

$$(3.3.16) \qquad B = X \begin{array}{c|c|c} & \overbrace{\overline{Y} \quad W}^{Y} & \\ \hline \overline{X} & \overline{B} & \multirow{2}{*}{0/1} \\ \cline{1-2} U & 0 & \end{array}$$

Matrix B displaying deletion minor $\overline{M}$

(iii) *Every representation matrix B of M displaying $\overline{M}$ via $\overline{B}$ is of the form given by* (3.3.16).

Addition, Expansion, Extension

Addition and *expansion* are the inverse operations of deletion and contraction. Thus, an addition (resp. expansion) corresponds to the adjoining of a column (resp. row) to a given B. An *extension* is an addition or expansion. We denote additions by "+" and expansions by "&." For example, a matrix for $\overline{M}\&\overline{U}+\overline{W}$ is obtained from one for $\overline{M}$ by adjoining rows indexed by $\overline{U}$ and columns indexed by $\overline{W}$. Suppose the matroid so created is a minor of some other matroid. Then without further specification, the entries in the added rows and columns are well defined. For example, suppose $\overline{M} = M/U \backslash W$ where U and W observe the circuit/cocircuit condition. Let $\overline{U} \subseteq U$ and $\overline{W} \subseteq W$. Then $\overline{M}\&\overline{U}+\overline{W}$ is taken to be $M/(U-\overline{U})\backslash(W-\overline{W})$. We use a simplified notation for extensions analogously to that for reductions. Thus, we may write $M\&U$, $M+W$, $M\&u$ when $U = \{u\}$, etc.

Deletion, Contraction, Addition, Expansion in Dual Matroid

By definition, deletions (resp. contractions) of M correspond to the removal of columns (resp. rows) from an appropriately chosen B. Recall that

B^t represents M^*. Thus, deletions (resp. contractions) of M correspond to contractions (resp. deletions) in M^*. Put differently, for any disjoint subsets U and W of E, $(M/U\backslash W)^* = M^*/W\backslash U$. Furthermore, additions (resp. expansions) in M correspond to expansions (resp. additions) in M^*.

Matroid Isomorphism

Two matroids are *isomorphic* if they become equal upon a suitable relabeling of the elements. Analogously to the case of graph minors, a given matroid $\overline{M}$ may be a minor of a matroid M, or may only be isomorphic to a minor of M. In the first situation, we say, as expected, that $\overline{M}$ is a minor of M, or that M has $\overline{M}$ as a minor. But in the second case, we frequently abbreviate "M has a minor isomorphic to $\overline{M}$" to "M has an $\overline{M}$ minor."

k-Separation and k-Connectivity

We turn to separations and the connectivity of M. Let B be partitioned as in (3.3.3), i.e.,

(3.3.17)
$$B = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & B^1 & D^2 \\ \hline X_2 & D^1 & B^2 \\ \hline \end{array}$$

Partitioned version of B

If for some $k \geq 1$,

(3.3.18)
$$\begin{aligned} &|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq k \\ &\text{GF(2)-rank } D^1 + \text{GF(2)-rank } D^2 \leq k-1 \end{aligned}$$

then $(X_1 \cup Y_1, X_2 \cup Y_2)$ is a *Tutte k-separation*, for short *k-separation*, of B and M. The k-separation is *exact* if the rank condition of (3.3.18) holds with equality. B and M are *Tutte k-separable*, for short *k-separable*, if they have a k-separation. For $k \geq 2$, B and M are *Tutte k-connected*, for short *k-connected*, if they have no l-separation for $1 \leq l < k$. When M is 2-connected, we also say that M is *connected*. Let M be the graphic matroid of a graph G. If M is connected (i.e., 2-connected), then by Corollary (3.2.29), G is 2-connected. Thus, connectedness of M is *not* the same as connectedness of G. We mentioned this problem previously in Section 3.2.

The two connectivity corollaries (3.2.31) and (3.2.32) for graphs and graphic matroids have easy extensions to the general binary case, as follows.

(3.3.19) Lemma. *Let M be a binary matroid with a representation matrix B. Then M is connected if and only if this is so for B.*

Proof. Via (3.3.17) and (3.3.18), it is easily checked that B is connected if and only if it is 2-connected. Thus, M is 2-connected, and hence connected, if and only if B is connected. □

(3.3.20) Lemma. *The following statements are equivalent for a binary matroid M with set E and a representation matrix B of M.*

(i) *M is 3-connected.*

(ii) *B is connected, has no parallel or unit vector rows and columns, and has no partition as in* (3.3.17) *with* GF(2)-rank $D^1 = 1$, $D^2 = 0$, *and* $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq 3$.

(iii) *Same as* (ii), *but* $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq 5$.

Proof. (i) $\Longleftrightarrow$ (ii) follows directly from the definition of 3-connectivity, and (ii) $\Longrightarrow$ (iii) is trivial. Finally, (iii) $\Longrightarrow$ (ii) is established as follows. In B of (3.3.17), assume GF(2)-rank $D^1 = 1$ and $D^2 = 0$. If the length of B^1 is 3 or 4, then B can be seen to have a zero column or row, or parallel or unit vector rows or columns. Any such case is already excluded by the first part of (ii). Thus, it suffices to require $|X_1 \cup Y_1| \geq 5$, and by duality, $|X_2 \cup Y_2| \geq 5$. □

Census of Small Binary Matroids

It is instructive that we include a census of small 3-connected binary matroids, say on $n \leq 8$ elements. In that census, we refer to graphic and cographic matroids and their graphic and cographic representation matrices. We have discussed such matroids in detail in Section 3.2. We also refer to regular and nonregular matroids, which are defined via a property of real matrices termed total unimodularity. Let us take up the latter concepts one by one. A real matrix A is *totally unimodular* if every square submatrix D of A has $\det_{\mathbb{R}} D = 0$ or ± 1. In particular, all entries of a totally unimodular matrix must be 0 or ± 1. A binary matroid M is *regular* if in some binary representation matrix B of M the 1s can be so signed so that a $\{0, \pm 1\}$ real totally unimodular matrix results.

We shall motivate these definitions in Chapter 9. For the time being, we just state without proof two of the many important properties of regular matroids. First, a binary matroid M is regular if and only if every representation matrix B can be signed to become a real totally unimodular matrix. Second, every graphic or cographic matroid is regular. Because of the first property, it makes sense that we define a binary matrix to be *regular* if its 1s can be signed so that a real totally unimodular matrix results. That property also implies that the dual matroid and every minor of a regular matroid are regular. Finally, we know from Section 3.2 that graphicness and cographicness are maintained under minor-taking.

Here is the promised census of the 3-connected binary matroids with $n \leq 8$ elements. It may be verified by straightforward enumeration of cases.

$n = 0$: M is the empty matroid.

$n = 1$: M consists of a loop or a coloop; the representation matrix B is the 0×1 or 1×0 trivial matrix, respectively.

$n = 2$: M consists of two parallel elements. We may also consider the two elements to be in series. At any rate, $B = [\,1\,]$.

$n = 3$: M is a triangle, i.e., a circuit with three elements, or a triad, i.e., a cocircuit with three elements. In the first case, $B = [\,1/1\,]$. In the second case, $B = [\,1\ 1\,]$.

$n = 4, 5$: There is no 3-connected binary matroid with four or five elements.

$n = 6$: M is the graphic matroid $M(W_3)$, where W_3 is the wheel with three spokes. Up to pivots, B is the matrix

(3.3.21)
$$\begin{array}{|ccc|} \hline 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ \hline \end{array}$$

Matrix representing graphic matroid $M(W_3)$

Note that up to this point, all matroids have been graphic and cographic.

$n = 7$: M is the *Fano matroid* F_7 given by

(3.3.22)
$$B^7 = \begin{array}{|cccc|} \hline 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ \hline \end{array}$$

Matrix representing Fano matroid F_7

or its dual F_7^* given by $(B^7)^t$. We claim that F_7, and hence F_7^*, are not regular. Indeed, if B^7 is signed so that all 2×2 submatrices have $\{0, \pm 1\}$ real determinants, then up to column and row scaling, B^7 is already that matrix. But the first three columns of B^7 define a 3×3 matrix with real determinant equal to 2. The name is based on the fact that the matroid is the Fano plane, which is the projective geometry PG(2,2). The seven elements of the matroid are the points of the geometry.

$n = 8$: If M is regular, then M is the graphic matroid of $M(W_4)$, where W_4 is the wheel with four spokes. Up to pivots, B is the matrix

$$\boxed{\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}} \tag{3.3.23}$$

Matrix representing graphic matroid $M(W_4)$

If M is nonregular, then M is one of two matroids given by

$$B^{8.1} = \boxed{\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}} \tag{3.3.24}$$

Matrix $B^{8.1}$ representing nonregular matroid with eight elements, case 1

and

$$B^{8.2} = \boxed{\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{array}} \tag{3.3.25}$$

Matrix $B^{8.2}$ representing nonregular matroid with eight elements, case 2

The matroid represented by $B^{8.2}$ is the affine geometry AG(2,3), which has eight points corresponding to the eight binary vectors with three entries each. A subset of the points is (affinely) GF(2)-mindependent if the corresponding subset of vectors has even cardinality and is linearly GF(2)-dependent, and if it is minimal with respect to these two conditions. For that matroid, contraction (resp. deletion) of any element produces the nonregular matroid F_7 (resp. F_7^*) as a minor. Thus, that matroid is highly nonregular. In contrast, deletion of any row or column from $B^{8.1}$ produces a matrix that represents a regular matroid.

So far, we have stressed what one might call the *matrix viewpoint* of binary matroids. As we shall see in Section 3.4, that notion can be extended to general matroids using matrices termed *abstract*. The main advantage of that notion is the fact that a binary matrix (or in general, an abstract matrix) displays numerous bases, circuits, cobases, and cocircuits of the matroid simultaneously. For this reason, we view matrices (binary, or over other fields, or abstract) to be an important tool of matroid theory.

There are, however, other and equally important viewpoints. One of them considers the rank function $r(\cdot)$ of M as a main tool. Recall that this function was defined via B of (3.3.17) as follows. For any subset $Z \subseteq E$, let $X_2 = Z \cap X$ and $Y_1 = Z \cap Y$. By (3.3.17), $X_1 = X - X_2$ and Y_1 index the submatrix B^1 of B. Then declare the rank of Z to be $r(Z) = |X_2| + \text{GF}(2)\text{-rank } B^1$. The utility of the rank function largely rests upon a property called *submodularity.* Section 2.3 includes a definition of submodularity for functions defined on matrices. There it is shown that the matrix rank function is submodular. Shortly, we define submodularity for functions that map the subsets of a set E to the nonnegative integers. We then prove the matroid rank function to be submodular.

The matrix viewpoint, the rank function viewpoint, as well as several others not mentioned so far (e.g., the geometric viewpoint, the lattice viewpoint — see the books cited in Chapter 1), have advantages and disadvantages. Intuitively speaking, matrices very conveniently display bases, circuits, cobases, and cocircuits that differ just by a few elements. On the other hand, the relationships between radically different bases, circuits, etc., are not well exhibited. The rank function approach, as well as several others, treats all such cases evenly. Indeed, for the solution of several problems involving radically different bases, circuits, etc., the rank function seems particularly suitable.

The results described in this book rely largely on the matrix viewpoint. But the reader should not be misled by this fact. It just turns out that the results of this book are nicely treatable by matrices. But there are a number of problems of matroid theory where other approaches, in particular ones relying on the rank function, are superior to the matrix technique employed here. Below, we describe two simple instances that exhibit the power of the rank function approach. But first we express the defining k-separation conditions of (3.3.18) in terms of $r(\cdot)$.

k-Separation Condition for Rank Function

(3.3.26) Lemma. *Let M be a binary matroid on a set E and with rank function $r(\cdot)$. Suppose E_1 and E_2 partition E. Then* (a) *and* (b) *below hold.*

(a) *(E_1, E_2) is a k-separation of M if and only if*

$$\begin{aligned} &|E_1|, |E_2| \geq k \\ &r(E_1) + r(E_2) \leq r(E) + k - 1 \end{aligned} \qquad (3.3.27)$$

(b) (E_1, E_2) *is an exact* k*-separation of* M *if and only if*

$$(3.3.28) \qquad \begin{aligned} &|E_1|, |E_2| \geq k \\ &r(E_1) + r(E_2) = r(E) + k - 1 \end{aligned}$$

Proof. (a) To establish the "only if" part, take any representation matrix B of M, say indexed by X and Y as before. Define for $i = 1, 2$, $X_i = E_i \cap X$ and $Y_i = E_i \cap Y$. Let these sets partition B as in (3.3.17). Since $|E_1|, |E_2| \geq k$, we have $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq k$. By the definition of $r(\cdot)$, we have for $i = 1, 2$, $r(E_i) = |X_i| + \text{GF(2)-rank } D^i$, and $r(E) = |X_1| + |X_2|$. Since $r(E^1) + r(E^2) \leq r(E) + k - 1$, we have $|X_1| + \text{GF(2)-rank } D^1 + |X_2| + \text{GF(2)-rank } D^2 \leq |X_1| + |X_2| + k - 1$, or $\text{GF(2)-rank } D^1 + \text{GF(2)-rank } D^2 \leq k - 1$. Thus, by (3.3.18), $(X_1 \cup Y_1, X_2 \cup Y_2) = (E_1, E_2)$ is a k-separation of M. For the proof of the "if" part, one reverses the above arguments.

(b) This follows from the proof of (a) by suitable replacement of some inequalities by equations. ☐

We now define the submodularity property and prove that the rank function $r(\cdot)$ has that property.

Submodularity of Rank Function

A function $f(\cdot)$ from the set of subsets of a finite set E to the nonnegative integers is *submodular* if for any subsets $S, T \subseteq E$,

$$(3.3.29) \qquad f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$$

(3.3.30) Lemma. $r(\cdot)$ *is a submodular function.*

Proof. Take any binary representation matrix of M. Let A^1, A^2, A^3, and A^4 be the column submatrices of $A = [I \mid B]$ corresponding to the column index sets S, T, $S \cup T$, and $S \cap T$, respectively. Evidently, A^4 is a submatrix of A^1, A^2, and A^3, and both A^1 and A^2 are submatrices of A^3. Let B^4 be a basis of A^4. For $i = 1, 2, 3$, extend B^4 to a basis of A^i, say by adjoining B^i. Evidently, the number of columns of $[B^4 \mid B^1]$, $[B^4 \mid B^2]$, $[B^4 \mid B^3]$, and B^4 is the matroid rank of S, T, $S \cup T$, and $S \cap T$, respectively. Thus, the submodularity inequality (3.3.29) holds if and only if B^1 and B^2 together have at least as many columns as B^3. The latter condition holds since by the construction, $[B^4 \mid B^3]$ is a basis of A^3, while $[B^4 \mid B^1 \mid B^2]$ spans all columns of A^3. ☐

We use submodularity of $r(\cdot)$ in the proof of the next result. Define Mⓒz to be the matroid obtained from M/z by deletion of the elements of each parallel class except for one representative of each class. Similarly, derive Mⓓz from $M \backslash z$ by contracting the elements of each series class except for one representative of each class.

(3.3.31) Lemma. *Let M be a 3-connected binary matroid on a set E. Take z to be any element of E. Then $M \textcircled{c} z$ or $M \textcircled{d} z$ is 3-connected.*

Proof. The arguments below rely repeatedly on three observations that follow directly from (3.3.10) and (3.3.11). First, if a set is not a circuit (resp. cocircuit) of M, then it cannot be a circuit (resp. cocircuit) in any minor derived from M by deletions only (resp. contractions only). In particular, the 3-connected matroid M has no parallel or series elements, and thus, for any element z, the minor $M\backslash z$ (resp. M/z) has no parallel (resp. series) elements. Second, if an element z is not a coloop (resp. loop) of M, then $M\backslash z$ (resp. M/z) has the same rank as M. Third, the rank of a set $Z \subseteq E$ drops at most by 1 when an element $z \notin Z$ is contracted, and stays the same when $z \notin Z$ is deleted.

Assume the lemma fails, i.e., for some 3-connected M and $z \in E$, both $M \textcircled{c} z$ and $M \textcircled{d} z$ are 2-separable. Let (P', Q') be a 2-separation of $M \textcircled{c} z$. We know that no two elements of $M \textcircled{c} z$ are in parallel. Thus, $|P'| \geq 3$ or P' contains two series elements. The obvious assignment of the parallel elements by which M/z and $M \textcircled{c} z$ differ converts (P', Q') to a 2-separation (P, Q) of M/z where $P \supseteq P'$ and $Q \supseteq Q'$. Since M is 3-connected, M/z has no series elements. If $|P'| = 2$ and $P' = P$, then P must be a set of two series elements in M/z, which is impossible. Thus, $|P'| = 2$ implies $|P| \geq 3$. We conclude $|P| \geq 3$ in general, and $|Q| \geq 3$ by symmetry. Using duality, $M \textcircled{d} z$ must have a 2-separation (R, S) with $|R|, |S| \geq 3$.

Assume $|P \cap R| \leq 1$. Then $|P \cap S|, |Q \cap R| \geq 2$. The same conclusion holds if $|Q \cap S| \leq 1$. Thus, we may assume $|P \cap R|, |Q \cap S| \geq 2$ or $|P \cap S|, |Q \cap R| \geq 2$. By symmetry, we may suppose that the former situation holds.

Denote by $r(\cdot)$, $r_{M/z}(\cdot)$, and $r_{M\backslash z}(\cdot)$ the rank functions of M, M/z, and $M\backslash z$, respectively. Since (P, Q) and (R, S) are 2-separations of M/z and $M\backslash z$, respectively, we have

$$
\begin{aligned}
r_{M/z}(P) + r_{M/z}(Q) &\leq r_{M/z}(E - \{z\}) + 1 \\
r_{M\backslash z}(R) + r_{M\backslash z}(S) &\leq r_{M\backslash z}(E - \{z\}) + 1
\end{aligned}
\tag{3.3.32}
$$

Now $r_{M/z}(P) \geq r(P \cup \{z\}) - 1$, $r_{M/z}(Q) \geq r(Q \cup \{z\}) - 1$, and $r_{M/z}(E - \{z\}) = r(E) - 1$. Also, $r_{M\backslash z}(R) = r(R)$, $r_{M\backslash z}(S) = r(S)$, and $r_{M\backslash z}(E - \{z\}) = r(E)$. We use these relationships to deduce from (3.3.32) the inequalities

$$
\begin{aligned}
r(P \cup \{z\}) + r(Q \cup \{z\}) &\leq r(E) + 2 \\
r(R) \qquad + r(S) \qquad &\leq r(E) + 1
\end{aligned}
\tag{3.3.33}
$$

By submodularity, we also have

$$
\begin{aligned}
r(P \cap R) + r(P \cup R \cup \{z\}) &\leq r(P \cup \{z\}) + r(R) \\
r(Q \cap S) + r(Q \cup S \cup \{z\}) &\leq r(Q \cup \{z\}) + r(S)
\end{aligned}
\tag{3.3.34}
$$

Add the two inequalities of (3.3.33). Similarly, add the two inequalities of (3.3.34). The resulting two inequalities imply

(3.3.35)
$$\begin{aligned} &[r(P \cap R) + r(Q \cup S \cup \{z\})] \\ &+ [r(Q \cap S) + r(P \cup R \cup \{z\})] \leq 2r(E) + 3 \end{aligned}$$

But then at least one of the pairs $(P \cap R, Q \cup S \cup \{z\})$ and $(Q \cap S, P \cup R \cup \{z\})$ must be a 2-separation of M, which is not possible. □

Along the same lines, but with much simpler arguments, one can prove the following lemma. We leave the proof to the reader.

(3.3.36) Lemma. *Let M be a connected binary matroid on a set E. Take z to be any element of M. Then M/z or $M \backslash z$ is connected.*

We conclude this section by proving that submodularity of the $r(\cdot)$ function and submodularity of the GF(2)-rank function are equivalent. To show this, we assume S and T to be subsets of E for a binary matroid M with a binary representation matrix B. Let the customary index sets X and Y of B be partitioned into X_0, X_1, X_2, X_3, and Y_0, Y_1, Y_2, Y_3, respectively, so that

(3.3.37)
$$\begin{aligned} X_0 &= X - (S \cup T) \\ X_1 &= X \cap (S - T) \\ X_2 &= X \cap (S \cap T) \\ X_3 &= X \cap (T - S) \\ Y_0 &= Y - (S \cup T) \\ Y_1 &= Y \cap (S - T) \\ Y_2 &= Y \cap (S \cap T) \\ Y_3 &= Y \cap (T - S) \end{aligned}$$

Let the partitions of X and Y induce the following partition of B.

(3.3.38)
$$\begin{array}{cc|c|c|c|c|} & & \multicolumn{4}{c}{Y} \\ & & Y_1 & Y_2 & Y_3 & Y_0 \\ \hline & X_3 & B^{11} & B^{12} & B^{13} & \\ \cline{3-5} X & X_0 & B^{21} & B^{22} & B^{23} & \\ \cline{3-5} & X_1 & B^{31} & B^{32} & B^{33} & \\ \cline{3-5} & X_2 & & & & 0/1 \\ \hline \end{array}$$

Partitioned version of B

Define submatrices D^1, D^2, D^3, D^4 of B as in (2.3.9), i.e.,

(3.3.39)
$$D^1 = \begin{array}{|c|c|}\hline B^{11} & B^{12} \\ \hline B^{21} & B^{22} \\ \hline\end{array} \;;\quad D^2 = \begin{array}{|c|c|}\hline B^{22} & B^{23} \\ \hline B^{32} & B^{33} \\ \hline\end{array}$$

$$D^3 = \begin{array}{|c|c|c|}\hline B^{21} & B^{22} & B^{23} \\ \hline\end{array} \;;\quad D^4 = \begin{array}{|c|}\hline B^{12} \\ \hline B^{22} \\ \hline B^{32} \\ \hline\end{array}$$

Submatrices D^1, D^2, D^3, D^4 of B

By the discussion following (3.3.3), the equations below relate the GF(2)-rank of the D^i to the matroid rank of S, T, $S \cup T$, and $S \cap T$.

(3.3.40)
$$\begin{aligned} r(S) &= |X_1| + |X_2| + \text{GF(2)-rank } D^1 \\ r(T) &= |X_2| + |X_3| + \text{GF(2)-rank } D^2 \\ r(S \cup T) &= |X_1| + |X_2| + |X_3| + \text{GF(2)-rank } D^3 \\ r(S \cap T) &= |X_2| + \text{GF(2)-rank } D^4 \end{aligned}$$

Then clearly the submodularity inequality for $r(\cdot)$, which is $r(S) + r(T) \geq r(S \cup T) + r(S \cap T)$, holds if and only if this is so for the submodularity inequality for the GF(2)-rank function, which is GF(2)-rank D^1 + GF(2)-rank $D^2 \geq$ GF(2)-rank D^3 + GF(2)-rank D^4.

In the next section, we move from binary matrices to abstract ones. Correspondingly, we obtain all matroids instead of just the binary ones.

3.4 Abstract Matrices Produce All Matroids

Binary matrices produce the binary matroids. In a natural extension, we could consider matrices over fields other than GF(2) and the matroids produced by them. We skip that step. Instead, we move in this section directly to abstract concepts of independence, bases, circuits, and rank, and in the process create the entire class of matroids. We also introduce abstract matrices as a generalization of matrices over fields, and we describe some of their features. Routine arguments prove that the abstract matrices generate all matroids, and that the matroids produce all abstract matrices. Indeed, one may view abstract matrices as one way of encoding matroids. Abstract matrices exhibit many features of linear algebra. They also display several properties of matroids rather conveniently, and we have found them to be very useful. In particular, they often help one to detect and prove new structural results that are hidden from view when, instead, one thinks of a matroid as a construction via certain sets, functions, geometries, or operators.

Abstract matrices behave to quite an extent like binary matrices. This fact explains why so many seemingly special results for binary matroids hold for the general case. To support that claim, toward the end of this section we list results of Section 3.3 for binary matroids that, sometimes after an elementary modification, hold for general matroids. We are now ready for the detailed discussion.

Definition of General Matroid

Let E be a set of vectors over some field $\mathcal{F}$. A central result of linear algebra says that for any given subset of vectors of E, the bases of that subset have the same cardinality. We abstract from this fact the axioms for general matroids as follows. A *matroid* M on a *ground set* E is a pair $(E, \mathcal{I})$, where $\mathcal{I}$ is a certain subset of the power set of E. The set $\mathcal{I}$ is the set of *independent* subsets of M. A subset of E that is not in $\mathcal{I}$ is called *dependent*. The set $\mathcal{I}$ must observe the following axioms.

(3.4.1)
- (i) The null set is in $\mathcal{I}$.
- (ii) Every subset of any set in $\mathcal{I}$ is also in $\mathcal{I}$.
- (iii) For any subset $\overline{E} \subseteq E$, the maximal subsets of $\overline{E}$ that are in $\mathcal{I}$ have the same cardinality.

The cardinality of any maximal independent subset of any $\overline{E} \subseteq E$ is called the *rank* of $\overline{E}$. A *base* of M is a maximal independent subset of E. A *circuit* is a minimal dependent subset of E. A *cobase* is the set $E - X$ for some base X. Let $\mathcal{I}^*$ be the set of cobases and their subsets. The pair $M^* = (E, \mathcal{I}^*)$ is a matroid, as is easily checked. It is called the *dual matroid* of M. A *cocircuit* of M is a circuit of M^*.

One can axiomatize matroids in terms of bases, circuits, and other subsets of E, or by certain functions, geometries, and operators. It is usually a simple, though at times tedious, exercise to prove equivalence of these systems. Here we just include the axioms that rely on bases, circuits, and the rank function.

Axioms Using Bases, Circuits, Rank Function

For bases, the axioms are as follows. Let $\mathcal{B}$ be a set of subsets of E. Suppose

$\mathcal{B}$ observes the following axioms.

$$(3.4.2)\qquad \begin{array}{rl} \text{(i)} & \mathcal{B} \text{ is nonempty.} \\ \text{(ii)} & \text{For any sets } B_1, B_2 \in \mathcal{B} \text{ and any } x \in (B_1 - B_2), \\ & \text{there is a } y \in (B_2 - B_1) \text{ such that } (B_1 - \{x\}) \cup \{y\} \\ & \text{is in } \mathcal{B}. \end{array}$$

Then $\mathcal{B}$ is the set of bases of a matroid on E.

Via circuits, we may define matroids as follows. Let $\mathcal{C}$ be the empty set, or be a set of nonempty subsets of E observing the following axioms.

$$(3.4.3)\qquad \begin{array}{rl} \text{(i)} & \text{For any } C_1, C_2 \in \mathcal{C},\ C_1 \text{ is not a proper subset of} \\ & C_2. \\ \text{(ii)} & \text{For any two } C_1, C_2 \in \mathcal{C} \text{ and any } z \in (C_1 \cap C_2), \\ & \text{there is a set } C_3 \in \mathcal{C} \text{ where } C_3 \subseteq (C_1 \cup C_2) - \{z\}. \end{array}$$

Then $\mathcal{C}$ is the set of circuits of a matroid on E.

With the rank function, we specify a matroid as follows. Let $r(\cdot)$ be a function from the power set of E to the nonnegative integers. Assume $r(\cdot)$ satisfies the following axioms for any subsets S and T of E.

$$(3.4.4)\qquad \begin{array}{rl} \text{(i)} & r(S) \leq |S|. \\ \text{(ii)} & S \subseteq T \text{ implies } r(S) \leq r(T). \\ \text{(iii)} & r(S) + r(T) \geq r(S \cup T) + r(S \cap T). \end{array}$$

Then $r(\cdot)$ is the rank function of a matroid on E.

We omit the proofs of equivalence of the systems. It is instructive, though, to express each one of $\mathcal{I}$, $\mathcal{B}$, $\mathcal{C}$, and $r(\cdot)$ in terms of the other ones. We do this next.

Suppose $\mathcal{I}$ is given. Then $\mathcal{B}$ is the set of $Z \in \mathcal{I}$ with maximum cardinality. $\mathcal{C}$ is the set of the minimal $C \subseteq E$ that are not in $\mathcal{I}$. For any $\overline{E} \subseteq E$, $r(\overline{E})$ is the cardinality of a maximal set $Z \subseteq \overline{E}$ that is in $\mathcal{I}$.

Suppose $\mathcal{B}$ is given. Then $\mathcal{I}$ is the set of all $X \in \mathcal{B}$ plus their subsets. $\mathcal{C}$ is the set of the minimal $C \subseteq E$ that are not contained in any $X \in \mathcal{B}$. For any $\overline{E} \subseteq E$, $r(\overline{E})$ is the cardinality of any maximal set $X \cap \overline{E}$ where $X \in \mathcal{B}$.

Suppose $r(\cdot)$ is given. Then $\mathcal{I}$ is the set of $\overline{E} \subseteq E$ where $r(\overline{E}) = |\overline{E}|$. $\mathcal{B}$ is the set of $Z \subseteq E$ for which $|Z| = r(E)$. $\mathcal{C}$ is the set of the minimal $C \subseteq E$ for which $r(C) = |C| - 1$.

Abstract Matrix

We take a detour to introduce *abstract matrices.* We want to acquire a good understanding of such matrices, since they not only represent matroids, but also display a lot of structural information about matroids that other ways do not.

An *abstract matrix* B is a $\{0,1\}$ matrix with row and column indices plus a function called *abstract determinant* and denoted by det. The function det associates with each square submatrix D of the $\{0,1\}$ matrix the value 0 or 1, i.e., $\det D$ is 0 or 1. Note that numerically identical square submatrices with differing row or column index sets may have different determinants. The reader should not be misled by the symbols 0 and 1. Indeed, for the moment, we do not view abstract matrices as part of some algebraic structure. It turns out, though, that 0 and 1 allow a rather appealing use of linear algebra terms. For example, we call D *nonsingular* if $\det D = 1$, and *singular* otherwise.

The function det must obey several conditions. First, if D is the 1×1 matrix $[\,0\,]$ (resp. $[\,1\,]$), then $\det D = 0$ (resp. $\det D = 1$).

Second, for any nonempty submatrix B^1 of B, the maximal nonsingular submatrices must have the same size. This condition may be rephrased as follows. Start with some nonsingular submatrix of B^1. Iteratively add a row and a column such that each time another nonsingular submatrix results. Stop when no further enlargement is possible. The above maximality condition demands that the order of the final nonsingular submatrix is the same regardless of the choice of the initial nonsingular submatrix and of the rows and columns added to it. The order of any such final nonsingular submatrix is called the *rank* of B^1. For the case where B^1 is trivial or empty, we declare rank B^1 to be 0. Upon deletion of a column or row, we demand that the rank drop at most by the rank of that row or column.

Third, the rank function of B must behave much like the rank function of matrices over fields. In particular, for any partition of any submatrix of B of the form

$$\begin{array}{|c|c|c|}\hline B^{11} & B^{12} & B^{13} \\ \hline B^{21} & B^{22} & B^{23} \\ \hline B^{31} & B^{32} & B^{33} \\ \hline \end{array} \tag{3.4.5}$$

Partitioned submatrix of B

the submatrices

$$D^1 = \begin{array}{|c|c|}\hline B^{11} & B^{12} \\ \hline B^{21} & B^{22} \\ \hline \end{array}\ ; \quad D^2 = \begin{array}{|c|c|}\hline B^{22} & B^{23} \\ \hline B^{32} & B^{33} \\ \hline \end{array}$$

$$D^3 = \begin{array}{|c|c|c|}\hline B^{21} & B^{22} & B^{23} \\ \hline \end{array}\ ; \quad D^4 = \begin{array}{|c|}\hline B^{12} \\ \hline B^{22} \\ \hline B^{32} \\ \hline \end{array} \tag{3.4.6}$$

Submatrices D^1, D^2, D^3, D^4

must observe

$$\text{rank } D^1 + \text{rank } D^2 \geq \text{rank } D^3 + \text{rank } D^4. \tag{3.4.7}$$

We call this property the *submodularity* of the rank function.

We summarize the above requirements as follows.

Axioms for Abstract Matrix

(3.4.8)

(i) If $D = [B_{xy}]$, then det $D = B_{xy}$.

(ii) For all submatrices B^1 of B: The maximal nonsingular submatrices of B^1 have the same size, called the rank of B^1. When a row or column is deleted from B^1, the rank drops at most by the rank of that row or column.

(iii) The rank function is submodular.

The *transpose* of an abstract matrix B is B^t with determinants defined as follows. For any square submatrix B^1 of B, det B^1 is the determinant value for the submatrix $(B^1)^t$ of B. By symmetry of the conditions of (3.4.8), B^t with its determinants is an abstract matrix, as expected.

We may create abstract matrices in several ways. In the simplest case, we start with a matrix A over some field $\mathcal{F}$. Then we declare B to be the support matrix of A. Thus, B is a $\{0,1\}$ matrix. We turn B into an abstract matrix as follows. We declare any square submatrix D of B to be nonsingular if the corresponding submatrix of A is $\mathcal{F}$-nonsingular, and to be singular otherwise. Well-known linear algebra results plus Lemma (2.3.11) imply that the axioms of (3.4.8) are satisfied.

Representation of Abstract Matrix

Suppose an abstract matrix B can be derived by the above construction from a matrix A over some field $\mathcal{F}$. We say that B is *represented* by A over $\mathcal{F}$. As an example, let A be the matrix

$$\begin{array}{c|cccc|} & e & f & g & h \\ \cline{2-5} a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 1 & 1 \\ c & 1 & 1 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \\ \cline{2-5} \end{array} \tag{3.4.9}$$

Matrix A producing an abstract matrix B

over GF(2). Then B is numerically identical to A, and the determinants for the submatrices of B are given by the GF(2)-determinants of A. For example, the submatrix D of B given by

(3.4.10)
$$D = \begin{array}{c|cc|} & g & h \\ \hline a & 1 & 1 \\ b & 1 & 1 \\ \hline \end{array}$$

Submatrix D of abstract matrix B

has $\det D = 0$, since the related submatrix of A has GF(2)-determinant 0.

The example may be modified to produce an abstract matrix that is not representable over any field. Let B and its determinants be as just defined. Then change the determinant of D of (3.4.10) from 0 to 1. One may check by enumeration that the new B observes the axioms of (3.4.8). For a proof that the new B is not representable, suppose A over some field $\mathcal{F}$ represents B. Then one readily shows that the rows and columns of A can be scaled so that the matrix of (3.4.9) results. For D of (3.4.10) as submatrix of the scaled A, we have $\det_{\mathcal{F}} D = 0$. But for D as submatrix of B, we have $\det D = 1$, a contradiction.

For certain abstract matrices, the axioms of (3.4.8) completely determine the rank. For example, by axiom (ii) of (3.4.8), zero matrices have rank 0. A more interesting instance is given in the next result.

(3.4.11) Lemma. *Let B be an $m \times m$ triangular abstract matrix. Then* $\det B = 1$ *if and only if the diagonal of B contains only* 1s.

Proof. The case $m = 1$ is immediate by (i) of (3.4.8). Thus, consider the case $m \geq 2$.

Assume that the diagonal of B has only 1s. For an inductive proof, we partition B as

(3.4.12)
$$B = \begin{array}{c|c|c|} & y & \overline{Y} \\ \hline x & 1 & 0 \\ \hline \overline{X} & 0/1 & \overline{B} \\ \hline \end{array}$$

Triangular B with 1s on the diagonal

where $\overline{B}$ is triangular and has only 1s on the diagonal. Apply the submodularity condition (3.4.7) as follows. Declare B^{22} (resp. B^{23}, B^{32}, B^{33}) to be the submatrix of B indexed by x and $\overline{Y}$ (resp. x and y, $\overline{X}$ and $\overline{Y}$, $\overline{X}$ and y); all other B^{ij} are trivial or empty. Then by (3.4.6), rank D^1 = rank $B^{22} = 0$, rank D^2 = rank B, rank $D^3 = 1$, and by induction, rank $D^4 = m - 1$. By submodularity, rank $B \geq 1 + (m-1) = m$, so $\det B = 1$.

Assume now that the diagonal of B contains a 0. Thus, we may partition B as

(3.4.13)
$$B = \begin{array}{c|c|c|c|} & Y_1 & y & Y_2 \\ \hline X_1 & B^1 & \multicolumn{2}{c|}{0} \\ \hline x & & \boxed{0} & \\ \hline X_2 & \multicolumn{2}{c|}{0/1} & B^2 \\ \hline \end{array}$$

Triangular B with a zero on the diagonal

where both B^1 and B^2 are square. By axiom (ii) of (3.4.8), the rows of B indexed by $X_1 \cup \{x\}$ have rank of at most $|Y_1|$. The remaining rows of B have rank of at most $|X_2|$. Thus, again by axiom (ii) of (3.4.8), the rank of B is at most $|Y_1| + |X_2| = |X_1| + |X_2| \leq m - 1$. Hence, $\det B = 0$. □

(3.4.14) Corollary. *The abstract matrices*

(3.4.15)
$$\boxed{0}\ ;\ \boxed{\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}}\ ;\ \boxed{\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}}\ ;\ \boxed{\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}}\ ;\ \boxed{\begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array}}$$

Small singular abstract matrices

are singular. The matrices

(3.4.16)
$$\boxed{1}\ ;\ \boxed{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}}\ ;\ \boxed{\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}}$$

Small nonsingular abstract matrices

are nonsingular. The matrix

(3.4.17)
$$\boxed{\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}}$$

Abstract matrix with determinant 0 or 1

may be singular or nonsingular.

Proof. Lemma (3.4.11) handles the cases of (3.4.15) and (3.4.16). A matrix over GF(3) with support given by (3.4.17) may be GF(3)-singular or GF(3)-nonsingular. This fact validates the claim about (3.4.17). □

Note that the GF(2)-determinants of the matrices of Corollary (3.4.14) agree with the abstract determinants, except possibly for the matrix of (3.4.17).

Abstract Matrices Encode Matroids

We claim that abstract matrices are nothing but one way of encoding matroids. For a proof, we first assume that an abstract matrix B with row index set X and column index set Y is at hand. We want to show that B encodes some matroid. Define a rank function $r(\cdot)$ on $E = X \cup Y$ as follows. Given $Z \subseteq E$, the sets $X_2 = Z \cap X$ and $Y_1 = Z \cap Y$ induce a partition of B of the form

(3.4.18)
$$B = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & B^1 & D^2 \\ \hline X_2 & D^1 & B^2 \\ \hline \end{array}$$

Partitioned version of B

Then define

(3.4.19)
$$r(Z) = |X_2| + \text{rank } B^1.$$

(3.4.20) Lemma. *$r(\cdot)$ is the rank function of a matroid.*

Proof. We verify the rank axioms (3.4.4). Let $S, T \subseteq E$. Clearly $r(S) \leq |S|$. A simple case analysis and axiom (ii) of (3.4.8) confirm that $S \subseteq T$ implies $r(S) \leq r(T)$. Finally, $r(S) + r(T) \geq r(S \cup T) + r(S \cap T)$ is argued as at the end of Section 3.3, except that we use the abstract rank function instead of the GF(2)-rank function. □

Matroids Produce Abstract Matrices

Given a matroid M on E with rank function $r(\cdot)$, we want to show that M may be encoded by some abstract matrix. Define X to be a base of M and $Y = E - X$. For any $y \in Y$, the set $X \cup \{y\}$ must contain at least one circuit. Indeed, there must be precisely one circuit in $X \cup \{y\}$, say C_y, since otherwise there is a base X' contained in $X \cup \{y\}$ with cardinality $|X'| \leq |X| - 1$. The circuit C_y is called the *fundamental circuit* that y creates with X. Declare the characteristic vectors of the fundamental circuits C_y, $y \in Y$, to be the columns of a matrix B. Thus, we have

(3.4.21)
$$\begin{array}{c|c|} & Y \\ \hline X & B \\ \hline \end{array}$$

Matrix B for matroid M with base X

where for $x \in X$ and $y \in Y$, we have $B_{xy} = 1$ if and only if $x \in C_y$. We endow B with determinants as follows. Let B^1 be a square submatrix of B, say indexed by $X_1 \subseteq X$ and $Y_1 \subseteq Y$ as in (3.4.18). Then declare $\det B^1 = 1$ if $X_2 \cup Y_1$ is a base of M, and to be 0 otherwise.

(3.4.22) Lemma. *B and its determinants constitute an abstract matrix.*

Proof. First, $\det[B_{xy}] = B_{xy}$ for all $x \in X$ and $y \in Y$, due to the equivalence of the following statements separated by semicolons: $B_{xy} = 1$; x is in the fundamental cycle C_y; $(X \cup \{y\}) - \{x\}$ does not contain a circuit; $(X \cup \{y\}) - \{x\}$ is a base of M; $\det[B_{xy}] = 1$.

Next, let B^1 be an arbitrary submatrix of B indexed by some $X_1 \subseteq X$ and $Y_1 \subseteq Y$. Define $X_2 = X - X_1$ and $Y_2 = Y - Y_1$. We prove that the maximal square nonsingular submatrices D of B^1 have same order. Any such matrix D corresponds to a base of M that contains X_2, that is contained in $X_1 \cup X_2 \cup Y_1$, and that, subject to these two conditions, intersects Y_1 as much as possible. By the independence axioms (3.4.1), the cardinality of any such maximal intersection is the same regardless of which independent subset of Y_1 one starts with. We conclude that the maximal nonsingular submatrices of B^1 have same order. That order is the rank of B^1, denoted by rank B^1.

The rank function $r(\cdot)$ of M is thus related to rank B^1 by $r(X_2 \cup Y_1) = |X_2| + \text{rank } B^1$. We apply the independence axioms of (3.4.1) to M and verify that removal of a column or row from B^1 reduces the rank at most by the rank of the removed column or row. Finally, the arguments at the end of Section 3.3 linking submodularity of $r(\cdot)$ with that of the GF(2)-rank function are easily adapted to prove that the rank function for B is submodular.

Thus, the axioms of (3.4.8) are satisfied, and B is an abstract matrix as claimed. □

The constructions of the proof of Lemmas (3.4.20) and (3.4.22) are inverses of each other in the following sense. Let M be a matroid with a base X. Define B to be the abstract matrix constructed from M in the proof of Lemma (3.4.22). The matroid deduced from B in the proof of Lemma (3.4.20) is M again. For this reason, we say that B *represents* M.

At this point, we have established an axiomatic link between abstract matrices and matroids. We could explore the ways in which matroid concepts manifest themselves in abstract matrices. But that discussion would largely duplicate the material of Section 3.3 about binary matroids. So we just sketch the definitions and relationships, and omit all proofs. Throughout, B is an abstract matrix with row index set X and column index set Y. The related matroid M on $E = X \cup Y$ has X as a base.

Column $y \in Y$ of B is the characteristic vector of $C_y - \{y\}$, where C_y is the *fundamental circuit* that y forms with X. Row $x \in X$ of B is the characteristic vector of $C^*_x - \{x\}$, where C^*_x is the *fundamental cocircuit* that x forms with Y. Two *parallel* (resp. *coparallel* or *series*) elements of M manifest themselves in two nonzero columns (resp. rows) of B of rank 1, or in a column (resp. row) unit vector. A *loop* (resp. *coloop*) of M is indicated by a column (resp. row) of 0s. The transpose of B represents the *dual matroid* M^* of M. Let U and W be disjoint subsets of E. Assume U contains no circuit and W contains no cocircuit. Then there is a B where $X \supseteq U$ and $Y \supseteq W$. Delete from B the rows of U and the columns of W, getting an abstract matrix $\overline{B}$. Correspondingly, *contraction* of U and *deletion* of W *reduce* M to the *minor* $\overline{M} = M/U \backslash W$, which is represented by $\overline{B}$. Furthermore, *expansion* by U and addition of W *extend* $\overline{M}$ to M.

Partition B as in (3.4.18). If for some $k \geq 1$, $|X_1 \cup Y_1|$, $|X_2 \cup Y_2| \geq k$ and $\text{rank } D^1 + \text{rank } D^2 \leq k-1$, then $(X_1 \cup Y_1, X_2 \cup Y_2)$ is a *k-separation* of B and M. Let $k \geq 2$. If B and M have no l-separation, $1 \leq l < k$, then they are *k-connected.* M is *connected* if it is 2-connected. Consistent with Section 2.3, B is *connected* if the graph $\text{BG}(B)$ is connected.

Abstract Pivot

We introduce pivots in abstract matrices. For comparison purposes, we rewrite the GF(2)-pivot of (3.2.12) as follows. Given is the pivot element $B_{xy} = 1$.

(3.4.23)

(3.4.23.1) We replace each B_{uw}, $u \in (X - \{x\})$, $w \in (Y - \{y\})$, by $\det_2 D^{uw}$, where D^{uw} is the submatrix of B given by

$$D^{uw} = \begin{array}{c|cc|} & y & w \\ \hline x & B_{xy} & B_{xw} \\ u & B_{uy} & B_{uw} \\ \hline \end{array}$$

(3.4.23.2) We exchange the indices x and y.

Let B be the abstract matrix for M as before. A pivot on B_{xy} is to produce the abstract matrix B' corresponding to the base $(X - \{x\}) \cup \{y\}$ of M. We claim that the following procedure generates the desired B'.

(3.4.24)

(3.4.24.1) We replace each B_{uw}, $u \in (X - \{x\})$, $w \in (Y - \{y\})$, by $\det D^{uw}$, where D^{uw} is the submatrix of B as given above.

(3.4.24.2) We exchange the indices x and y. Let B' be the resulting matrix.

(3.4.24.3) We endow the square submatrices D' of B' with determinants. Let U' be the row index set of one such D', and W' be the column index set. Then $\det D' = \det D$ for the submatrix D of B indexed by U and W as specified below.

If $y \in U'$, $x \in W'$: $U = U' - \{y\}$, $W = W' - \{x\}$.
If $y \in U'$, $x \notin W'$: $U = (U' - \{y\}) \cup \{x\}$, $W' = W$.
If $y \notin U'$, $x \in W'$: $U = U'$, $W = (W' - \{x\}) \cup \{y\}$.
If $y \notin U'$, $x \notin W'$: $U = U' \cup \{x\}$, $W = W' \cup \{y\}$.

Validity of the procedure is argued as follows. By step (3.4.24.2), $X' = (X - \{x\}) \cup \{y\}$ and $Y' = (Y - \{y\}) \cup \{x\}$ index the rows and columns, respectively, of B'. By step (3.4.24.1), for $u \in (X - \{x\})$ and $w \in (Y - \{y\})$, $B'_{uw} = \det B^{uw}$. Now $\det D^{uw} = 1$ if and only if $(X - \{x, u\}) \cup \{y, w\}$ is a base of M. The latter set is $(X' - \{u\}) \cup \{w\}$, so B'_{uw} is correctly computed. The entries B'_{yw} $(= B_{xw})$, $w \in (Y - \{y\})$, are correct since $(X - \{x\}) \cup \{w\} = (X' - \{y\}) \cup \{w\}$. Similarly, the entries B'_{ux} $(= B_{uy})$, $u \in (X - \{x\})$, are correct since $(X - \{u\}) \cup \{y\} = (X' - \{u\}) \cup \{x\}$. Finally, B'_{yx} $(= B_{yx} = 1)$ since $(X' - \{y\}) \cup \{x\} = X$, the assumed base of M. Analogous arguments involving the sets U and W, instead of the elements u and w, validate (3.4.24.3). We conclude that B' is the abstract matrix corresponding to the base $X' = (X - \{x\}) \cup \{y\}$ of M as claimed.

Except for the computationally tedious step (3.4.24.3), the pivot is almost identical to the GF(2)-pivot of (3.4.23). Indeed, for 2×2 matrices, Corollary (3.4.14) establishes an almost complete agreement between GF(2)-determinants and abstract determinants. Thus, the step (3.4.24.1) of an abstract pivot looks very much like the step (3.4.23.1) of a GF(2)-pivot. Informally, one is tempted to say that general matroids behave locally to quite an extent like binary matroids.

There remains, of course, the cumbersome step (3.4.24.3). But we can always imagine that this step is implicitly carried out, without our actually having to write down the list of determinant values. When we take that attitude, the pivot operation becomes simple and useful.

Some Matroid Results

The similarity of local behavior of binary and general matroids is easily demonstrated. We do this here by listing a number of results for general matroids that by trivial modifications may be obtained from the results for binary matroids proved in Section 3.3. In each case, the proof as given in Section 3.3 suffices, or in that proof one simply substitutes an abstract matrix whenever a binary one is employed. With each result, we cite in parentheses the related result of Section 3.3.

(3.4.25) Lemma (see Lemma (3.3.6)). *Let C (resp. C^*) be a circuit (resp. cocircuit) of a matroid M. Then $|C \cap C^*| \neq 1$.*

(3.4.26) Lemma (see Lemma (3.3.12)). *Let M be a matroid with a minor $\overline{M}$, and $\overline{B}$ be an abstract representation matrix of $\overline{M}$. Then M has an abstract representation matrix B that displays $\overline{M}$ via $\overline{B}$ and thus makes the minor $\overline{M}$ visible.*

(3.4.27) Lemma (see Lemma (3.3.13)). *The following statements are equivalent for any matroid M and any minor $\overline{M}$ of M. Let $\overline{B}$ be an abstract representation matrix of $\overline{M}$.*

(i) *$\overline{M}$ is a contraction minor of M.*

(ii) *M has an abstract representation matrix B displaying $\overline{M}$ via $\overline{B}$, where B is of the form*

(3.4.28)
$$B = \begin{array}{cc|c|c|} & & \multicolumn{2}{c}{Y} \\ & & \overline{Y} & W \\ \cline{3-4} X & \overline{X} & \overline{B} & 0 \\ \cline{3-4} & U & \multicolumn{2}{c|}{0/1} \\ \cline{3-4} \end{array}$$

Matrix B displaying contraction minor $\overline{M}$

(iii) *Every abstract representation matrix B of M displaying $\overline{M}$ via $\overline{B}$ is of the form given by (3.4.28).*

(3.4.29) Lemma (see Lemma (3.3.15)). *The following statements are equivalent for any matroid M and any minor $\overline{M}$ of M. Let $\overline{B}$ be an abstract representation matrix of $\overline{M}$.*

(i) *$\overline{M}$ is a deletion minor of M.*

(ii) *M has an abstract representation matrix B displaying $\overline{M}$ via $\overline{B}$, where B is of the form*

(3.4.30)
$$B = \begin{array}{cc|c|c|} & & \multicolumn{2}{c}{Y} \\ & & \overline{Y} & W \\ \cline{3-4} X & \overline{X} & \overline{B} & 0/1 \\ \cline{3-3} & U & 0 & \\ \cline{3-4} \end{array}$$

Matrix B displaying deletion minor $\overline{M}$

(iii) *Every abstract representation matrix B of M displaying $\overline{M}$ via $\overline{B}$ is of the form given by (3.4.30).*

(3.4.31) Lemma (see Lemma (3.3.19)). *Let M be a matroid with an abstract representation matrix B. Then M is connected if and only if this is so for B.*

(3.4.32) Lemma (see Lemma (3.3.20)). *The following statements are equivalent for a matroid M with set E and an abstract representation matrix B.*

(i) *M is 3-connected.*

(ii) *B is connected, has no parallel or unit vector rows or columns, and has no partition as in (3.4.18) with* rank $D^1 = 1$, $D^2 = 0$, *and* $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq 3$.

(3.4.33) Lemma (see Lemma (3.3.26)). *Let M be a matroid on a set E and with rank function $r(\cdot)$. Suppose E_1 and E_2 partition E. Then* (a) *and* (b) *below hold.*

(a) *(E_1, E_2) is a k-separation of M if and only if*

$$\begin{aligned} |E_1|, |E_2| &\geq k \\ r(E_1) + r(E_2) &\leq r(E) + k - 1 \end{aligned} \tag{3.4.34}$$

(b) *(E_1, E_2) is an exact k-separation of M if and only if*

$$\begin{aligned} |E_1|, |E_2| &\geq k \\ r(E_1) + r(E_2) &= r(E) + k - 1 \end{aligned} \tag{3.4.35}$$

(3.4.36) Lemma (see Lemma (3.3.31)). *Let M be a 3-connected matroid on a set E. Take z to be any element of E. Then $M Ⓒ z$ or $M Ⓓ z$ is 3-connected.*

(3.4.37) Lemma (see Lemma (3.3.36)). *Let M be a connected matroid on a set E. Take z to be any element of E. Then M/z or $M\backslash z$ is connected.*

Aspects of Representability

In the remainder of this section and in the next one, we examine several aspects of the representability of abstract matrices. Recall that an abstract matrix B is represented by a matrix A of the same size and over some field $\mathcal{F}$ if the following holds. For every square submatrix of B, the determinant of that submatrix is 1 if and only if the $\mathcal{F}$-determinant of the corresponding submatrix of A is nonzero. We want to establish a direct connection between the matroid M, defined by B, and any matrix A representing B over some field $\mathcal{F}$. To this end, we partition such A in agreement with the partition of B of (3.4.18). Thus,

$$A = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A^1 & C^2 \\ \hline X_2 & C^1 & A^2 \\ \hline \end{array} \tag{3.4.38}$$

Partitioned version of matrix A over field $\mathcal{F}$

Evidently, the submatrix A^1 of A corresponds to B^1 of B. Assume these submatrices to be square. We know that $\det_{\mathcal{F}} A^1 \neq 0$ if and only if $\det B^1 = 1$. We also know that $\det B^1 = 1$ if and only if $X_2 \cup Y_1$ is a base of M. Thus,

$\det A^1 \neq 0$ if and only if $X_2 \cup Y_1$ is a base of M. When this relationship holds, we also say that A over $\mathcal{F}$ *represents* M.

We have seen that some abstract matrices are not representable over any field. By definition, the same holds for some matroids. In particular, the abstract matrix deduced from (3.4.9) by modifying the determinant of D of (3.4.10) produces a matroid that is not representable over any field.

Deciding whether or not a matroid is representable over some field is a nontrivial problem. Usually, one assumes that the matroid is given by a *black box* or *oracle* that in unit time tells whether or not a subset of E is independent in the matroid. No additional information about the matroid is available. Under these assumptions, even representability over GF(2) cannot be tested in polynomial time. Indeed, the same conclusion can be drawn for a great many representability questions. There is one extraordinary exception. One can test in polynomial time whether or not a matroid is representable over *every* field. That representability problem is intimately linked to the problem of deciding whether a matrix is totally unimodular. Details are covered in Chapter 11.

Suppose an abstract matrix B and the related matroid M are representable over some field $\mathcal{F}$. Let A be a matrix over $\mathcal{F}$ proving this fact. Then every matrix B' derivable from B by one or more pivots is also representable over $\mathcal{F}$. For a proof, one carries out a pivot in B, say on $B_{xy} = 1$, and a second $\mathcal{F}$-pivot in A on A_{xy}. Let B' and A' result. It is easily seen that A' establishes B' to be representable over $\mathcal{F}$. Combine this result with the trivial observation that every proper submatrix of B and the transpose of B are representable over $\mathcal{F}$. We conclude that every minor of M and the dual M^* of M are representable over $\mathcal{F}$.

It is easy to check that all matroids with at most three elements are representable over every field. Couple this observation with the fact that the taking of minors maintains representability. Evidently, a matroid M not representable over a given field $\mathcal{F}$ must have a minor that also is not representable over $\mathcal{F}$, but all of whose proper minors are representable over $\mathcal{F}$. We call such a minor a *minimal violator* of representability over $\mathcal{F}$. Not much is known about the minimal violators for the various fields $\mathcal{F}$. Complete lists of the minimal violators exist only for the fields GF(2) and GF(3). There is also a complete description for the case when representability over every field is demanded. Beyond these cases, only incomplete results are known.

For the case of GF(2), there is just one minimal violator. It is a matroid on four elements called U_4^2. In the next section, we define that matroid and prove the claim we just made. In Chapter 9, we determine the minimal violators for GF(3) and for the case of representability over every field. In both cases, there are just two minimal violators plus their duals. As we shall see, abstract matrices and abstract pivots are useful for the proof of the results for GF(2) and GF(3).

3.5 Characterization of Binary Matroids

In this section, we characterize the binary matroids in several ways. In particular, we prove that a certain matroid on four elements called U_4^2 is the only minimal violator of representability over GF(2). Accordingly, a matroid is binary if and only if it has no U_4^2 minors. To begin, we define, for $1 \leq m \leq n$, U_n^m to be the matroid on n elements where every subset of cardinality m is a base. It is easily checked that U_n^m is indeed a matroid. It is the *uniform matroid* of rank m on n elements.

Let M be a matroid on a set E. For an arbitrary base X of M, compute the abstract matrix B. Suppose M is representable over $\mathcal{F}$. By definition, there is a matrix A over $\mathcal{F}$ with B as support matrix, such that for all corresponding submatrices D of B and D' of A, $\det D = 1$ if and only if $\det_{\mathcal{F}} D' \neq 0$. If $\mathcal{F}$ is GF(2), evidently A is numerically identical to B.

Consider U_n^m, for $2 \leq m \leq n-2$. Since every set of cardinality m is a base of U_n^m, every abstract matrix B for that matroid is of size $m \times n$, contains only 1s, and has only nonsingular square submatrices. In the related binary matrix A, all square submatrices of order at least 2 are GF(2)-singular. Thus, U_n^m is nonbinary. The smallest nonbinary case has $2 = m = n-2$, i.e., the matroid is U_4^2. Representability over any field $\mathcal{F}$ is maintained under minor-taking. Thus, a binary matroid cannot possibly have U_4^2 minors. We now prove that absence of U_4^2 minors implies representability over GF(2).

Let M be a nonbinary matroid all of whose proper minors are binary. Select any abstract representation B for M, and let A be the associated binary matrix. Then there are minimal submatrices D of B and D' of A in the same position such that exactly one of $\det D$ and $\det_2 D'$ is 0. Since every proper minor of M is binary, we must have $D = B$ and $D' = A$. Since the entries of B agree with those of A, the order of B must be at least 2.

If B is a zero matrix, then both $\det B$ and $\det_2 A$ are zero, a contradiction. Thus, B contains a 1, say $B_{xy} = 1$. If the order of B is greater than 2, we perform a pivot in B and the corresponding GF(2)-pivot in A. In both cases, we delete the pivot row and pivot column. Let B' and A' result. By the minimality assumptions on B' and A' and the rules (3.4.24.1) and (3.4.23.1), the two matrices must agree numerically. By (3.4.24.3) and the analogous rule of linear algebra for A, exactly one of $\det B'$ and $\det_2 A'$ is zero. Thus, we have proved that a proper minor of M is not binary, a contradiction. Hence, B is a 2×2 matrix. By Corollary (3.4.14), there is only one choice for B and A. That is, B is the abstract matrix of (3.4.17), and A is also that matrix when viewed as binary. We include the two matrices below in (3.5.1). Since exactly one of the determinants of B and A is nonzero and $\det_2 A = 0$, we must have $\det B = 1$.

(3.5.1) $$B = \boxed{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}}\ ; \quad A = \boxed{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}}$$

Abstract matrix B of minimal nonbinary matroid
and related binary matrix A

Evidently, M has four elements, and every 2-element subset is a base. Thus, M is isomorphic to U_4^2. We record this conclusion and state and prove a related corollary for future reference.

(3.5.2) Theorem. *A matroid M is binary if and only if M does not have U_4^2 minors.*

(3.5.3) Corollary. *Let an abstract matrix B represent a nonbinary matroid M. Let N be the binary matroid represented by the binary matrix A that is numerically identical to B. Then M has a base that is not a base of N.*

Proof. Carry out the earlier proof, except that the proper minors of M are not assumed to be binary. Accordingly, D and D' may be proper submatrices of B and A, respectively. By the pivot argument, we know that $\det D = 1$ and $\det_2 D' = 0$. This implies that the base of M corresponding to D is not a base of N. □

The proof of Theorem (3.5.2) implies the following statement. A matroid M is nonbinary if and only if M has an abstract representation matrix B with a 2×2 submatrix D that is nonsingular and that contains only 1s. This fact leads to an elementary proof of the following theorem.

(3.5.4) Theorem. *The following statements are equivalent for a matroid M on a set E.*

(i) *M is binary.*
(ii) *For any circuit C and cocircuit C^*, $|C \cap C^*|$ is even.*
(iii) *The symmetric difference of two circuits is a disjoint union of circuits.*
(iv) *The symmetric difference of two disjoint unions of circuits is a disjoint union of circuits.*
(v) *The symmetric difference of two distinct circuits contains a circuit.*
(vi) *Given any distinct circuits C_1 and C_2, and any two elements $e, f \in (C_1 \cap C_2)$, there is a circuit $C_3 \subseteq [(C_1 \cup C_2) - \{e, f\}]$.*
(vii) *For any base X and $Y = E - X$, any circuit C is the symmetric difference of the fundamental circuits C_y corresponding to X and with $y \in (C \cap Y)$.*

Proof. Statement (i) plus Lemmas (3.3.6), (3.3.7), (3.3.8) imply (ii)–(vi). Statement (vii) follows from (i) by the previous characterization of circuits in terms of column submatrices of the representation matrix B. We prove the converse implications by contradiction. Thus, we assume M to be a nonbinary matroid. For M, we select an abstract matrix B with the 2×2 submatrix D as described above and show by inspection and routine arguments that none of (ii)–(vii) holds. □

3.6 References

The basic material on graphic, binary, and general matroids relies on Whitney (1935), and Tutte (1965), (1966b), (1971). Theorem (3.2.25) is proved in Truemper (1987b). Corollary (3.2.29) was first established in Tutte (1966b); a short proof appears in Cunningham (1981), together with other connectivity concepts (see also Inukai and Weinberg (1981), Oxley (1981a), and Wagner (1985b)). Corollary (3.2.31) and its generalization to general matroids are included in Cunningham (1973); the results also appear in Duchamp (1974) and Krogdahl (1977).

The 2-isomorphism results (Lemma (3.2.33) and Theorem (3.2.36)) are due to Whitney (1933a). Shorter proofs are given in Truemper (1980a), Wagner (1985a), and Kelmans (1987), together with an upper bound of $n-2$ switchings for 2-connected graphs with n vertices. The generalization to directed graphs is covered in Thomassen (1989). For a variation of the 2-isomorphism problem, define a matroid from the node/edge incidence matrix of a graph as in Section 3.2, except that the matrix is viewed to be over $\mathbb{R}$ instead of GF(2). Just as in the GF(2) case, several graphs may produce the same matroid. A partial analysis of the graphs generating the same matroid is given in Wagner (1988). Another variation, called vertex 2-isomorphism, is treated in Swaminathan and Wagner (1992).

The graphicness testing subroutine is due to Löfgren (1959). Well implemented, it has led to the presently most efficient algorithms for that problem (see Fujishige (1980), and Bixby and Wagner (1988)). Other relevant references are Gould (1958), Auslander and Trent (1959), (1961), Tutte (1960), (1964), Iri (1968), Tomizawa (1976a), Bixby and Cunningham (1980), Wagner (1983), and Bixby (1984a). The first polynomial test for graphicness of binary matroids was given by Tutte (1960). An efficient graphicness test for matroids not known to be binary is described in Seymour (1981c); see also Bixby (1982a), and Truemper (1982a).

The notion of submodularity of the rank function as one of the central tools of matroid theory is due to Edmonds (see, e.g. Edmonds (1970)). In this book, we use the submodularity concept rather infrequently. An excellent survey of the many facets and applications is given in Fujishige (1984). For optimization and decomposition results, see, e.g., Fujishige (1983), (1989), (1991), and Frank (1993b). Seymour (1988) analyses an unusual way of specifying a matroid. Define the *connectivity function* of a matroid on a set E and with rank function $r(\cdot)$ to be the function $c(X) = r(X) + r(E - X) - r(E)$, $X \subseteq E$. Clearly, the connectivity function determines the matroid at most up to duality. In Seymour (1988), it is shown that the connectivity function generally does not determine the matroid up to duality, but that this is so when the matroid is binary.

Lemma (3.3.31) and its generalization (Lemma (3.4.36)) have also been independently proved by others (Seymour (1981b), Bixby (1982b)). The special case of graphs was first treated in Seymour (1980b).

The concept of abstract matrices is based on that of partial representation matrices of Truemper (1984). The example of a nonrepresentable abstract matrix is taken from that reference. Early examples of nonrepresentable matroids are in MacLane (1936), Lazarson (1958), Ingleton (1959), (1971), and Vamos (1968). Additional references about the representability problem are included in Section 9.5.

Theorem (3.5.2) is one of the key contributions of Tutte (1958). That theorem set the stage for and motivated a number of subsequent results on representability. The proof and Corollary (3.5.3) are taken from Truemper (1982b). The conditions of Theorem (3.5.4) are from Whitney (1935), Rado (1957), Lehman (1964), Tutte (1965), Minty (1966), and Fournier (1981). Additional characterizations of the binary matroids may be found in Bixby (1974), Duchamp (1974), and White (1987).

For additional material on binary matroids or matroids representable over fields with characteristic 2, see Gerards (1988), Hassin (1988), (1990a), (1990b), (1991), Oxley (1990b), Lemos (1991), and Ziegler (1991).