Chapter 11

Regular Matroids

11.1 Overview

Building upon the material on graphic matroids of Chapter 10, we analyze in this chapter the class of regular matroids. Already, we know that class quite well, or so at least it seems. By Theorem (9.3.2), a binary matroid is regular if and only if it has no F_7 or F_7^* minors. By Corollary (9.2.12), every graphic matroid is regular. There are also regular matroids that are not graphic and not cographic. If such a matroid is 3-connected, then by Theorem (10.4.1), it has an R_{10} or R_{12} minor. These results are interesting; indeed, the first and third one are profound. But they do not tell us how to construct regular matroids, or how to test binary matroids efficiently for regularity. That gap in our knowledge is filled by the extraordinary decomposition theorem of regular matroids due to Seymour. The entire chapter is devoted to that theorem and to some of its ramifications and applications.

We proceed as follows. In Section 11.2, we prove that 1-, 2-, and 3-sum compositions produce regular matroids when the components are regular. As a lemma for the 3-sum case, we also establish that the ΔY exchange defined in Section 4.4 maintains regularity.

Section 11.3 contains the main result of this chapter, the regular matroid decomposition theorem. It essentially says that every regular matroid can be produced by 1-, 2-, and 3-sums where the building blocks are graphic matroids, cographic matroids, and copies of R_{10}. Furthermore, only regular matroids can be generated by this process.

In Section 11.4, we develop efficient tests for regularity of binary matroids and for total unimodularity of real matrices. We obtain the regularity test by combining the method for finding 1-, 2-, and 3-sums of Section 8.4 with the graphicness test of Section 10.6. That scheme is then rather easily extended to an efficient test for total unimodularity.

The uses and implications of the regular matroid decomposition theorem are far-ranging. In Section 11.5, we describe representative applications. In the final section, 11.6, we list extensions and references.

The chapter requires knowledge of Chapters 2–10.

11.2 1-, 2-, and 3-Sum Compositions Preserve Regularity

In this section, we show that any 1-, 2-, or 3-sum with regular components is regular. This result is the comparatively easy part of the regular matroid decomposition theorem. We also prove that the ΔY matroids of Section 4.4 are regular.

The reader may wonder why we confine ourselves to 1-, 2-, and 3-sums, and do not treat general k-sum compositions. It turns out that the 1-, 2-, 3-sum cases have simple proofs and actually are the only k-sums needed for the regular matroid decomposition theorem. On the other hand, for k-sums with $k \geq 4$, the situation becomes much more complicated. In Section 11.6 we sketch what is known about that case.

Recall from Section 9.2 that a real matrix is totally unimodular if all of its determinants are $0, \pm 1$. Furthermore, a binary matrix is regular if it can be signed to become a totally unimodular real matrix. By Lemma (9.2.6) and Corollary (9.2.7), the signing is unique up to scaling by $\{\pm 1\}$ factors. Furthermore, the signing can be accomplished by signing one arbitrarily selected row or column at a time.

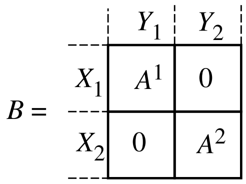
We need some elementary facts about the real $\{0, \pm 1\}$ matrices that are not totally unimodular, but each of whose proper submatrices has that property. We call such a matrix a *minimal violation matrix of total unimodularity*, for short *minimal violation matrix*. First, a minimal violation matrix is obviously square, and its real determinant is different from $0, \pm 1$. This fact implies that a 2×2 minimal violation matrix contains four ± 1s. Next, let a minimal violation matrix have order $k \geq 3$. Suppose we perform a real pivot in that matrix, then delete the pivot row and column. A simple cofactor argument proves that the resulting matrix is also a minimal violation matrix.

We are prepared for the proofs of this section. The first lemma deals with the case of 1- and 2-sums.

(11.2.1) Lemma. *Any 1- or 2-sum of two regular matroids is also regular.*

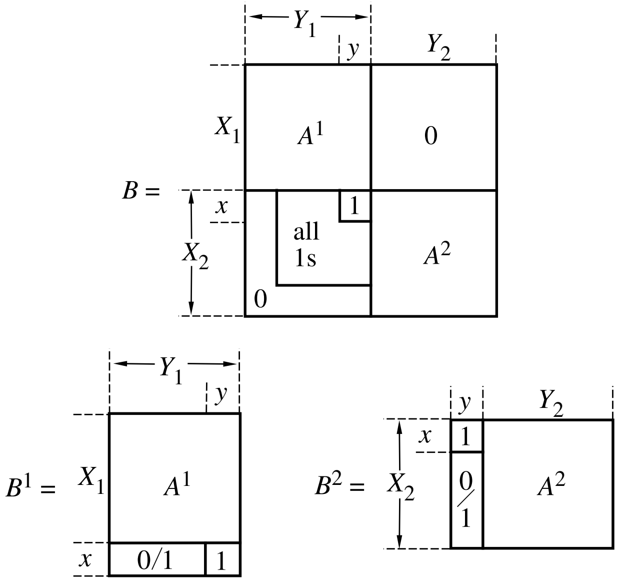
Proof. Let M_1 and M_2 be the given regular components of a 1- or 2-sum M . We rely on the matrices of (8.2.1), (8.2.3), and (8.2.4), repeated below for convenient reference.

(11.2.2)



Matrix B for 1-sum M

(11.2.3)



Matrices B, B^1, and B^2 for 2-sum M
with components M_1 and M_2

In the 1-sum case, M_1 and M_2 are represented by A^1 and A^2 of (11.2.2). Since M_1 and M_2 are regular, we can convert A^1 and A^2 by signing to totally unimodular real matrices, say $\tilde{A}^1$ and $\tilde{A}^2$. Derive a matrix $\tilde{B}$ from B of (11.2.2) by replacing A^1 and A^2 by $\tilde{A}^1$ and $\tilde{A}^2$. Evidently, $\tilde{B}$ is a totally unimodular signed version of B. Thus, M is regular.

The 2-sum case is slightly more complicated. Define D to be the submatrix of B of (11.2.3) indexed by X_2 and Y_1. We sign B^1 and B^2 of (11.2.3) so that totally unimodular matrices $\tilde{B}^1$ and $\tilde{B}^2$ result. The signing converts the submatrices A^1 and A^2 of B^1 and B^2 to, say, $\tilde{A}^1$ and $\tilde{A}^2$. Next we compute a signed version $\tilde{D}$ of D by the formula

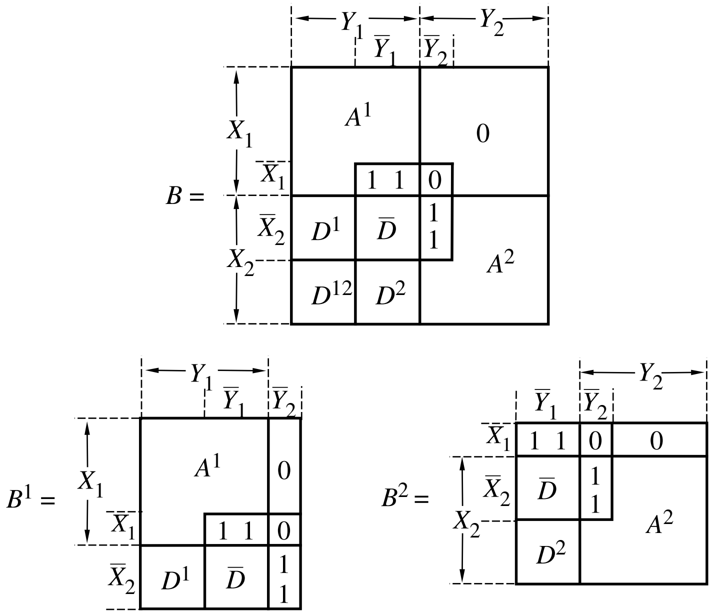
(11.2.4) $$\tilde{D} = (\text{column } y \text{ of } \tilde{B}^2) \cdot (\text{row } x \text{ of } \tilde{B}^1)$$

Then $\tilde{A}^1$, $\tilde{A}^2$, and $\tilde{D}$ define a signed version $\tilde{B}$ of the matrix B for M. By the construction, the submatrices $[\tilde{A}^1/\tilde{D}]$ and $[\tilde{D} \mid \tilde{A}^2]$ of $\tilde{B}$ are totally unimodular.

We complete the proof by showing that the entire matrix $\tilde{B}$ is totally unimodular. Suppose it is not. Then $\tilde{B}$ contains a minimal violation matrix V that intersects $\tilde{A}^1$, $\tilde{A}^2$, and $\tilde{D}$. Thus, V also intersects the 0 submatrix of $\tilde{B}$ indexed by X_1 and Y_2, and accordingly must have order of at least 3. We perform a real pivot in $\tilde{B}$ on a ± 1 that is in both $\tilde{A}^1$ and V. The resulting real matrix $\tilde{B}'$ contains a smaller minimal violation matrix. The pivot changes $\tilde{A}^1$ and $\tilde{D}$, say to $\tilde{A}^{1'}$ and $\tilde{D}'$, but leaves $\tilde{A}^2$ unchanged. Since $[\tilde{A}^1/\tilde{D}]$ is totally unimodular, and since, by Lemma (9.2.2), pivots do not destroy total unimodularity, the matrix $[\tilde{A}^{1'}/\tilde{D}']$ is totally unimodular. If we perform the corresponding GF(2)-pivot in B, we thus get an unsigned version of $\tilde{B}'$. Furthermore, each column of $\tilde{D}'$ is a scaled version of a column of $\tilde{D}$. Thus, $[\tilde{D}' \mid \tilde{A}^2]$ is totally unimodular. A suitable repetition of the preceding reduction process eventually produces the contradictory case of a minimal violation matrix contained in a totally unimodular matrix. ∎

We turn to the 3-sum case. By (8.3.10) and (8.3.11), we may assume M, M_1, and M_2 to be represented by the matrices B, B^1, and B^2, respectively, as follows.

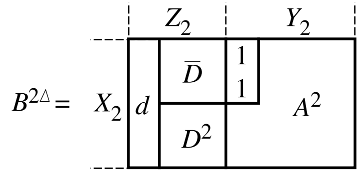
(11.2.5)



Matrices B, B^1, and B^2 for 3-sum M
with components M_1 and M_2

We may convert the component M_2 by a ΔY exchange of Section 4.4 to a matroid $M_{2\Delta}$ that is represented by the following matrix $B^{2\Delta}$, taken from (8.5.3).

(11.2.6)



Matrix $B^{2\Delta}$ for $M_{2\Delta}$

We first introduce a lemma that links regularity of M_2 to regularity of $M_{2\Delta}$.

(11.2.7) Lemma. *M_2 of (11.2.5) is regular if and only if $M_{2\Delta}$ of (11.2.6) has that property.*

Proof. For the "only if" part, let $\tilde{B}^2$ be a totally unimodular version of B^2 so that the two columns of $\tilde{B}^2$ indexed by $\overline{Y}_1$ do not contain any -1. This is possible, since we may begin the signing process with these two columns. Denote by $\tilde{\overline{D}}$ and $\tilde{D}^2$ the submatrices of $\tilde{B}^2$ that correspond to the submatrices $\overline{D}$ and $\tilde{D}^2$ of B. Declare $\tilde{d}$ to be the real difference of the two $\{0,1\}$ columns of $[\tilde{\overline{D}}/\tilde{D}^2]$. Thus, $\tilde{d}$ is a $\{0,\pm 1\}$ vector that, together with the submatrices of $\tilde{B}^2$, defines a signed version $\tilde{B}_{2\Delta}$ of $B^{2\Delta}$. We are done once we prove $\tilde{B}_{2\Delta}$ to be totally unimodular. If this is not the case, then $\tilde{B}_{2\Delta}$ contains a minimal violation matrix V. By the construction, the latter matrix must intersect $\tilde{d}$. The two columns of $[\tilde{\overline{D}}/\tilde{D}^2]$ and the vector $\tilde{d}$ are $\mathbb{R}$-dependent, so V intersects $[\tilde{\overline{D}}/\tilde{D}^2]$ in at most one column. But in $\tilde{B}^2$, we can produce a scaled version of V as submatrix by a real pivot on one of the explicitly shown 1s in the first row, as may be readily checked. The latter fact contradicts the total unimodularity of $\tilde{B}^2$.

We prove the "if" part using duality. First we observe that M_2^* may be derived from $M_{2\Delta}^*$ by replacing a triad by a triangle. We just established that such a change maintains regularity. Thus, M_2^* is regular if $M_{2\Delta}^*$ is regular. $\square$

Lemma (11.2.7) implies the following result.

(11.2.8) Corollary. *ΔY exchanges maintain regularity.*

We are ready for the 3-sum case.

(11.2.9) Lemma. *Any 3-sum of two regular matroids is regular.*

Proof. We start with B, B^1, and B^2 of (11.2.5) for the assumed 3-sum M with regular components M_1 and M_2. By Lemma (11.2.7), the matrix $B^{2\Delta}$ of (11.2.6) is regular. Let d be the column vector displayed in $B^{2\Delta}$. Then up to indices, $[d \mid D \mid A^2]$ is the regular matrix $B^{2\Delta}$ plus possibly parallel columns. Thus, $[d \mid D \mid A^2]$ may be signed to become a totally unimodular matrix $[\tilde{d} \mid \tilde{D} \mid \tilde{A}^2]$. By the regularity of B^1 and duality, $[A^1/D]$ is also regular, and thus can be signed, starting with the submatrix D, to a totally unimodular matrix $[\tilde{A}^1/\tilde{D}]$. Clearly, the matrices $\tilde{A}^1$, $\tilde{A}^2$, and $\tilde{D}$ define a signed version $\tilde{B}$ of B.

We are done once we show $\tilde{B}$ to be totally unimodular. We accomplish this by essentially the same arguments as for the 2-sum case. Thus, if $\tilde{B}$ is not totally unimodular, then it has a minimal violation matrix V that intersects $\tilde{A}^1$, $\tilde{A}^2$, and $\tilde{D}$. The order of V must be at least 3. In $\tilde{B}$, we pivot on a ± 1 that is in both $\tilde{A}^1$ and V. The pivot changes $[\tilde{A}^1/\tilde{D}]$ to another totally unimodular matrix, say $[\tilde{A}^{1\prime}/\tilde{D}']$. It is easy to verify that the columns of $\tilde{D}'$ are nothing but scaled versions of the columns of $[\tilde{d} \mid \tilde{D}]$. Thus, the matrix $[\tilde{D}' \mid \tilde{A}^2]$ is up to scaling and parallel columns a submatrix of the previously defined $[\tilde{d} \mid \tilde{D} \mid \tilde{A}^2]$, and hence is totally unimodular. By a suitable repetition of the above reduction process, we eventually get the contradiction that a minimal violation matrix is contained in a totally unimodular matrix. $\square$

For future reference, we combine Lemmas (11.2.1) and (11.2.9) to the following theorem.

(11.2.10) Theorem. *Any 1-, 2-, or 3-sum of two regular matroids is regular.*

We return to a class of binary matroids introduced in Section 4.4, the class of ΔY matroids. Each of these matroids is constructed from the matroid represented by $B = [\,1\,]$ by repeated SP (= series-parallel) and ΔY exchanges. We do not repeat the details of these operations here. Thus, the reader may want to review Section 4.4 before proceeding. There the following result is claimed but not proved. We supply the proof next.

(11.2.11) Theorem. *The ΔY matroids are regular.*

Proof. We use induction on the number of SP extensions and ΔY exchanges used in the construction of a binary ΔY matroid M. The binary matroid represented by $B = [\,1\,]$ is regular. By Lemma (8.2.6), any SP extension of M can be viewed as a 2-sum composition where M is one of the components, and where the second component is a regular matroid with three elements. By Theorem (11.2.10), a 2-sum is regular if its components are regular. Thus, SP extensions maintain regularity. By Corollary (11.2.8), the same conclusion holds for ΔY exchanges. $\square$

Section 8.5 contains two variations of the 3-sum, called Δ-sum and Y-sum. The latter sum is the dual of the Δ-sum. A Δ-sum decomposition is obtained from a 3-sum $M_1 \oplus_3 M_2$ by replacing M_2 by $M_{2\Delta}$. For the Δ-

sum composition, we invert this process. Theorem (11.2.10) and Corollary (11.2.8) thus imply the following corollary for Δ-sums and Y-sums.

(11.2.12) Corollary. *Any Δ-sum or Y-sum M of two regular matroids is regular.* □

In Section 4.4, it is mentioned that the class of ΔY matroids includes 3-connected matroids that are nongraphic and noncographic. By Lemma (4.4.10), every connected minor of a ΔY matroid is also a ΔY matroid. By (10.2.8), the regular matroid R_{10} has no triangles or triads, and thus is not a ΔY matroid. Indeed, R_{10} is 4-connected. We conclude that no binary matroid with an R_{10} minor is a ΔY matroid. The situation is different for R_{12}. In (10.2.9), the latter matroid is represented by a graph plus a node subset T. With the aid of that representation, one easily proves R_{12} to be a ΔY matroid, a fact already claimed in Section 4.4. Specifically, a triangle of that graph is a triangle of R_{12}, and a 3-star that is not a T node is a triad. Because of these facts, a ΔY sequence reducing R_{12} to the matroid represented by $B = [\,1\,]$ can be computed by graph operations. First, one reduces R_{12} while retaining the T set until a graphic or cographic matroid is attained. Second, one switches representations by selecting a suitable graph and finds the remaining reductions. We leave the details to the reader.

Theorems (11.2.10) and (11.2.11) enable us to construct many regular matroids from the matroids we already know to be regular, which are the graphic matroids, their duals, and R_{10} and R_{12}. In the next section, we see that we can construct all regular matroids that way. In fact, we require only a subset of the above initial matroids and a subset of the above construction steps to produce all regular matroids.

11.3 Regular Matroid Decomposition Theorem

In this section, we prove Seymour's profound decomposition theorem for regular matroids. It essentially says that every regular matroid can be obtained by 1-, 2-, and 3-sums where the building blocks are graphic matroids, cographic matroids, and matroids isomorphic to R_{10}. The theorem thus provides an elegant and useful construction for the entire class of regular matroids.

For the proof of the decomposition theorem, we rely on two ingredients. The first one we already know. It is Theorem (10.4.1). That result says that any 3-connected regular nongraphic and noncographic matroid has an R_{10} or R_{12} minor. The second ingredient consists of decomposition results of Sections 6.3 and 7.2, specifically Corollary (6.3.24) and the splitter Theorem (7.2.1). We could cast the results of this section in terms of the recursive construction of decomposition theorems introduced for graphs in Section 10.5. We will not do so. But we should mention that finding decomposition theorems such as the one of this section is facilitated by that recursive construction scheme. We discuss this aspect further in Chapter 13.

First, we analyze the influence of R_{10} and R_{12} minors. So assume R_{10} to be a minor of a regular matroid. For the analysis, we need the splitter Theorem (7.2.1), which we repeat next for convenient reference.

(11.3.1) Theorem (Splitter Theorem). *Let $\mathcal{M}$ be a class of binary matroids that is closed under isomorphism and under the taking of minors. Let N be a 3-connected matroid of $\mathcal{M}$ on at least six elements.*

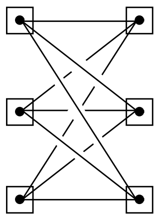
(a) *If N is not a wheel, then N is splitter of $\mathcal{M}$ if and only if $\mathcal{M}$ does not contain a 3-connected 1-element extension of N.*
(b) *If N is a wheel, then N is a splitter of $\mathcal{M}$ if and only if $\mathcal{M}$ does not contain a 3-connected 1-element extension of N and does not contain the next larger wheel.*

The matroid N is called a splitter of the class $\mathcal{M}$ of matroids. Here is the result for R_{10}.

(11.3.2) Theorem. *R_{10} is a splitter of the class $\mathcal{M}$ of regular matroids.*

Proof. By Theorem (11.3.1), we only need to show that every 3-connected 1-element extension of R_{10} is nonregular. Since R_{10} is isomorphic to its dual, it suffices that we consider 1-element additions. The case checking is conveniently accomplished when we represent R_{10} by a graph plus a T set as in (10.2.8), i.e., by

(11.3.3)

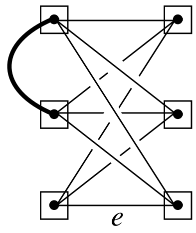


Graph plus T set representing R_{10}

Up to isomorphism, just three distinct 3-connected 1-element additions are possible.

In the first case, we join two nonadjacent nodes of the graph of (11.3.3) by an edge shown in bold below.

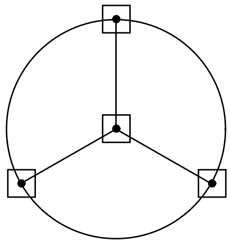
(11.3.4)



1-element addition of R_{10}, case 1

We contract edge e. The two T endpoints of e become a non-T node. Evidently, the resulting graph contains a subdivision of

(11.3.5)



Graph plus T set representing F_7

which by (10.2.4) represents the Fano matroid F_7. Thus, that extension of R_{10} is nonregular.

The remaining two cases involve 1-element additions that are nongraphic even when one deletes from R_{10} the element represented by the set T. For both cases, we depict the additional element by a subset T' of the node set with $|T'| = 4$. The two possible ways are easily reduced to an instance of (11.3.5), and thus are also nonregular. □

Now we assume R_{12} to be a minor of a regular matroid. This time we rely on Corollary (6.3.24). We need a bit of preparation before we can restate that result. Let N be a binary matroid with a k-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ given by

(11.3.6)
$$B^N = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A^1 & 0 \\ \hline X_2 & D & A^2 \\ \hline \end{array}$$

Matrix B^N for N with k-separation

Consider the following three ways of extending N as depicted by the representation matrices below.

(a) The 1-element expansions represented by

$$(11.3.7)\qquad \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A^1 & 0 \\ \hline x & e & f \\ \hline X_2 & D & A^2 \\ \hline \end{array}$$

Matrix of 1-element expansion of N

In row x, e is not spanned by the rows of D, and f is nonzero.

(b) The 1-element additions given by

$$(11.3.8)\qquad \begin{array}{c|c|c|c|} & Y_1 & y & Y_2 \\ \hline X_1 & A^1 & g & 0 \\ \hline X_2 & D & h & A^2 \\ \hline \end{array}$$

Matrix of 1-element addition of N

In column y, g is nonzero, and h is not spanned by the columns of D.

(c) The 2-element extensions with representation matrix

$$(11.3.9)\qquad \begin{array}{c|c|c|c|} & Y_1 & y & Y_2 \\ \hline X_1 & A^1 & g & 0 \\ \hline x & e & \alpha & f \\ \hline X_2 & D & h & A^2 \\ \hline \end{array}$$

Matrix of 2-element extension of N

Either (c.1) or (c.2) below holds for e, f, g, h, and α.

(c.1) e is not spanned by the rows of D; $f = 0$; $g = 0$; $h \neq 0$; $\alpha = 1$; e is not parallel to a row of A^1. If column $z \in Y_1$ of A^1 is nonzero, then e is not a unit vector with 1 in column z.

(c.2) $g \neq 0$; h is spanned by the columns of D; e is spanned by the rows of D; $f = 0$ implies $e \neq 0$; $[e \mid \alpha]$ is not spanned by the rows of $[D \mid h]$. If $\overline{D}$, the matrix obtained from D by deletion of a column $z \in Y_1$, has the same GF(2)-rank as D, then $[g/h]$ is not parallel to column z of $[A^1/D]$. If the rows of D do not span a row $z \in X_1$ of A^1, then $[g/h]$ is not a unit vector with 1 in row z.

We restate Corollary (6.3.24).

(11.3.10) Theorem. *Let $\mathcal{M}$ be a class of binary matroids that is closed under isomorphism and under the taking of minors. Suppose that N given by B^N of* (11.3.6) *is in $\mathcal{M}$, but that the* 1- *and* 2-*element extensions of N given by* (11.3.7), (11.3.8), (11.3.9), *and the accompanying conditions are not in $\mathcal{M}$. Assume that a matroid $M \in \mathcal{M}$ has an N minor. Then any k-separation of any such minor that corresponds to $(X_1 \cup Y_1, X_2 \cup Y_2)$ of N under one of the isomorphisms induces a k-separation of M.*

We intend to use Theorem (11.3.10) with R_{12} as N and with the class of regular matroids as $\mathcal{M}$. For the detailed arguments, we need the binary representation matrix B^{12} of (9.2.14) for R_{12}. We include that matrix below. According to that matrix, the pair $(X_1 \cup Y_1, X_2 \cup Y_2)$ constitutes a 3-separation of R_{12}.

(11.3.11)

$$B^{12} = \begin{array}{c|cccc|cc|}
 & & Y_1 & & & Y_2 & \\ \hline
\multirow{2}{*}{X_1} & 1 & 0 & 1 & 1 & 0 & 0 \\
 & 0 & 1 & 1 & 1 & 0 & 0 \\ \hline
\multirow{4}{*}{X_2} & 1 & 0 & 1 & 0 & 1 & 1 \\
 & 0 & 1 & 0 & 1 & 1 & 1 \\
 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline
\end{array}$$

Matrix B^{12} for R_{12}

Theorem (11.3.10) leads to the following decomposition result for the regular matroids with R_{12} minor.

(11.3.12) Theorem. *Let M be a regular matroid with an R_{12} minor. Then any* 3-*separation of that minor corresponding to the* 3-*separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ of R_{12} (see* (11.3.11)*) under one of the isomorphisms induces a* 3-*separation of M.*

Proof. We verify the sufficient conditions of Theorem (11.3.10). As a preparatory step, we calculate all 3-connected regular 1-element additions of R_{12}. By the symmetry of B^{12} of (11.3.11), and thus by duality, this result effectively gives us all 3-connected 1-element expansions as well. The addition cases are collected as columns in the following matrix C.

(11.3.13)

$$C = \begin{array}{c|cccc|}
 & & Y_1 & & \\ \hline
\multirow{2}{*}{X_1} & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 \\ \hline
\multirow{4}{*}{X_2} & 1 & 1 & 0 & 1 \\
 & 1 & 0 & 1 & 1 \\
 & 0 & 1 & 0 & 1 \\
 & 0 & 0 & 1 & 1 \\ \hline
\end{array}$$

Matrix C of 3-connected regular additions to R_{12}

Verification of the claim about C involves simple but somewhat tedious case checking. We omit the details, but should mention that the representation of R_{12} by a graph plus T set as in (10.2.9) greatly simplifies the task. The added element can be represented by a subset T' of the node set.

We verify the conditions of Theorem (11.3.10). The cases depicted in (11.3.13) rule out (11.3.7) and (11.3.8). They also narrow down the case analysis for (c.1) and (c.2) of (11.3.9), as follows.

According to (c.1), e is not spanned by the rows of D, $f = 0$, $g = 0$, $h \neq 0$, $\alpha = 1$, and e is not parallel to a row of A^1. If column $z \in Y_1$ of A^1 is nonzero, then e is not a unit vector with 1 in column z. We apply these conditions to B^{12}. By (11.3.13) and symmetry of B^{12}, we see that e must be the vector [0 0 1 1]. Furthermore, h must be a unit vector, or must be parallel to a column of A^2 or D, or must be the subvector of any column of C of (11.3.13) indexed by X_2. All such choices lead to nonregular matroids, as desired.

Suppose (c.2) applies. We ignore the conditions on e, f, and α. The remaining conditions are as follows. The vector g is nonzero, and h is spanned by the columns of D. If $\overline{D}$, the matrix obtained from D by deletion of a column $z \in Y_1$, has the same GF(2)-rank as D, then $[g/h]$ is not parallel to column z of $[A^1/D]$. If the rows of D do not span a row $z \in X_1$ of A^1, then $[g/h]$ is not a unit vector with 1 in row z. We apply these conditions to B^{12}. Thus, we determine that the matrix of (11.3.9) minus row x, which is B^{12} of (11.3.11) plus the column vector $[g/h]$, represents a 3-connected 1-element addition of R_{12}. By (11.3.13), any such addition with $g \neq 0$ is nonregular.

Thus, all conditions of Theorem (11.3.10) are satisfied. The conclusion of Theorem (11.3.10) then proves the result. □

At long last, we have completed all preparations for the regular matroid decomposition theorem.

(11.3.14) Theorem (Regular Matroid Decomposition Theorem)**.** *Every regular matroid M can be decomposed into graphic and cographic matroids and matroids isomorphic to R_{10} by repeated 1-, 2-, and 3-sum decompositions.*

Specifically, if M is 3-connected and not graphic and not cographic, then M is isomorphic to R_{10} or has an R_{12} minor. In the latter case, any 3-separation of that minor corresponding to the 3-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ of R_{12} (see (11.3.11)) under one of the isomorphisms, induces a 3-separation of M.

Conversely, every binary matroid produced from graphic matroids, cographic matroids, and matroids isomorphic to R_{10} by repeated 1-, 2-, and 3-sum compositions is regular.

Proof. Let a regular matroid M be given. Assume M to be nongraphic and noncographic. If M is 1-separable, it is a 1-sum. If M is 2-separable, then it is by Lemma (8.2.6) a 2-sum. Hence, assume M to be 3-connected. By Theorem (10.4.1), M has an R_{10} or R_{12} minor. In the first case, M must by Theorem (11.3.2) be isomorphic to R_{10}. In the second case, M has by Theorem (11.3.12) an induced 3-separation, as claimed. By Lemma (8.3.12), M is a 3-sum. Finally, Theorem (11.2.10) establishes the converse part. □

In Theorem (11.3.14), one may want to rely on Δ-sums or Y-sums instead of 3-sums. The next corollary supports that substitution.

(11.3.15) Corollary. *The claims of Theorem* (11.3.14) *remain valid when instead of* 3*-sums one specifies* Δ*-sums or* Y*-sums.*

Proof. By Section 8.5, any 3-sum decomposition may be converted to a Δ-sum or Y-sum decomposition by one ΔY exchange involving one of the components. By Lemma (11.2.8), such an exchange maintains regularity. Indeed, if M is 3-connected, then by Lemma (8.5.6), the components of any Δ-sum or Y-sum decomposition are isomorphic to minors of M. At any rate, the above arguments prove that we may substitute Δ-sums or Y-sums for 3-sums in Theorem (11.3.14). □

Recall that the graph decomposition theorems (10.5.14) and (10.5.21) imply a construction of certain 2-connected graphs via 2- and Δ-sums. Each time, one of the components is a member of a class of building blocks, and the second component is a graph obtained by prior construction steps. We establish the analogous result for the connected regular matroids using Theorem (11.3.14). We treat Δ-sums as well as Y-sums and 3-sums. The terminology is adapted from that of Section 10.5.

(11.3.16) Theorem. *Any connected regular matroid is graphic, cographic, or isomorphic to* R_{10}*, or may be constructed recursively by* 2*-sums and* Δ*-sums (or* Y*-sums, or* 3*-sums) using as building blocks graphic matroids, cographic matroids, or matroids isomorphic to* R_{10}*.*

The proof of Theorem (11.3.16) is similar to that of Theorem (10.5.15). We confine ourselves to compositions involving 2-sums and Δ-sums. The Y-sum case is handled by duality, and the 3-sum case is proved by a simple adaptation of the proof for Δ-sums. We begin with two lemmas.

(11.3.17) Lemma. *Let* M *be a* 3*-connected, regular, nongraphic, and noncographic matroid that is not isomorphic to* R_{10}*. Assume* M *to have a triangle* C*. Then* M *has a* 3*-separation* (E_1, E_2) *where* $|E_1|, |E_2| \geq 6$ *and where one of* E_1*,* E_2 *contains* C*.*

Proof. By Theorem (11.3.14), M has a 3-separation (E_1, E_2) induced by the 3-separation of an R_{12} minor. The latter 3-separation corresponds to the one of (11.3.11) for R_{12}, and thus has six elements on each side. Hence, $|E_1|, |E_2| \geq 6$. If C is contained in E_1 or E_2, we are done. Otherwise, we shift one element of C from one side of the 3-separation (E_1, E_2) to the other one, to get a 3-separation (E_1', E_2') where C is contained in E_1' or E_2'. It is easily checked that the 3-separation of R_{12} depicted in (11.3.11) becomes an exact 4-separation when any one element is shifted from one side to the other. Thus, the element of C that we have shifted from E_1 or E_2 to the other set cannot be an element of the R_{12} minor inducing the 3-separation (E_1, E_2). We conclude that $|E_1'|, |E_2'| \geq 6$, as desired. □

(11.3.18) Lemma. *Let M be a 3-connected, regular, nongraphic, and noncographic matroid that is not isomorphic to R_{10}. Assume M to have either a designated triangle C or a designated element y. Then M is a Δ-sum $M_1 \oplus_\Delta M_2$ where M_1 contains C or y, whichever applies, and where M_2 is graphic or cographic.*

Proof. We prove the case for the triangle C and leave the easier situation with the element y to the reader. We use induction. The smallest case involves an M isomorphic to R_{12}. By Lemma (11.3.17), the matroid R_{12} is a Δ-sum where one component contains C, and where the second component has nine elements and thus is graphic or cographic. Thus, we are done. For larger M, we again apply Lemma (11.3.17). Thus, M is a Δ-sum $M_1 \oplus_\Delta M_2$ where C is part of the component M_1, and where both M_1 and M_2 have at least nine elements. If the second component M_2 is graphic or cographic, we are done. Otherwise, we may assume M_2 to be 3-connected. We define C' to be the triangle of M_2 involved in the Δ-sum. Because of the presence of the triangle C', or by the splitter result for R_{10}, M_2 cannot be isomorphic to the 4-connected R_{10}. By induction, M_2 has a Δ-sum decomposition $M_{21} \oplus_\Delta M_{22}$ where M_{21} contains C', and where M_{22} is graphic or cographic. Via a representation matrix for M displaying the 3-separations involved in the Δ-sums $M_1 \oplus_\Delta M_2$ and $M_{21} \oplus_\Delta M_{22}$, we readily verify that M is a Δ-sum with $M_1 \oplus_\Delta M_{21}$ and M_{22} as components. That Δ-sum has the desired properties. □

Proof of Theorem (11.3.16). Let M be any connected, regular, nongraphic, and noncographic matroid that is not isomorphic to R_{10}. If G is 3-connected, the result follows from Lemma (11.3.18). Otherwise, G is a 2-sum. Choose the 2-sum decomposition, say $M_1 \oplus_2 M_2$, so that M_2 has a minimal number of elements. Evidently, any 2-separation of M_2 contradicts the minimality assumption, so M_2 is 3-connected. If M_2 is graphic, cographic, or isomorphic to R_{10}, we are done. Otherwise, let y be the element of M_2 that together with an element of M_1 defines the 2-sum. By Lemma (11.3.18), M_2 is a Δ-sum $M_{21} \oplus_\Delta M_{22}$ where M_{21} contains the element y, and where M_{22} is graphic or cographic. Via a representation matrix for M, we confirm that M is a Δ-sum where one component is $M_1 \oplus_2 M_{21}$ and where the second component is the graphic or cographic M_{22} as demanded in the theorem. □

An easy generalization of Theorems (11.3.14) and (11.3.16) is possible by the following splitter result.

(11.3.19) Lemma. *F_7 (resp. F_7^*) is a splitter of the binary matroids without F_7^* (resp. F_7) minors.*

Proof. According to the census of small 3-connected binary matroids in Section 3.3, there are just two 3-connected nonregular matroids on eight elements, with representation matrices given by (3.3.24) and (3.3.25). Clearly, both matroids have F_7 and F_7^* minors. Indeed, the matroids are selfdual. Thus, every 3-connected 1-element extension of F_7 (resp. F_7^*) has both F_7 and F_7^* minors. The result then follows from the splitter Theorem (11.3.1). □

We leave it to the reader to rewrite Theorems (11.3.14) and (11.3.16) so that they become results for the matroids without F_7 minors, or for the matroids without F_7^* minors.

One may concatenate matroid decomposition theorems of this section and graph decomposition theorems of Section 10.5. Later in this chapter, in Section 11.5, we meet one such case. At any rate, such concatenation is easy, and the reader may want to try his/her hand at producing potentially useful theorems.

In the next section, we use Theorem (11.3.14) to assemble efficient algorithms to decide regularity of binary matroids and total unimodularity of real matrices.

11.4 Testing Matroid Regularity and Matrix Total Unimodularity

Prior to the introduction of the regular matroid decomposition Theorem (11.3.14), no efficient algorithm was known for testing a binary matroid for regularity, or for deciding total unimodularity of real matrices. We know from Chapter 9 that these two tests are intimately linked. In fact, the results of Section 9.2 imply that an efficient test for one of the two problems can be easily converted to one for the other problem.

In this section, we construct the desired tests, relying, of course, on the regular matroid decomposition Theorem (11.3.14). In addition, we invoke the algorithm of Section 8.4 for finding 1-, 2-, and 3-sums, as well as the graphicness test of Section 10.6.

We begin with the regularity test for binary matroids. Let M be the binary matroid for which regularity is to be decided. We first apply the polynomial algorithm of Theorem (8.4.1) to determine whether or not M is a 1-, 2-, or 3-sum. In the affirmative case, we decompose M into two components. Then we apply the algorithm to each component, etc., until we eventually have a collection of binary matroids, say M_1, $M_2, \ldots, M_n$, none of which is a 1-, 2-, or 3-sum. It is not difficult to prove that n, the number of such matroids, is bounded by a function that is linear in the number of elements of M.

The regular matroid decomposition Theorem (11.3.14) says that M is regular if and only if each one of the matroids M_1, $M_2, \ldots, M_n$ is graphic, or cographic, or isomorphic to R_{10}. We settle graphicness, and if necessary, cographicness, of each one of the matroids with the polynomial algorithm of Theorem (10.6.1). Deciding whether or not a matroid is isomorphic to R_{10} is trivial. If one of the matroids is found to be nongraphic, noncographic, and not isomorphic to R_{10}, then M is not regular. Otherwise, M has been proved to be regular.

The preceding algorithm is clearly polynomial. It can be made very efficient by an appropriate implementation. The algorithm is readily adapted to test total unimodularity of real matrices as follows. Given is a real matrix A. If A is not a $\{0, \pm 1\}$ matrix, we declare A to be not totally unimodular. So assume A to be a $\{0, \pm 1\}$ matrix. Let B be the support matrix of A. We view B to be binary. With the algorithm just described, we test the matroid M represented by B for regularity. If M is not regular, we know A to be not totally unimodular. So assume M, and hence B, to be regular. With the signing algorithm of Corollary (9.2.7), we deduce from B a totally unimodular matrix A'. By Lemma (9.2.6), A is totally unimodular if and only if A' can be obtained from A by scaling of some rows and columns by -1. Implicit in the proof of the lemma is an algorithm that finds the appropriate scaling factors, or determines that A' cannot be scaled to become A. Accordingly, we declare A to be totally unimodular or not.

We just have encountered one very important application of Theorem (11.3.14). We introduce several others in the next section.

11.5 Applications of Regular Matroid Decomposition Theorem

The uses of the regular matroid decomposition Theorem (11.3.14) range from the obvious to the unexpected. In this section, we cover representative instances.

Construction of Totally Unimodular Matrices

We begin with the most obvious case, the construction of the real totally unimodular matrices. These matrices represent precisely the regular matroids. To find a construction for them, we only need to translate the

3-sum version of the construction Theorem (11.3.16) into matrix language. First, we identify the matrix building blocks. We know that (10.2.8) and (10.4.5) provide the two possible GF(2)-representation matrices for R_{10}. With the scheme of Corollary (9.2.7), we sign these two matrices to obtain the following totally unimodular representation matrices.

$$B^{10.1} = \begin{array}{|ccccc|} \hline 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} \; ; \qquad B^{10.2} = \begin{array}{|ccccc|} \hline -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \\ \hline \end{array} \tag{11.5.1}$$

The two totally unimodular matrices representing R_{10}

By Lemma (9.2.6), $B^{10.1}$ and $B^{10.2}$ are, up to scaling of rows and columns with $\{\pm 1\}$ factors, the unique totally unimodular representation matrices for R_{10}.

By the proof of Corollary (9.2.12), all graphic matroids are minors of the graphic matroids that are represented by the binary node/edge matrices. The latter matroids are also represented by the real totally unimodular node/arc incidence matrices, where each nonzero column has exactly one $+1$ and one -1. The minor-taking translates to $\mathbb{R}$-pivots in node/arc incidence matrices and to the deletion of rows and columns. Call any matrix so produced a *network matrix.* The effect of the $\mathbb{R}$-pivots can be readily established by graph operations. We leave it to the reader to work out the details.

At this point, we have the matrix building blocks that correspond to the matroid building blocks of Theorem (11.3.16). We now must understand the 2-sum and 3-sum decomposition/composition. The task is made complicated by the fact that we must interpret these operations in GF(2)-representation matrices without the use of GF(2)-pivots. Only that way can we deduce decomposition rules for totally unimodular matrices that do not involve $\mathbb{R}$-pivots. So let us assume B to be an arbitrary GF(2)-representation matrix of a regular 2- or 3-sum M with components M_1 and M_2. The underlying 2- or 3-separation is $(X_1 \cup Y_1, X_2 \cup Y_2)$. We assume M, and hence B, to be connected.

We start with the 2-sum case. The 2-separation manifests itself in B as depicted below, up to a switching of the roles of $X_1 \cup Y_1$ and $X_2 \cup Y_2$. The submatrix D of B has GF(2)-rank 1.

$$B = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A^1 & 0 \\ \hline X_2 & D & A^2 \\ \hline \end{array} \tag{11.5.2}$$

Matrix B with 2-separation

Correspondingly, the component matroids M_1 and M_2 are, by (8.2.3) and (8.2.4), represented by B^1 and B^2 below once appropriate indices are assigned to the row vector a of B^1 and the column vector b of B^2. These two vectors are nonzero vectors of D. Since GF(2)-rank $D = 1$, the two vectors are unique up to indices. Indeed, because of permutations of rows and columns in D, we may assume $D = b \cdot a$.

(11.5.3)

$$B^1 = \begin{array}{c|c|} & Y_1 \\ \hline X_1 & A^1 \\ \hline & a \\ \hline \end{array} \quad ; \qquad B^2 = \begin{array}{c|c|c|} & & Y_2 \\ \hline X_2 & b & A^2 \\ \hline \end{array}$$

Matrices B^1 and B^2 of 2-sum

We sign B, B^1, and B^2 to get totally unimodular matrices. To simplify the notation, we now assume the matrices B, B^1, B^2 to be already such signed totally unimodular versions. We have the following 2-sum matrix composition rule.

(11.5.4) Matrix 2-Sum Rule. *Given are B^1 and B^2 of* (11.5.3). *Then we derive B of* (11.5.2) *from B^1 and B^2 by letting $D = b \cdot a$ (in $\mathbb{R}$).*

The 3-sum situation is more complicated, but yields to the same approach. At the outset, we assume all matrices to be binary. The underlying 3-separation manifests itself in one of two ways, up to a switching of the role of $X_1 \cup Y_1$ and $X_2 \cup Y_2$. Below, we indicate by B and $\tilde{B}$ the two cases, where GF(2)-rank $D = 2$ and GF(2)-rank $R =$ GF(2)-rank $S = 1$.

(11.5.5)

$$B = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A^1 & 0 \\ \hline X_2 & D & A^2 \\ \hline \end{array} \quad ; \qquad \tilde{B} = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A^1 & S \\ \hline X_2 & R & A^2 \\ \hline \end{array}$$

Matrices B and $\tilde{B}$ with 3-separation

We claim that, correspondingly, the component matroids M_1 and M_2 are represented below by B^1 and B^2, or by $\tilde{B}^1$ and $\tilde{B}^2$, once the vectors a, b, c, d, e, f, g, h of these matrices are appropriately defined and missing indices are added. Specifically, the vectors a and b of B^1 (resp. c and d of B^2) are two arbitrarily selected GF(2)-independent row (resp. column) vectors of D. Let $\overline{D}$ be the 2×2 submatrix of D created by the intersection of the row vectors a and b of D with the column vectors c and d of D. By Lemma (2.3.14), $\overline{D}$ is nonsingular. Because of (8.3.5)–(8.3.7) and row and column permutations, we may assume $D = [c \mid d] \cdot (\overline{D})^{-1} \cdot [a/b]$ (in GF(2)). We define the vector e of $\tilde{B}^1$ (resp. g of $\tilde{B}^2$) to be a nonzero vector of R, and f of $\tilde{B}^1$ (resp. h of $\tilde{B}^2$) to be a nonzero vector of S. Since GF(2)-rank R = GF(2)-rank $S = 1$, the vectors e, g, f, h are unique up to indices. Furthermore, because of row and column permutations in R and S, we may assume $R = g \cdot e$ and $S = f \cdot h$.

(11.5.6)

$$B^1 = \begin{array}{c|c|c|} & Y_1 & \\ \hline X_1 & A^1 & 0 \\ \hline & a & 1 \\ \hline & b & 1 \\ \hline \end{array}\;; \qquad B^2 = \begin{array}{|c|c|c|} & & Y_2 \\ \hline 1 & 1 & 0 \\ \hline c & d & A^2 \\ \hline \end{array}\; X_2$$

$$\tilde{B}^1 = \begin{array}{c|c|c|c|} & Y_1 & & \\ \hline X_1 & A^1 & f & f \\ \hline & e & 0 & 1 \\ \hline \end{array}\;; \qquad \tilde{B}^2 = \begin{array}{|c|c|c|} & & Y_2 \\ \hline 1 & 0 & h \\ \hline g & g & A^2 \\ \hline \end{array}\; X_2$$

Matrices B^1, B^2 as well as $\tilde{B}^1$, $\tilde{B}^2$ of 3-sum

The case of B, B^1, and B^2 is our customary way of displaying 3-sums. The second one, with $\tilde{B}$, $\tilde{B}^1$, and $\tilde{B}^2$, we have not presented before. We show it to be correct by GF(2)-pivots that transform it to the first case. Specifically, suppose the submatrix S of $\tilde{B}$ has a 1 in the lower left-hand corner. Correspondingly, the last element of the leftmost vector f in $\tilde{B}^1$ and the first element of h in $\tilde{B}^2$ are 1s. Perform GF(2)-pivots on these 1s in the respective matrices. Each such GF(2)-pivot exchanges a row index against a column index. After the pivots, one readily confirms that the 3-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ of the new $\tilde{B}$ corresponds to the first case, and that the new $\tilde{B}^1$ and new $\tilde{B}^2$ are the component matrices for that case.

Consider the case of B, B^1, and B^2 of (11.5.5) and (11.5.6). We may sign B to obtain a totally unimodular matrix. In fact, by (11.5.5) and (11.5.6), the signing may be selected so that in the corresponding signing of B^1 and B^2 the explicitly shown 1s are not negated. The same signing convention may be followed for $\tilde{B}$, $\tilde{B}^1$, and $\tilde{B}^2$. As a matter of convenience, let us now assume that B, B^1, B^2, $\tilde{B}$, $\tilde{B}^1$, and $\tilde{B}^2$ are the signed totally unimodular matrices.

The above discussion validates the following matrix 3-sum construction.

(11.5.7) Matrix 3-Sum Rule. *Given are B^1 and B^2, or $\tilde{B}^1$ and $\tilde{B}^2$, of* (11.5.6). *In the first case, we calculate $D = [c \mid d] \cdot (\overline{D})^{-1} \cdot [a/b]$ (in $\mathbb{R}$) to determine B of* (11.5.5). *In the second case, we compute $R = g \cdot e$ and $S = f \cdot h$ (in $\mathbb{R}$) to find $\tilde{B}$ of* (11.5.5).

The next result summarizes the above discussion.

(11.5.8) Lemma. *The matrix* 2- *and* 3-*sum composition rules* (11.5.4) *and* (11.5.7) *correspond precisely to the regular matroid* 2- *and* 3-*sum compositions.*

The desired construction theorem for totally unimodular matrices can now be stated and proved.

(11.5.9) Theorem. *Any connected totally unimodular matrix is up to row and column indices and scaling by $\{\pm 1\}$ factors a network matrix, or is the transpose of such a matrix, or is the matrix $B^{10.1}$ or $B^{10.2}$ of* (11.5.1)*, or may be constructed recursively by matrix* 2*-sums and* 3*-sums. The rules are given by* (11.5.4) *and* (11.5.7)*. The building blocks are network matrices, their transposes, and the matrices $B^{10.1}$ and $B^{10.2}$ of* (11.5.1).

Proof. The conclusion follows directly from Theorem (11.3.16) and Lemma (11.5.8). □

Construction of $\{0, 1\}$ Totally Unimodular Matrices

Closely related to the construction of totally unimodular matrices is that of $\{0, 1\}$ totally unimodular matrices. So suppose B is a regular matrix that requires no signing to achieve total unimodularity. The matrix 2-sum rule obviously can be adopted without modification. The two 3-sum cases with B and $\tilde{B}$ of (11.5.5) are more troublesome. Even though B or $\tilde{B}$ requires no signing, the matrices B^1, B^2, $\tilde{B}^1$, or $\tilde{B}^2$ of (11.5.6) may not be totally unimodular without signing of some entries. By Corollary (9.2.7), it is easy to see that such signing can be confined to the explicitly shown 1s of the matrices of (11.5.6). Suppose such signing is needed for B^1 of (11.5.6). We may assume that the 1 in the lower right corner of B^1 must become a -1. From now on, let B^1 denote that real totally unimodular matrix, with the just-defined -1. We modify the last column of B^1 and add one additional row and column to get the following real matrix $\hat{B}^1$.

(11.5.10)
$$\hat{B}^1 = \begin{array}{c|c|c|c|} & Y_1 & & \\ \hline X_1 & A^1 & 0 & 0 \\ \hline & a & 1 & 0 \\ \cline{2-4} & b & 0 & 1 \\ \cline{2-4} & 0 & 1 & 1 \\ \hline \end{array}$$

Matrix $\hat{B}^1$ derived from B^1

We claim that the $\hat{B}^1$ is totally unimodular. For a proof, we perform an $\mathbb{R}$-pivot on the 1 of $\hat{B}^1$ in the lower right hand corner. We obtain B^1 as submatrix. Indeed, we see by the pivot that $\hat{B}^1$ represents a 3-sum. One component is the matroid of B^1. The second component is isomorphic to $M(W_4)$, the graphic matroid of the wheel with four spokes. Similarly, if needed, we modify B^2 to

(11.5.11)
$$\hat{B}^2 = \begin{array}{c|ccc|c|} & & & & Y_2 \\ \hline & 1 & 1 & 0 & 0 \\ & 1 & 0 & 1 & 0 \\ \hline X_2 & 0 & c & d & A^2 \\ \hline \end{array}$$

Matrix $\hat{B}^2$ derived from B^2

Consider the 3-sum case involving $\tilde{B}^1$ and $\tilde{B}^2$. Assume the explicitly shown 1 in one or both of these matrices requires signing. The modifications are as follows, where $\hat{\tilde{B}}^1$ (resp. $\hat{\tilde{B}}^2$) corresponds to $\tilde{B}^1$ (resp. $\tilde{B}^2$).

(11.5.12)
$$\hat{\tilde{B}}^1 = \begin{array}{c|c|ccc|} & Y_1 & & & \\ \hline X_1 & A^1 & f & f & 0 \\ \hline & e & 0 & 0 & 1 \\ & 0 & 0 & 1 & 1 \\ \hline \end{array} \qquad \hat{\tilde{B}}^2 = \begin{array}{c|ccc|c|} & & & & Y_2 \\ \hline & 1 & 1 & 0 & 0 \\ & 1 & 0 & 0 & h \\ \hline X_2 & 0 & g & g & A^2 \\ \hline \end{array}$$

Matrix $\hat{\tilde{B}}^1$ and $\hat{\tilde{B}}^2$ derived from $\tilde{B}^1$ and $\tilde{B}^2$

We leave it to the reader to verify by $\mathbb{R}$-pivots that the modifications are appropriate. Indeed, $\hat{\tilde{B}}^1$ (resp. $\hat{\tilde{B}}^2$) of (11.5.12) represents a 3-sum; one component is represented by $\tilde{B}^1$ (resp. $\tilde{B}^2$) of (11.5.6), and the second component is isomorphic to $M(W_4)$.

The reader likely anticipates that the analogue of Theorem (11.5.9) holds for $\{0, 1\}$ totally unimodular matrices, where this time we permit for 3-sums the cases of (11.5.10)–(11.5.12) besides the ones of (11.5.6). That guess is correct. But the proof requires some care. For instance, we must show that the matrices of (11.5.10)–(11.5.12) are always smaller than the matrix of (11.5.5) produced by them. We must also establish that any one of the matrices of (11.5.10)–(11.5.12) is graphic or cographic if the related matrix of (11.5.6) is graphic or cographic. The latter task may seem very difficult. Indeed, in general it cannot be accomplished, as we find out in the proof of the next theorem.

(11.5.13) Theorem. *Any connected $\{0, 1\}$ totally unimodular matrix is up to row and column indices a $\{0, 1\}$ network matrix, or the transpose of such a matrix, or the matrix $B^{10.1}$ of (11.5.1), or may be constructed recursively by matrix 2-sums and 3-sums. The 2-sum rule is given by (11.5.4). The 3-sums are specified by the 3-sum rule (11.5.7), except that we also allow $\hat{B}^1$, $\hat{B}^2$, $\hat{\tilde{B}}^1$, $\hat{\tilde{B}}^2$ of (11.5.10)–(11.5.12) as component matrices. The building blocks are $\{0, 1\}$ network matrices, their transposes, and the matrix $B^{10.1}$ of (11.5.1).*

Proof. We invoke Theorem (11.5.9) and the above derivation of $\hat{B}^1$, $\hat{B}^2$, $\hat{\hat{B}}^1$, $\hat{\hat{B}}^2$. The case of $B^{10.2}$ is not possible since that matrix cannot be scaled to become a $\{0,1\}$ matrix. Thus, we are done, unless $\hat{B}^1$, $\hat{B}^2$, $\hat{\hat{B}}^1$, or $\hat{\hat{B}}^2$ is needed, and is of the same size as B or is not graphic or cographic. Because of pivots, symmetry, and duality, we may select any one of the above matrices, say $\hat{B}^1$, to eliminate these concerns.

First, we prove that $\hat{B}^1$ is smaller than B. By the proof of Theorem (11.3.16), the 3-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ of B inducing the 3-sum that in turn produces $\hat{B}^1$ plus B^2 or $\hat{B}^2$ satisfies $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq 6$. But $\hat{B}^1$ has length $|X_1 \cup Y_1| + 5$ and thus is smaller than B.

The second part is more complicated. We want to show that B^1 graphic or cographic implies $\hat{B}^1$ graphic or cographic. This goal turns out to be attainable in all cases but one. That exception we handle by a switch to a different 3-separation of B.

Suppose B^1 is graphic. As stated above, $\hat{B}^1$ represents a 3-sum. One component is the matroid of B^1. The second component is isomorphic to $M(W_4)$. We claim that 3-sum to be graphic. Indeed, any graph G for B^1 can be converted to one for $\hat{B}^1$ as follows. By (11.5.6), the edges of G not indexed by $X_1 \cup Y_1$ form a triangle C, say with vertices u, v, w. Then on an appropriately selected edge of C, say with endpoints u and v, we place a midpoint and connect that new node with w by a new edge. One easily verifies that this construction does produce a graph for $\hat{B}^1$.

We examine the second possibility, where B^1 is cographic and not graphic. Thus, B^1 is nonplanar. Let H be any graph for the transpose of B^1. By (11.5.6), the edges of H corresponding to the unlabeled rows and columns of B^1 form a cocircuit C^* of H of cardinality 3.

Assume C^* to be a 3-star of H. Analogously to the earlier case, one confirms $\hat{B}^1$ to be cographic, as desired.

Assume C^* to be not a 3-star of H. Thus, removal of the three edges of C^* transforms H to two connected nonempty graphs H_1 and H_2. Accordingly, the minor $H \backslash C^*$ is not 2-connected. That minor corresponds to the transpose of the submatrix A^1 of B^1. Thus, the transpose of A^1, and hence A^1, are not connected. Indeed, A^1 is a 1-sum of two matrices A^{11} and A^{12} where A^{11} corresponds to H_1, and A^{12} to H_2. We derive from H a graph $\tilde{H}_1$ (resp. $\tilde{H}_2$) by contracting the edges of H_2 (resp. H_1). In $\tilde{H}_1$ and $\tilde{H}_2$, the set C^* is a 3-star. If both $\tilde{H}_1$ and $\tilde{H}_2$ are planar, then one easily verifies H to be planar as well, a contradiction of the fact that B^1 is cographic and not graphic. Thus, we may assume $\tilde{H}_1$ to be nonplanar, and hence to have at least nine edges. We now redefine the 3-separation of B by shifting the submatrix A^{12} of A^1 to A^2. For the new 3-separation, the new B^1 corresponds to $\tilde{H}_1$, and thus is still cographic and not graphic. But C^* is now a 3-star of $\tilde{H}_1$, so the earlier case applies. $\square$

So far, we have seen two applications that involve a rather obvious translation of Theorem (11.3.16). The next application makes a more sophisticated use of that theorem.

Characterization of Cycle Polytope

Given is a connected binary matroid M with groundset E. To each element $e \in E$, a real weight w_e is assigned. We want to find a disjoint union C of circuits of M so that $\sum_{e \in C} w_e$ is maximized. This problem occurs in a number of settings. We describe one, and reference others in Section 11.6.

Let G be a connected graph with edge set E and real weights w_e, $e \in E$. We want to partition the vertex set V of G into sets V_1 and V_2 such that the sum of the weights of the set C^* of edges connecting V_1 and V_2 is maximized. The set C^* is readily seen to be a disjoint union of the cocircuits of the graphic matroid $M(G)$ of G. Conversely, any such disjoint union of cocircuits of $M(G)$ is the set C^* for some partition of V. Thus, the graph problem, which usually is called the *max cut problem*, becomes the earlier matroid problem with $M = M(G)^*$.

The matroid problem can be easy or difficult, depending on M and on the sign of the weights. In general, the problem is known to be $\mathcal{NP}$-hard. We sketch a polyhedral approach for a special subclass. Let x^C be the characteristic vector for a disjoint union C of circuits of M. Thus, x^C is indexed by E, and the entry in position $e \in E$ of x^C is equal to 1 if $e \in C$, and equal to 0 otherwise. We view the x^C vectors to be in $\mathbb{R}^E$. Define $P(M)$ to be the convex hull of the x^C vectors. Usually, $P(M)$ is called the *cycle polytope* of M. By results of polyhedral combinatorics, we can solve the given problem efficiently if we can determine whether or not a vector $x \in \mathbb{R}^E$ is in $P(M)$, and if for $x \notin P(M)$ we can provide a hyperplane that separates x from $P(M)$. As a first step, we thus strive to find linear inequalities that are satisfied by all points of $P(M)$. We say that such inequalities are *valid* for $P(M)$. Notationally, for any subset $\overline{E} \subseteq E$, let $x(\overline{E}) = \sum_{e \in \overline{E}} x_e$.

Obviously, the inequalities

$$(11.5.14) \qquad 0 \leq x_e \leq 1;\ e \in E$$

are valid for $P(M)$. We derive additional valid inequalities as follows. By Lemma (3.3.6), each circuit of M intersects each cocircuit in an even number of elements. Let C be a disjoint union of circuits, C^* be a cocircuit, and F be a subset of C^* of odd cardinality. We claim that the following inequality is valid for $P(M)$.

$$(11.5.15) \qquad x(F) - x(C^* \setminus F) \leq |F| - 1;\ F \subseteq C^* = \text{ cocircuit; } |F| \text{ odd.}$$

Since $P(M)$ is the convex hull of the x^C vectors, we may establish validity of (11.5.15) by proving it for $x = x^C$, C being an arbitrary disjoint union of circuits of M. If F is not contained in C, then $x(F) \leq |F| - 1$, and (11.5.15) clearly holds. So assume $F \subseteq C$. We know that $F \subseteq C^*$. Since $|F|$ is odd and $|C \cap C^*|$ is even, C has at least one element in $C^* \setminus F$. Thus, once more (11.5.15) holds. We conclude that (11.5.15) is valid.

Define $Q(M)$ to be the subset of $\mathbb{R}^E$ defined by the inequalities of (11.5.14) and (11.5.15). When is $P(M) = Q(M)$? We sketch the answer to this question and its proof. First, one shows that $P(M) \neq Q(M)$ if M has a minor isomorphic to F_7^*, $M(K_5)^*$, or R_{10}. Second, one establishes that $P(M) = Q(M)$ if M is graphic or isomorphic to F_7, $M(K_{3,3})^*$, or $M(G_8)^*$, where G_8 is given by (10.5.9). Third, one proves that $P(M) = Q(M)$ if M is a 2-sum or Y-sum of two matroids M_1 and M_2 for which $P(M_1) = Q(M_1)$ and $P(M_2) = Q(M_2)$. Fourth and last, one concatenates the dual version of Theorem (10.5.15) and Theorem (11.3.16) to the following result.

(11.5.16) Theorem. *Let M be a connected binary matroid having no F_7^* or $M(K_5)^*$ minors. Then M is graphic or isomorphic to F_7, $M(G_8)^*$, $M(K_{3,3})^*$ or R_{10}, or may be constructed recursively by 2-sums and Y-sums. The building blocks are graphic matroids and matroids isomorphic to F_7, $M(G_8)^*$, $M(K_{3,3})^*$, and R_{10}.*

The four ingredients clearly imply the following conclusion.

(11.5.17) Theorem. *Let M be a connected binary matroid. Then $P(M) = Q(M)$ if and only if M has no F_7^*, $M(K_5)^*$, or R_{10} minors.*

References for additional material about $P(M)$ and $Q(M)$ are included in Section 11.6.

We describe some additional applications of the regular matroid decomposition theorem without proofs. The first one concerns the number of nonzeros in the rows of totally unimodular matrices having fewer rows than columns.

Number of Nonzeros in Totally Unimodular Matrices

Let A be a totally unimodular matrix, say of size $m \times n$ with $m \leq n$. For $i = 1, 2, \ldots, m$, let p_i be the number of nonzeros in row i of A. Define $p^* = \min p_i$. Trivially, p^* is bounded from above by n, the number of columns of A. That upper bound is attained by the $m \times n$ matrix A containing only 1s. The example matrix has parallel columns. Thus, one may conjecture that a tighter upper bound on p^* may exist in the absence of parallel columns. The next theorem confirms this conjecture by proving m, the number of rows of A, to be an upper bound on p^* when A has no parallel columns.

(11.5.18) Theorem. *Let A be a totally unimodular matrix of size $m \times n$ with $m \leq n$. For $i = 1, 2, \ldots, m$, let p_i be the number of nonzeros in row i of A, and define $p^* = \min p_i$. If A has no parallel columns, then $p^* \leq m$.*

Triples in Circuits

The next theorem addresses the following question. Given are three elements e, f, g of a binary matroid M. When does some circuit of M include them all?

It is not difficult to reduce the problem to the case where the matroid is sufficiently connected. For that situation the answer is as follows.

(11.5.19) Theorem. *Let e, f, g be distinct elements of a 3-connected binary matroid M. Assume that M does not have a 3-separation with at least four elements on each side. Then there is no circuit of M containing e, f, and g if and only if $\{e, f, g\}$ is a cocircuit of M or the following condition holds. M is graphic, and in the corresponding graph, the edges e, f, and g are edges with a common endpoint.*

Odd Cycles

Given is an undirected graph G. What is the structure of G if every two cycles of G, each of odd length, have a node in common? Here, too, it is not difficult to reduce the problem to the situation where the graph is sufficiently connected. The following theorem provides the answer for that case.

(11.5.20) Theorem. *Let G be a 3-connected graph on at least six vertices and without parallel edges. Assume that G does not have a 3-separation with at least four edges on each side. Then every two cycles of G, each with an odd number of edges, have a common vertex if and only if G observes* (i) *or* (ii) *below.*

(i) *Deletion of some vertex or deletion of the edges of a triangle from G results in a bipartite graph.*
(ii) *G can be drawn in the projective plane so that every region is bounded by an even number of edges.*

Another application concerns the construction of the connected, undirected, and signed graphs without so-called odd-K_4 minors. Relevant definitions and details of the construction are included in Chapter 13.

In the last section, we point out some extensions and identify relevant references.

11.6 Extensions and References

The main reference for the entire chapter is Seymour (1980b), which contains Seymour's decomposition theorem for regular matroids.

Lemmas (11.2.1) and (11.2.9) constitute the easy part of the regular matroid decomposition theorem. They are due to Brylawski (1975). For $k \geq 4$, k-sums with regular components are not necessarily regular. An example for $k = 4$ is given in Truemper (1985b). Intuitively, one is tempted to argue that regularity of k-sums must be assured when the connecting matroid is sufficiently large and structurally rich. Indeed, this notion can be made precise by sufficient conditions that, for any $k \geq 4$, assure regularity of k-sums with regular components. Because of space limitations, we omit a detailed treatment.

All decomposition results of Section 11.3 either are taken directly from Seymour (1980b), or are implied by that reference. We have described them using 3-, Δ-, and Y-sums. Seymour (1980b) relies on Δ-sums.

A very efficient version of the regularity/total unimodularity test of Section 11.4 is presented in Truemper (1990). Indeed, the algorithm described there has the currently best worst-case bound of all known schemes. Other polynomial algorithms, using different ideas, are given in Cunningham and Edmonds (1978), Bixby, Cunningham, and Rajan (1986), and Rajan (1986). We should mention that Seymour (1980b) already contains implicitly a polynomial, though not very efficient, testing algorithm. A polynomial test for deciding regularity of matroids *a priori* not known to be binary is given in Truemper (1982a).

Theorems (11.5.9) and (11.5.13), which are matrix versions of Theorem (11.3.16), are given without proof in Seymour (1985a), and Nemhauser and Wolsey (1988). Seymour (1985a) uses the hypergraph terminology of Berge (1973). The polytope question $P(M) \stackrel{?}{=} Q(M)$ answered by Theorem (11.5.17) is just one example of numerous questions concerning flows, circuits, and cutsets in matroids. Seymour (1981a) contains a wide-ranging investigation of these issues. One of them concerns the *sum of circuits property* first defined in Seymour (1979b). It is shown in Seymour (1981a) that this property holds for a matroid M if and only if M is binary and has no F_7^*, $M(K_5)^*$, or R_{10} minors. That result, Theorems (10.5.15) and (11.3.16), and an amazing symmetry of $P(M)$ allow one to completely resolve the $P(M) \stackrel{?}{=} Q(M)$ question by Theorem (11.5.17), which is due to Barahona and Grötschel (1986). The proof sketched here is from Grötschel and Truemper (1989b), which contains additional results about $P(M)$ and treats computational aspects. Earlier results for special matroid classes and applications are described in Orlova and Dorfman (1972), Edmonds and Johnson (1973), Hadlock (1975), Barahona (1983), Barahona and Mahjoub (1986), and Barahona, Grötschel, Jünger, and Reinelt (1988). Additional results for the cycle polytope are given in Grötschel and Truemper (1989a). Theorem (11.5.18) is proved in Bixby and Cunningham (1987). Theorem (11.5.19) is from Seymour (1986a). Theorem (11.5.20) is due to Lovász and is included in Gerards, Lovász, Schrijver, Seymour, and Truemper (1991). References for the odd-K_4 result are given in Chapter 13. Additional applications of the regular matroid decomposition theorem are described in Seymour (1981d), (1981f). Bland and Edmonds (1978) have used the decomposition to reduce linear programs with totally unimodular constraint matrix to a sequence of maximum flow and shortest route problems.