Chapter 10

Graphic Matroids

10.1 Overview

At this point, we are ready to investigate the first complicated class of binary matroids treated in this book: the class of graphic matroids. Recall the following definitions and results. K_n is the complete graph on n nodes, and $K_{m,n}$ is the complete bipartite graph with m nodes on one side and n nodes on the other side. For any graph G, the corresponding graphic matroid is regular and is denoted by $M(G)$. F_7 denotes the nonregular Fano matroid. Finally, the asterisk is used as dualizing operator.

In this chapter, we first identify certain minimal regular matroids that are not graphic, or that are not graphic and not cographic. Specifically, in Section 10.2, we characterize the planar regular matroids, i.e., the matroids produced by planar graphs. In Section 10.3, we investigate the behavior of nongraphic regular matroids with $M(K_{3,3})$ minors. We build upon these results in Section 10.4. There we prove two profound theorems of matroid theory: Tutte's theorem that $M(K_5)^*$ and $M(K_{3,3})^*$ are the minimal regular matroids that are not graphic, and Seymour's theorem that the two matroids R_{10} and R_{12} defined in Section 9.2 are the minimal regular, 3-connected matroids that are not graphic and not cographic. The latter theorem is one of the two central ingredients in the proof of Seymour's profound decomposition theorem for regular matroids. We take up the latter theorem and its proof in Chapter 11.

In Section 10.5, we introduce a simple but very useful decomposition scheme that will be used repeatedly in Chapters 11–13. Indeed, the scheme is the second ingredient in the proof of the just-mentioned decomposition theorem for regular matroids. In Section 10.5, we employ the scheme to deduce some graph decomposition theorems, among them Wagner's famous decomposition theorem for the graphs without K_5 minors.

In Section 10.6, we present an efficient algorithm for deciding graphicness of binary matroids and for deciding whether or not a real $\{0, \pm 1\}$ matrix A is the coefficient matrix of a network flow problem. Finally, in Section 10.7, we indicate extensions and provide references.

The chapter requires knowledge of Chapters 2, 3, and 5–8. We also make use of the easy part of Theorem (9.3.2), according to which the Fano matroid is nonregular.

10.2 Characterization of Planar Matroids

In this section, we prove that a regular matroid is planar if and only if it has no $M(K_{3,3})$, $M(K_{3,3})^*$, $M(K_5)$, or $M(K_5)^*$ minors. The result constitutes one of two preparatory steps toward proofs of the theorems by Tutte and Seymour cited in the introduction to this chapter. The latter results are established in Section 10.4.

Definition of Graph with T Nodes

For the arguments of this section and the next one, we need a convenient way to encode and manipulate 1-element binary additions of any graphic matroid N. Let $\overline{B}$ be a binary representation matrix of N, say with row index set X and column index set Y. Thus, the matroid $M = N{+}z$ is represented by the following matrix B.

(10.2.1)

$$B = \begin{array}{c|c|c|} & Y & z \\ \hline X & \overline{B} & b \\ \hline \end{array}$$

Matrix B for matroid $M = N{+}z$

Let G be any connected graph for N, i.e., $M(G)$ is N. The graph G need not be unique. The row index set X of B is a tree of G. Suppose we premultiply the matrix $[I \mid \overline{B} \mid b]$ with the node/edge incidence matrix of the tree X. By the results of Section 3.2, that multiplication turns the submatrix $[I \mid \overline{B}]$ into the node/edge incidence matrix, say F, of G. The column vector b becomes some vector, say d. Evidently, F and $[F \mid d]$ have the same GF(2)-rank. Accordingly, since every column of F has exactly two 1s or none, the vector d must have an even number of 1s. Since each row of F corresponds to a node of G, we can associate the 1s of d with a node subset T of G. Each node of T we call a T *node*. In drawings of G, we denote each T node by a square box. By the construction, M is completely represented by G and the set T.

Section 3.2 contains the following alternate way of representing M via G. Each 1 of the vector b of (10.2.1) corresponds to an edge of the tree X. To single out these edges, we temporarily paint them red. In general, the red edges form a red subgraph $\overline{X}$ of G without cycles. The following lemma links that subgraph to the set T.

(10.2.2) Lemma. *A node of G is a T node if and only if the node has an odd number of red edges incident.*

Proof. In the matroid M, the element z forms a fundamental circuit C with X. Indeed, $C-\{z\}$ is nothing but the red subgraph $\overline{X}$. That subgraph indexes a column submatrix $\overline{F}$ of F. Thus, the columns of $[\overline{F} \mid d]$, which are indexed by C, are GF(2)-mindependent, i.e., the columns are GF(2)-dependent, but any proper subset of the columns is GF(2)-independent. This is so if and only if the 1s of d are in the rows of $\overline{F}$ with an odd number of 1s. Equivalently, a node of G is in T if and only if the node has an odd number of red edges incident. $\square$

Lemma (10.2.2) implies a convenient method for determining the T nodes. Suppose we have B of (10.2.1) and a graph G for the matroid N of $\overline{B}$. For each 1 in the vector b, say in row $x \in X$ of B, we temporarily paint the edge x of G red. Then we define the nodes of G with odd number of red edges incident to be the T nodes. Finally, we declare the red edges to be unpainted again.

We claim that M is graphic if $|T| = 2$. Indeed, the red subgraph $\overline{X}$ is then a red path, say from node u of G to node v. We thus may add an edge z connecting u and v to get a graph representing M.

Example Graphs with T Nodes

Below, we carry out the derivation of T for a few nongraphic matroids that are important for our purposes. We depict each instance using the following scheme.

(10.2.3)

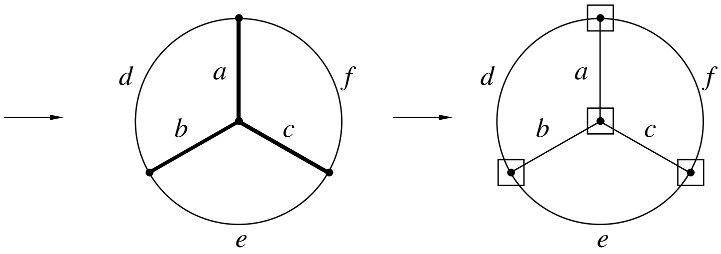
$B = \begin{array}{c\|c\|c\|} & Y & z \\ \hline X & \overline{B} & b \\ \hline \end{array}$ Matrix B	$\longrightarrow$	Graph G with red subgraph $\overline{X}$ (indicated by bold edges)	$\longrightarrow$	Graph G with T nodes (indicated by squares enclo- sing nodes

Representation of M by matrix B, by graph G with red edges, and by graph G with T nodes

Here are the example matroids M.

(10.2.4)

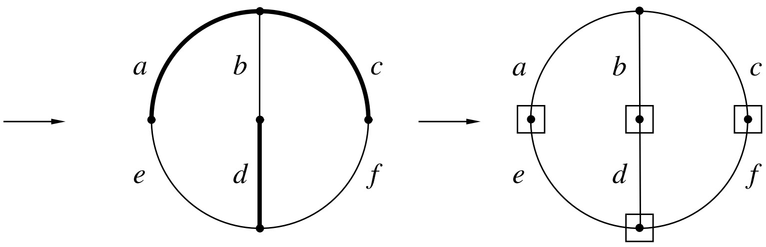
$$\begin{array}{c|ccc|c|} & d & e & f & z \\ \hline a & 1 & 0 & 1 & 1 \\ b & 1 & 1 & 0 & 1 \\ c & 0 & 1 & 1 & 1 \\ \hline \end{array}$$



Fano Matroid F_7 (defined by (9.3.1))

(10.2.5)

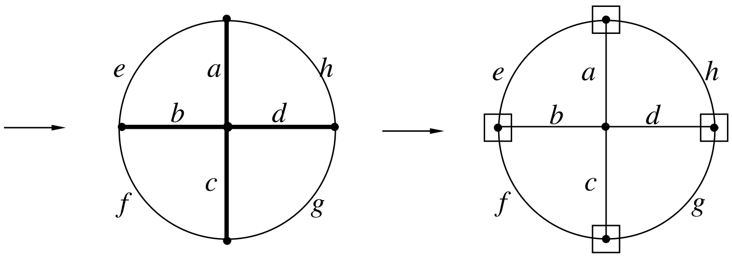
$$\begin{array}{c|cc|c|} & e & f & z \\ \hline a & 1 & 0 & 1 \\ b & 1 & 1 & 0 \\ c & 0 & 1 & 1 \\ d & 1 & 1 & 1 \\ \hline \end{array}$$



Fano Dual F_7^*

(10.2.6)

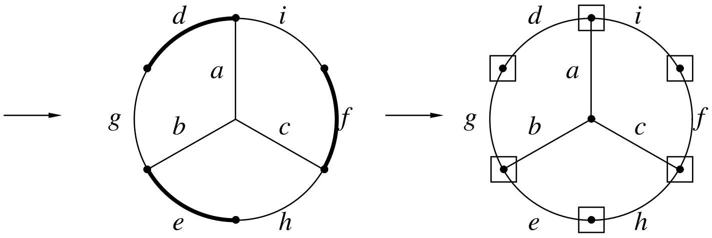
$$\begin{array}{c|cccc|c|} & e & f & g & h & z \\ \hline a & 1 & 0 & 0 & 1 & 1 \\ b & 1 & 1 & 0 & 0 & 1 \\ c & 0 & 1 & 1 & 0 & 1 \\ d & 0 & 0 & 1 & 1 & 1 \\ \hline \end{array}$$



$M(K_{3,3})^*$ (defined by (3.2.46))

(10.2.7)

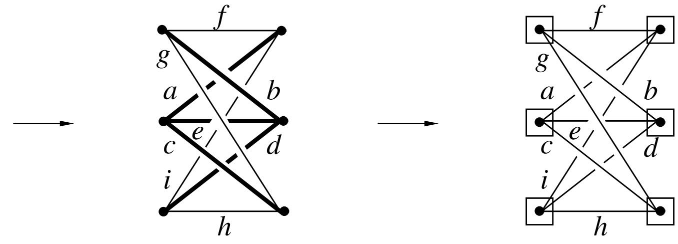
$$\begin{array}{c|ccc|c|} & g & h & i & z \\ \hline a & 1 & 0 & 1 & 0 \\ b & 1 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 1 & 0 & 0 & 1 \\ e & 0 & 1 & 0 & 1 \\ f & 0 & 0 & 1 & 1 \\ \hline \end{array}$$



$M(K_5)^*$ (defined by (3.2.44))

(10.2.8)

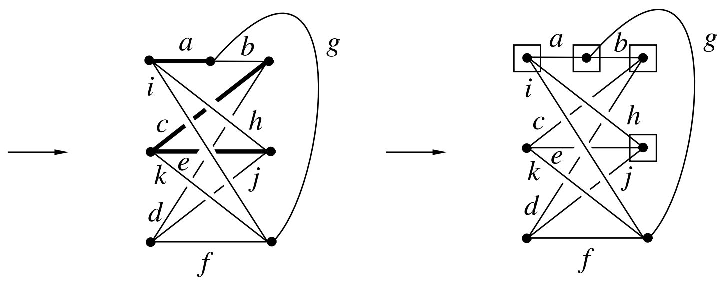
$$\begin{array}{c|cccc|c|} & f & g & h & i & z \\ \hline a & 1 & 0 & 0 & 1 & 1 \\ b & 1 & 1 & 0 & 0 & 1 \\ c & 0 & 1 & 1 & 0 & 1 \\ d & 0 & 0 & 1 & 1 & 1 \\ e & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$



R_{10} (defined by (9.2.13))

(10.2.9)

$$\begin{array}{c|c|ccccc|} & z & g & h & i & j & k \\ \hline a & 1 & 0 & 1 & 1 & 0 & 0 \\ b & 0 & 1 & 1 & 1 & 0 & 0 \\ c & 1 & 0 & 1 & 0 & 1 & 1 \\ d & 0 & 1 & 0 & 1 & 1 & 1 \\ e & 1 & 0 & 1 & 0 & 1 & 0 \\ f & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}$$



R_{12} (defined by (9.2.14); note that column z is the first column)

In each of the example cases, we have established the set T via a particular tree X of G. Since T was originally defined via the matrix $[F \mid d]$, any other tree of G would have produced the same set T. For this reason, we may always select a particularly suitable tree X when discussing some matroid operation and its impact on G and T. We have three such operations in mind: deletion of an element $y \neq z$ of M that is not a coloop of G, contraction of an element $x \neq z$ of M that is not a loop of G, and the switching operation of Section 3.2. We take up these operations next.

Deletion, Contraction, and Switching in Graphs with T Nodes

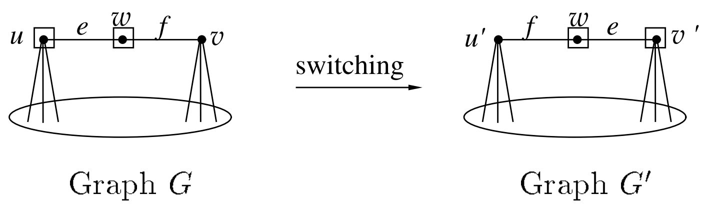
Suppose we delete from M an element $y \neq z$ that is not a coloop of G. Thus, G has a tree X that is also a tree of $G\backslash y$. Then $G\backslash y$ with the original T node labels represents $M\backslash y$.

Suppose in M we contract an element $x \neq z$ that is not a loop. We may assume that the tree X of G contains x. Then $X - \{x\}$ is a tree of G/x and a base of M/x. We claim that the red edges of $X - \{x\}$ plus z

constitute the fundamental circuit that z forms with $X - \{x\}$ in M/x. For a proof, we delete row $x \in X$ from the matrix B to obtain a representation matrix for M/x. Inspection of the column z of the latter matrix verifies the claim. Thus, we may use the red edges of $X - \{x\}$ to represent the element z of M/x by a node subset T' of G/x analogously to the use of the red edges of X to define the node subset T of G. We deduce T' directly from T by the following rules. Let i be any node of G different from the two endpoints u and v of x. Then i is in T' if and only if i is in T. Define w to be the node of G/x created from u and v of G when x is contracted, i.e., $w = (u \cup v) - \{x\}$. Then w is in T' if and only if exactly one of u and v is in T. Validity of these rules follows directly from the just-mentioned fact about the red edges of $X - \{x\}$, and from a simple parity argument involving the red edges of G incident at nodes u or v.

Recall the switching operation of Section 3.2. On hand must be a 2-separation of G, say involving subgraphs G_1 and G_2. The graph G_1 is removed, turned over, and reattached to G_2. The resulting graph G' is 2-isomorphic to G. Thus, G' also represents $M\backslash z$. We want to deduce the set T' for G' from the set T of G. That is, G' and T' are to represent M analogously to G and T. Evidently, any tree X of G is one of G', and the red edges of X may be used to deduce T' for G'. Thus, any node different from the two nodes joining G_1 and G_2 is in T' if and only if it is in T. The rules for the latter two nodes are also quite simple. For the general situation, we leave their derivation to the reader. Instead, we just examine the special case where the graph G has two series edges e and f with a common endpoint $w \in T$ with degree 2. Let u be the second endpoint of edge e, and let v be that of edge f. Assume $u \neq v$. The switching operation resequences e and f. Thus, u becomes $u' = (u - \{e\}) \cup \{f\}$ and v becomes $v' = (v - \{f\}) \cup \{e\}$. For the derivation of T' of G' from T of G, we may suppose that the tree X of G includes both e and f. Since $w \in T$, exactly one of the edges e and f is red. Correspondingly, the parity of the number of red edges at u (resp. v) in G is different from the parity of the number of red edges at u' (resp. v') in G'. Thus, u' (resp. v') is in T' if and only if u (resp. v) is not in T. An example case is depicted below. As before, nodes of T are indicated by squares.

(10.2.10)



Effect of switching on T nodes

The set T' produced by any switching may have cardinality different from that of T. Thus, we are justified in assuming for convenience that $|T|$ is minimal under switchings. In that case, M is graphic if and only if $|T|$ is 0 or 2. The 0 case corresponds to z being a loop of M. It cannot occur when M is connected.

Characterization of Planar Matroids

We now have sufficient machinery to prove the main result of this section, which characterizes planarity of regular matroids in terms of excluded minors.

(10.2.11) Theorem. *A regular matroid M is planar if and only if M has no $M(K_{3,3})$, $M(K_{3,3})^*$, $M(K_5)$, or $M(K_5)^*$ minors.*

Proof. The "only if" part holds since planarity is maintained under minor-taking and since $M(K_{3,3})$, $M(K_{3,3})^*$, $M(K_5)$, and $M(K_5)^*$ are not planar. For the proof of the nontrivial "if" part, let M be a regular matroid all of whose proper minors are planar. Thus, M is minimally nonplanar with respect to the taking of minors. We must show that M is isomorphic to $M(K_{3,3})$, $M(K_{3,3})^*$, $M(K_5)$, or $M(K_5)^*$.

If M is graphic or cographic, then the desired conclusion is provided by Theorem (7.4.1), which characterizes nonplanar graphs by exclusion of $K_{3,3}$ and K_5 minors. Thus, we may assume from now on that M is not graphic and not cographic.

We claim that M is 3-connected. If that is not the case, then M has a 1- or 2-separation. By Lemma (8.2.2) or (8.2.6), M is a 1- or 2-sum. In either case, the components of the sum are proper minors of M and thus planar. But by Lemma (8.2.2) or (8.2.7), the latter conclusion implies M to be planar as well, a contradiction.

By the census of Section 3.3, every 3-connected regular matroid on at most eight elements is planar. Thus, M has at least nine elements.

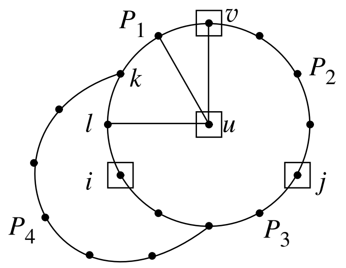
We apply the binary matroid version of the wheel Theorem (7.3.3) to the 3-connected, nongraphic, and noncographic M on at least nine elements. Accordingly, M must have an element z so that at least one of the minors M/z and $M\backslash z$ is 3-connected. If the M/z case applies, we replace M by its dual. Since all assumptions made so far for M are invariant under dualizing, this change does not affect the proof. Thus, we may assume that $M\backslash z$ is 3-connected. By the minimality of M, the 3-connected minor $M\backslash z$ is planar. Let G be the corresponding planar graph. We extend G to a representation of M by selecting an appropriate subset T of nodes of G for the element z. Recall that the cardinality of T is necessarily even. Since M is not graphic, $|T| \geq 4$.

Suppose G is a wheel. It is easily verified that one can delete spokes from G and contract rim edges so that the wheel with four spokes and with four T nodes results. Then we either have, up to indices, the $M(K_{3,3})^*$ case of (10.2.6) and are done, or we can delete one spoke and contract one rim edge to obtain an instance of the F_7 case of (10.2.4), which contradicts the regularity of M.

We are left with the case where G is not a wheel. By the wheel Theorem (7.3.3), G has an edge e so that G/e or $G\backslash e$ is 3-connected. Assume the latter case. The deletion of the edge e from G leaves the number of T nodes unchanged. Note that $G\backslash e$ plus these T nodes represents $M\backslash e$. Since $G\backslash e$ is 3-connected and $|T| \geq 4$, the minor $M\backslash e$ must be 3-connected and nongraphic, a contradiction of the minimality of M. Thus, the case of a 3-connected G/e must be at hand. We claim that the contraction of the edge e in G must reduce the number of T nodes to 2. If this is not the case, then arguments analogous to those for $M\backslash e$ prove M/e to be 3-connected and nongraphic, a contradiction. Since G/e has exactly two T nodes, the graph G must have exactly four T nodes, two of which must be the endpoints of e, say u and v. Define i and j to be the other two T nodes of G.

By the 3-connectedness of G and Menger's theorem, there is a path P from i to u and a second path Q from j to u such that these paths have only node u in common and do not involve node v. Imagine G drawn in the plane. Then deletion of node u would create a new face. The boundary would be a cycle, say C. Clearly, v has become a node of C. If in G the T node i does not lie on C, then we contract the edge of the path P incident at i. After suitable repetition of this process, the T node has become a node of C. Then we declare that T node to be i again. Similarly, we make the T node j a node of C. At that time, the edges incident at node u and those of the cycle C constitute a subdivision of a wheel with at least three spokes. The rim of that wheel subdivision contains the T nodes v, i, and j. The latter nodes induce a partition of the rim into three paths, say P_1 from i to v, P_2 from j to v, and P_3 from i to j. Simple case checking confirms that the arguments made earlier for the wheel case of G apply here fully unless every edge incident at node u has its second endpoint in just one of the paths P_1 or P_2, say P_1. Below, we show a typical instance of that exceptional case, together with an additional path P_4. That path is nothing but a portion of the previously defined path Q that in G connected nodes j and u. In the general case, the path P_4 connects an interior node of the path $P_2 \cup P_3$ with an interior node k of the path P_1.

(10.2.12)



Wheel subdivision plus path P_4

The drawing does not show any edges of the previously defined path P. But there are still enough edges left from that path so that node u can be reached from node i while avoiding all nodes of P_4. This fact and the planarity of G imply that an interior node l of the subpath of P_1 from i to k must be connected with node u, as depicted in (10.2.12).

Evidently, we can contract enough edges of the paths P_1–P_4 and delete some edges incident at node u so that node k becomes the center node of a wheel graph with four spokes and with four T nodes on its rim. Thus, up to indices, we have an instance of the $M(K_{3,3})^*$ case of (10.2.6). $\square$

Related to Theorem (10.2.11) is the following pioneering combinatorial characterization of planar graphs due to Whitney. It constitutes the first matroid result about graph planarity.

(10.2.13) Corollary. *A graph G is planar if and only if $M(G)^*$ is graphic.*

Proof. The "only if" part is clear. For proof of the "if" part, let $M = M(G)$. Since $M^* = M(G)^*$ is graphic, M^* has no $M(K_5)^*$ or $M(K_{3,3})^*$ minors. Thus, M has no $M(K_5)$ or $M(K_{3,3})$ minors. Since M is graphic, M has no $M(K_5)^*$ or $M(K_{3,3})^*$ minors. Thus, by Theorem (10.2.11), M and G are planar. $\square$

One may derive Corollary (10.2.13) directly from Theorem (7.4.1), without use of Theorem (10.2.11). Indeed, Theorem (7.4.1) is the graph version of Corollary (10.2.13). Whitney's contribution is the deduction of this result from Kuratowski's original planarity characterization, which involved subdivisions of K_5 and $K_{3,3}$.

In the preceding chapters, we took great care when we dualized planar graphs. Each time, we embedded a given planar graph in the plane, then dualized that plane graph. Corollary (10.2.13) frees us from this adherence to planar embeddings. We now may take the following viewpoint. Given a planar graph G, let H be any graph for $M(G)^*$. Thus, the matroids $M(G)$ and $M(H)$ are duals of each other. We declare H to be a *dual graph* of G. For our purposes, any H satisfying $M(H)^* = M(G)$ will do. By this definition, any two such graphs are 2-isomorphic. Thus, by Theorem (3.2.36), H is unique if it, or equivalently G, is 3-connected, the case typically of interest to us. In that situation, we are justified to call H *the* dual of G. Suppose H is 2-connected. By Theorem (3.2.36), any other graph for $M(G)^*$ is related to H by switchings. We leave it to the reader to explore this issue further. Whitney first recognized these relationships among embeddings, 2-isomorphism, and switchings.

We move on to the next section, where we investigate regular matroids with $M(K_{3,3})$ minors.

10.3 Regular Matroids with $M(K_{3,3})$ Minors

The title of this section may seem strange. Why would one be interested in $M(K_{3,3})$ minors of regular matroids? Early in this section, we give a partial answer in the form of two lemmas. More satisfactory as answer are the arguments of the next section, which prove $M(K_{3,3})$ to play a central role in the analysis of regular matroids.

Following the two lemmas and some other preparatory material, we introduce the main theorem of this section. It says that any 3-connected, regular, nongraphic and noncographic matroid with an $M(K_{3,3})$ minor has a minor isomorphic to one of the nongraphic and noncographic matroids R_{10} and R_{12}. This profound result is due to Seymour. In the next section we combine it with Theorem (10.2.11) to obtain two important characterizations of nongraphic regular matroids by Seymour and Tutte.

To start, we recall the splitter definition of Section 7.2. Let $\mathcal{M}$ be a class of binary matroids that is closed under isomorphism and under the taking of minors. Let N be a 3-connected matroid of $\mathcal{M}$ on at least six elements. Then N is a *splitter* of $\mathcal{M}$ if every connected matroid $M \in \mathcal{M}$ with a proper N minor is 2-separable. Intuitively and informally speaking, the presence of an N minor forces M to split. By Theorem (7.2.11), the graph K_5 is a splitter of the graphs without $K_{3,3}$ minors. That result has the following matroid extension.

(10.3.1) Lemma. *$M(K_5)$ is a splitter of the regular matroids without $M(K_{3,3})$ minors.*

Proof. By Theorem (7.2.1), we only need to show that every 3-connected regular 1-element extension of $M(K_5)$ has an $M(K_{3,3})$ minor. This is accomplished by a straightforward case analysis. To assist the reader, we sketch one way of checking.

By (3.2.38), the matrix

$$B = \boxed{\begin{array}{cccccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}} \tag{10.3.2}$$

Matrix B for $M(K_5)$

represents $M(K_5)$. Suppose we adjoin a column vector b that represents an added element z. If b has at most two 1s, then z is a coloop or a parallel element. If b has three or four 1s, then $[B \mid b]$ contains a submatrix representing the Fano matroid F_7, which is nonregular. Thus, no 3-connected regular 1-element addition is possible.

Now assume we adjoin a row vector c that represents a 1-element regular and 3-connected expansion by an element e. We may view B as the node/edge incidence matrix of a graph H that is isomorphic to K_4. It is convenient to encode each 1 of the row vector c by a red edge of H. Since $M(K_5)\&e$ is 3-connected, H must have at least two red edges. We analyze the possible configurations of red edges.

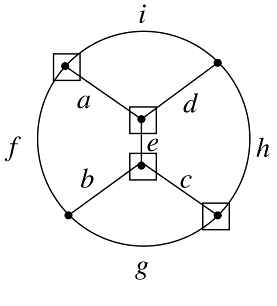
Suppose that H contains exactly two red edges, and that these edges share an endpoint. Then $M(K_5)\&e$ is easily confirmed to be graphic, with an $M(K_{3,3})$ minor. Next, suppose H contains exactly four red edges that form a cycle C. Let y and z be the edges of H that are not in C. Then $(M(K_5)\&e)\backslash\{y,z\}$ is isomorphic to $M(K_{3,3})$.

For the remaining configurations of red edges, one proves the presence of an F_7^* minor. Specifically, if the red edges form a triangle, then the columns of $[B/c]$ corresponding to that triangle establish the presence of an F_7^* minor. The other cases are slightly more difficult to prove. As an example case, let us examine the 1-element expansion M of $M(K_5)$ by the element e given by

(10.3.3)
$$\begin{array}{c|cccccc|} & f & g & h & i & j & k \\ \hline a & 1 & 0 & 0 & 1 & 1 & 0 \\ b & 1 & 1 & 0 & 0 & 0 & 1 \\ c & 0 & 1 & 1 & 0 & 1 & 0 \\ d & 0 & 0 & 1 & 1 & 0 & 1 \\ \hline e & 1 & 0 & 1 & 0 & 0 & 0 \\ \hline \end{array}$$

Matrix of 1-element expansion M of $M(K_5)$

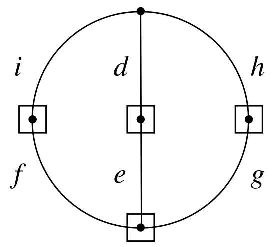
It turns out that we can prove nonregularity without using the last column k. So let us consider that column deleted. One readily verifies that the remaining matrix is represented by

(10.3.4)


Graph plus T nodes for $M\backslash k$

where the T nodes correspond to the element j. From the graph of (10.3.4), we delete the edges a and c, and contract the edge b. The resulting graph with adjusted T set is given by (10.3.5) below. By (10.2.5), that graph represents an F_7^* minor. Thus, the matroid given by the matrix of (10.3.3) is nonregular.

(10.3.5)



Graph plus T nodes for $M/\{b\}\backslash\{a,c,k\}$

The remaining open cases are handled the same way. $\square$

For later reference, we include another lemma about $M(K_{3,3})$. We omit the elementary case analysis via graphs plus T sets.

(10.3.6) Lemma. *Every 3-connected binary 1-element expansion of $M(K_{3,3})$ is nonregular.*

The proof of the next result about $M(K_{3,3})$ requires a bit of preparation. Recall from Section 3.3 the following definition. Let M be a binary matroid and z be an element of M. Then $Mⓒz$ is the matroid obtained from M/z by deletion of all elements but one from each parallel class. The minor $Mⓓz$ is derived from $M\backslash z$ by contraction of all elements but one in each series class. For convenient reference, we restate Lemma (3.3.31), which links $Mⓒz$ and $Mⓓz$ to 3-connectedness.

(10.3.7) Lemma. *Let M be a 3-connected binary matroid on a set E. Take z to be any element of E. Then $Mⓒz$ or $Mⓓz$ is 3-connected.*

Define a *line* of a graph to be a path of maximal length where all internal nodes have degree 2. A *corner node* of a graph is a node of degree at least 3. Let G be a subdivision of a 3-connected graph with at least four corner nodes. Evidently, each line of G is a series class and vice versa. We emphasize the latter fact, since generally a series class of a graph need not be a line. This is due to the definition of Section 2.2, where two edges are declared to be in series if they form a cocycle. *A priori*, the same subtle point needs to be considered when one specializes Lemma (10.3.7) to graphs. In that case, the notation $Gⓒz$ and $Gⓓz$ is interpreted analogously to that of $Mⓒz$ and $Mⓓz$.

We now show that the just-mentioned complications concerning series edges do not arise when we apply Lemma (10.3.7) to graphs. Let G be 3-connected, and let z be an edge of G, say with endpoints u and v. Recall that u and v are edge subsets. By Lemma (10.3.7), one of $Gⓒz$, $Gⓓz$ is 3-connected. Let $Gⓒz$ be that graph. The contraction of z may introduce parallel edges only at the new vertex $(u \cup v) - \{z\}$. Thus, $Gⓒz$ is readily determined. Now let $Gⓓz$ be 3-connected. We claim that $G\backslash z$ contains series classes with more than one edge only if u or v has degree 3, and that such series classes correspond to paths with two edges. Indeed, if u has degree 3, then $u - \{z\}$ is one such class. Similarly, v may produce a second series class. No other series class is possible, since 2-connected series expansions of a 3-connected graph can only produce a subdivision of that graph. Thus, Gⓓz is readily determined.

The above conclusion is valid only if Gⓓz is 3-connected. Indeed, if Gⓓz is not 3-connected, then $G\backslash z$ may have a series class that is not a line. Fortunately, we never deal with the latter case since we make use of Gⓓz only when that graph is 3-connected.

Here is a second preparatory lemma about Gⓒz and Gⓓz.

(10.3.8) Lemma. *Let H be a subdivision of a 3-connected graph. Assume H has at least four corner nodes. Let G be any graph derived from H by the addition of nonloop edges. No such added edge is to connect two nodes of a line of H. Also, G is not allowed to have parallel or series edges. Then* (a)–(c) *below hold.*

(a) *G is 3-connected.*
(b) *Gⓓz is 3-connected for every arc of G that is not in H.*
(c) *Gⓒz is 3-connected for every arc of H both of whose endpoints have degree 2 in H.*

Proof. Clearly, G of part (a) is 2-connected. Suppose G has a 2-separation. We know that H is a subdivision of a 3-connected graph. Thus, any 2-separation of H has on one side a subset of one line of H. Assume that the 2-separation of G induces one in H. Then G has series edges, or G has an edge that is not in H and that connects two nodes of one line of H. Both cases contradict the assumptions. If the 2-separation of G does not induce one in H, then G must have parallel edges, again a contradiction. Thus, G is 3-connected.

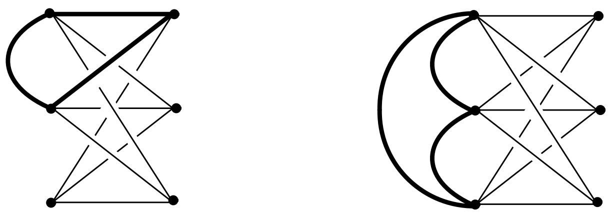
Under the given assumptions, Gⓓz and Gⓒz of parts (b) and (c) satisfy the construction rules imposed on G. By (a), these minors are 3-connected. □

The next result concerns graphs with $K_{3,3}$ minors.

(10.3.9) Theorem. *Let G be a 3-connected graph with a $K_{3,3}$ minor.*

(a) *If G contains a triangle formed by edges e, f, and g, then G has one of the graphs of* (10.3.10) *below as a minor. The bold edges denote e, f, and g.*
(b) *If G has a node u of degree 3, then G has as subgraph a subdivision of $K_{3,3}$ that has u as corner node.*

(10.3.10)



Two extensions of $K_{3,3}$ containing a triangle

Proof. First we show part (a). Due to minor-taking, we may assume that every proper minor of G is not 3-connected, or does not have a $K_{3,3}$ minor, or does not contain the triangle $\{e,f,g\}$. We say that G is *minimal* to denote this fact. Let u, v, and w be the nodes of the triangle. Denote by H any subgraph of G that is a subdivision of $K_{3,3}$.

Suppose u is not a node of some H, say of H_1. Thus, u has an arc $z \neq e,f,g$ incident that is not in H_1. By Lemma (10.3.7), one of $G \textcircled{c} z$, $G \textcircled{d} z$ is 3-connected. Assume $G \textcircled{c} z$ to be 3-connected. In G/z we can delete parallel edges so that the triangle $\{e,f,g\}$ is retained. Now suppose $G \textcircled{d} z$ is 3-connected. If u has degree 3, then in $G \backslash z$ two edges of e, f, g are in series, and $G \textcircled{d} z$ has two edges of e, f, g in parallel, a contradiction. Thus, u has degree of at least 4, and $G \textcircled{d} z$ contains the triangle $\{e,f,g\}$. Clearly, both $G \textcircled{c} z$ and $G \textcircled{d} z$ have $K_{3,3}$ minors. But these facts contradict the minimality of G.

We conclude that u, v, and w are nodes of every H. If all three nodes occur on one line of some H, then there exists another H that avoids one of the three nodes. Therefore, at most two of the nodes occur on any one line of any H. Accordingly, we can always delete and contract edges in G such that e, f, g are retained and such that their endpoints become corner nodes of some $K_{3,3}$ minor. That process produces one of the graphs of (10.3.10).

For part (b), we once more define H to be any subgraph of G that is a subdivision of $K_{3,3}$. Suppose the given degree 3 node u of G is not a corner node of some H. Then by a ΔY exchange (see Section 4.3), the 3-star u can be replaced by a triangle $\{e,f,g\}$. It is easily seen that upon deletion of edges parallel to e, f, or g, we have a 3-connected graph with a $K_{3,3}$ minor. Apply part (a) to the latter graph. Thus, that graph has as minor one of the graphs of (10.3.10). Now replace $\{e,f,g\}$ by the 3-star u again. The resulting graph is a minor of G and is readily verified to have a $K_{3,3}$ minor $\overline{G}$ with u as a corner node. In straightforward fashion, we extend $\overline{G}$ to a subgraph of G that is a subdivision of $K_{3,3}$ and that has u as a corner node. $\square$

We are ready to state and prove the main result of this section.

(10.3.11) Theorem. *Let M be a 3-connected regular matroid with an $M(K_{3,3})$ minor. Assume that M is not graphic and not cographic, but that each proper minor of M is graphic or cographic. Then M is isomorphic to R_{10} or R_{12}.*

Proof. Let M be a smallest regular matroid that satisfies the assumptions of the theorem but not its conclusion. By Lemma (10.3.6), M does not have a 3-connected 1-element expansion of any $M(K_{3,3})$ minor. Take N to be any $M(K_{3,3})$ minor of M. Apply Theorem (7.3.4) to M and N. That theorem establishes the existence of a certain nested sequence of minors. In particular, the result implies that M has an element z for which the minor $M\backslash z$ is a series extension of a 3-connected matroid with an $M(K_{3,3})$ minor. The proof of Theorem (10.3.11) consists of a thorough analysis of $M\backslash z$ and of the role of the element z. We begin with two simple claims about $M\backslash z$.

First, we claim that each series class of $M\backslash z$ has at most two elements. Suppose otherwise. Thus, the connected $M\backslash z$ has a representation matrix with three parallel rows. Adjoin a column z to that matrix to get a representation matrix for M. Regardless of the entries of column z, the matrix for M has two parallel rows. Thus, M is not 3-connected, a contradiction.

Second, we claim that $M\backslash z$ is graphic. This is so since each proper minor of M is graphic or cographic, and since $M\backslash z$ has an $M(K_{3,3})$ minor, which is not cographic.

By the first claim, a graph G for $M\backslash z$ is obtained from a 3-connected graph by subdividing each edge at most once. Hence, each line of G has one or two edges. We represent M by G plus a node subset T that handles the extra element z. Recall that $|T|$ is even. Choose G so that $|T|$ is minimal. Since M is not graphic, we have $|T| \geq 4$.

The strategy in the remainder of the proof is as follows. We attempt to reduce G and T to G' and T', where G' has fewer edges than G or $|T'| < |T|$. The graph G' is to have an $M(K_{3,3})$ minor, and $|T'| \geq 4$ is to hold. Indeed, G' and T' must represent a nongraphic matroid. At times, we simply say that we *reduce* G to denote this process. Evidently, any such reduction contradicts the minimality of M or T, and thus is not possible. The reduction attempts reveal enough structural information about M to prove that matroid to be isomorphic to R_{10} or R_{12}, contrary to the initial assumption. We present the details.

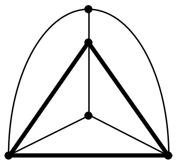
Claim 1. G is 3-connected.

Proof. Suppose G is not 3-connected. We know that G is a subdivision of a 3-connected graph. Indeed, each line of G with at least two edges has exactly two edges. Let G have m such lines. If a midpoint of one of these lines is not in T, then in M the elements corresponding to the two edges of that line are in series. Thus, M is not 3-connected, a contradiction. Hence, the midpoints of the m lines are in T.

If $m \geq 4$, we contract in one of the m lines an edge. By switchings, the set of T nodes may now be reduced, say to T'. But the midpoints of the remaining $m - 1$ lines must remain in T', so $|T'| \geq 4$. Thus, G has been reduced. Similarly, one may handle the cases $m = 1$ and 2, and also $m = 3$ when the three lines do not form a cycle.

Consider the remaining case, where $m = 3$ and where the three lines form a cycle. If any node other than those of the three lines is in T, again we can reduce G. Thus, all such nodes are not in T. Temporarily contract one edge of each line, getting a graph $\overline{G}$. The three lines have become a triangle. According to Theorem (10.3.9), the graph $\overline{G}$ has a minor isomorphic to one of the two graphs of (10.3.10). The bold lines of those graphs are those of the triangle. We can further reduce each one of the two graphs to

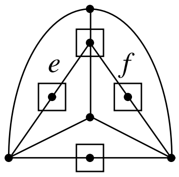
(10.3.12)



Minor of the two graphs of (10.3.10)

where again the bold lines indicate the triangle. The graph of (10.3.12) is still a minor of G if we replace each triangle edge by the corresponding line with appropriate designation of T nodes. By the minimality of $|T|$, this substitution must result in

(10.3.13)



Minor of G with three lines

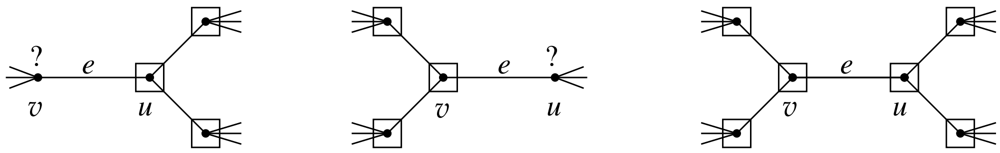
which is not graphic. Contract the edges labeled e and f in (10.3.13). The resulting G' and T' has $|T'| = 2$ and represents a matroid with an $M(K_{3,3})$ minor. Thus, the matroid of (10.3.13) is not cographic. But that matroid is a proper minor of M since the graph of (10.3.12) is a proper minor of the two graphs of (10.3.10). Thus, we have a contradiction of the minimality of M. Q. E. D. Claim 1

Denote by H any subgraph of G that is a subdivision of $K_{3,3}$. Recall that $G\textcircled{c}e$ is obtained from G by contraction of edge e and deletion of parallel edges. Similarly, $G\textcircled{d}e$ is produced by deletion of edge e and contraction of each line with two edges to just one edge. Indeed, in $G\backslash e$, at most two lines of length 2 may exist. In each such line, we may choose the edge to be contracted as is convenient. This aspect is important when we want to preserve T nodes. As was done before, we use squares to denote nodes in T. We assign a question mark to a node if that node may or may not be in T. We say that a node is *cubic* if it is a 3-star.

Claim 2. Every node v of G that is not part of some H is cubic, and that node and two of its neighbors are in T. Furthermore, $|T| = 4$ in that case.

Proof. Let e be an edge incident at v, and u be its other endpoint. If Gⓓe is 3-connected, one of the situations below must prevail; otherwise a reduction is possible, as is readily checked.

(10.3.14)



Case (i)
u cubic
$|T| = 4$

Case (ii)
v cubic
$|T| = 4$

Case (iii)
Both u, v cubic
$|T| = 6$

If Gⓒe is 3-connected, then $u, v \in T$ and $|T| = 4$, since otherwise Gⓒe with adjusted T set is a smaller case. We call the last situation case (iv). Thus, we have a total of four possible cases for the edge e.

Suppose $v \notin T$. Apply the above arguments to every edge e incident at v. In each instance, case (i) of (10.3.14) must apply, since cases (ii), (iii), and (iv) demand v to be in T. Thus, $|T| = 4$, and all neighbors of v and their neighbors (except v) must be in T. In addition, the neighbors of v must be cubic. A simple case analysis shows that these conditions imply G to be 2-separable. Thus, $v \in T$.

Suppose v is not cubic. Again by arguments for each edge e incident at v, we have $|T| = 4$, and v and all neighbors of v must be in T. By assumption, there are at least four neighbors, so this is not possible.

So far we know v to be cubic and to be in T. If $|T| > 4$, then case (iii) of (10.3.14) must hold for every edge e incident at v, and $|T| = 6$. Once more, a case analysis shows that G is not 3-connected. Thus, $|T| = 4$.

Suppose v and its three neighbors are in T. Then no other node of G is in T. Take any node w different from these four nodes. There exist three internally node-disjoint paths in G from v to w. Then clearly G and T can be reduced to a graph isomorphic to that of (10.2.5). The latter graph represents the nonregular matroid F_7^*, a contradiction. Q. E. D. Claim 2

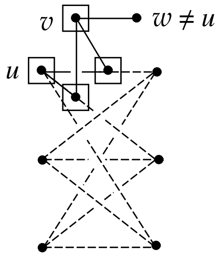
Claim 3. There exists an H such that

(i) G has no node beyond those of H, and

(ii) no edge of G that is not an edge of H connects two nodes of a line of H.

Proof. Suppose some H violates (ii). We then can find another one violating (i). Thus, G has a node u not in some H. By Claim 2, u is cubic, u and precisely two of its neighbors are in T, and $|T| = 4$. By Theorem (10.3.9), there is another H with u as corner node. Pick a minimal such H, say H_1. Clearly, H_1 satisfies (ii) of Claim 3. We now prove that (i) holds as well. If not, then G has, again by Claim 2, a cubic node v not in H_1 such that v and precisely two of its neighbors are in T. Thus, G has the following subgraph, where dashed lines represent internally node-disjoint paths.

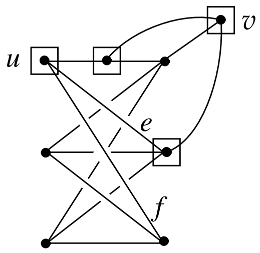
(10.3.15)



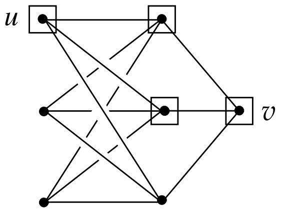
Graph H plus node v

By Menger's Theorem, there is a path from w to a non-T node of the dashed graph. Thus, we can reduce G to produce one of the following graphs.

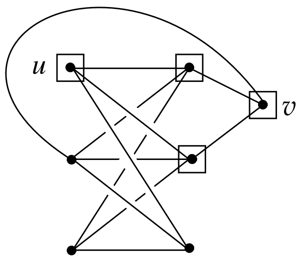
(10.3.16)



Graph G_1



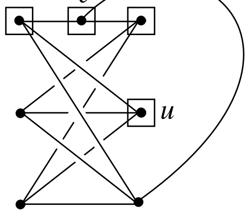
Graph G_2



Graph G_3

From G_1, delete the edge e and contract the edge f. This produces the graph

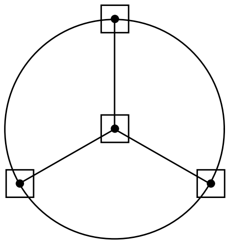
(10.3.17)



Graph for R_{12}

which by (10.2.9) represents R_{12}. It is easily checked that both G_2 and G_3 can be reduced to the graph

(10.3.18)



Graph for F_7

which by (10.2.4) represents the nonregular F_7. Q. E. D. Claim 3

Claim 4. $|T| \leq 6$. Take any H that satisfies (i) and (ii) of Claim 3. Suppose G has an edge e that is not in H. Then at least one endpoint v of e is cubic, and v and the neighbors linked to v by edges of H are in T.

Proof. Suppose an edge e exists that is not in H. By part (b) of Lemma (10.3.8), Gⓓe is 3-connected. If $|T| \geq 8$, or if the remaining conclusions of Claim 4 do not hold, then Gⓓe provides a smaller case. If G has no edges beyond those of H, then G is isomorphic to $K_{3,3}$, and $|T| \leq 6$ holds trivially. Q. E. D. Claim 4

Claim 5. There is an H satisfying (i) and (ii) of Claim 3 such that H contains all edges of G but at most one.

Proof. Let H be the graph of Claim 3. Simple case checking confirms that H must contain all edges of G except for possibly two edges. Assume there are two such edges. By Claim 4, one of these arcs has a cubic endpoint, say u, such that u and the neighbors linked to u by arcs of H are in T.

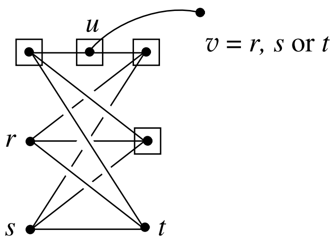
Now select a minimal H, say H_2, that has u as corner node. By the proof of Claim 3, H_2 may be assumed to satisfy (i) and (ii) of that claim. Suppose again there are two edges in G, say e and f, that are not in H_2. Let v and w be the cubic endpoints of e and f that have the properties described in Claim 4. Clearly $|T| = 6$. If v and w are on one line of H_2, we can contract an intermediate arc of that line, and by part (c) of Lemma (10.3.8) have a smaller case. Otherwise, one endpoint of e or f, say of e, is not in T, and Gⓓe produces a smaller case. Thus, H_2 is the desired graph. Q. E. D. Claim 5

Let H be the graph of Claim 5. If every edge of G is in H, then G and H are isomorphic to $K_{3,3}$. If $|T| = 6$, we have the graph of (10.2.8), and M is isomorphic to R_{10}. If $|T| = 4$, then G can be reduced to the nonregular instance of (10.3.18).

Finally, assume just one edge of G, say e, is not in H. By Claim 4, at least one endpoint of e, say u, is cubic, and u and the neighbors linked to

u by arcs of H are in T. If the second endpoint v of e is cubic or $|T| \geq 6$, then we can reduce G to a smaller case. If $v \in T$, then G can be reduced to the nonregular case of (10.3.18). Thus, G must be

(10.3.19)



Graph H plus one arc

If $v = r$ or s, then the nonregular case of (10.3.18) can be produced. If $v = t$, we have by (10.2.9) an instance of R_{12}. □

We are prepared for the next section, where we prove two profound excluded minor theorems of Seymour and Tutte.

10.4 Characterization of Graphic Matroids

We prove profound characterizations of two classes of regular matroids in terms of excluded minors. The first characterization is due to Tutte. It says that a regular matroid is graphic if and only if it has no $M(K_{3,3})^*$ or $M(K_5)^*$ minors. The second characterization is due to Seymour. According to that result, a 3-connected regular matroid is graphic or cographic if and only if it has no R_{10} or R_{12} minors. The latter matroids are defined by (10.2.8) and (10.2.9). In this section, we prove these characterizations and deduce some related material. Given the results of the preceding two sections, it is advantageous for us to start with the second characterization.

(10.4.1) Theorem. *A 3-connected regular matroid is graphic or cographic if and only if it has no R_{10} or R_{12} minors.*

Proof. The graphs, T sets, and representation matrices of (10.2.8) and (10.2.9) for R_{10} and R_{12} prove these matroids to be nongraphic and isomorphic to their respective duals. Thus, R_{10} and R_{12} are also not cographic. These observations establish the easy "if" part.

For proof of the converse, let M be a 3-connected regular matroid that is not graphic and not cographic. Thus, M is not planar, and by Theorem (10.2.11) has a minor isomorphic to $M(K_5)$, $M(K_{3,3})$, $M(K_5)^*$, or $M(K_{3,3})^*$. By Lemma (10.3.1), $M(K_5)$ is a splitter for the regular matroids without $M(K_{3,3})$ minors. By these results, M has a minor isomorphic to $M(K_{3,3})$ or $M(K_{3,3})^*$, or M is isomorphic to $M(K_5)$ or $M(K_5)^*$. The latter case is a contradiction. Thus, M or M^* has an $M(K_{3,3})$ minor. By Theorem (10.3.11), M or M^* has an R_{10} or R_{12} minor. Since R_{10} and R_{12} are isomorphic to their duals, M itself has an R_{10} or R_{12} minor. □

We turn to the characterization of regular matroids that are graphic.

(10.4.2) Theorem. *A regular matroid is graphic (resp. cographic) if and only if it has no $M(K_5)^*$ or $M(K_{3,3})^*$ minors (resp. $M(K_5)$ or $M(K_{3,3})$ minors).*

Proof. Lemma (3.2.48) implies the easy "only if" part. For proof of the converse, let M be a nongraphic regular matroid all of whose proper minors are graphic. If M is 1- or 2-separable, then arguments analogous to those of the proof of Theorem (10.2.11) establish M to be graphic, a contradiction. Thus, M is 3-connected.

Suppose M is not cographic. By Theorem (10.4.1), M has an R_{10} or R_{12} minor. The drawings of (10.2.8) and (10.2.9) clearly establish that both R_{10} and R_{12} have proper $M(K_{3,3})$ minors. Now R_{10} and R_{12} are isomorphic to their duals. Thus, M has an $M(K_{3,3})^*$ minor.

Now consider M to be cographic, i.e., consider M^* to be graphic. If M^* is planar, then M is planar, and hence graphic, a contradiction. If M^* is nonplanar, then by Theorem (7.4.1), M^* has an $M(K_5)$ or $M(K_{3,3})$ minor. Thus, M has an $M(K_5)^*$ or $M(K_{3,3})^*$ minor, as desired.

The parenthetic claim of the theorem follows by duality. □

We complete this section with two corollaries. The first one effectively restates Corollary (10.2.13).

(10.4.3) Corollary. *A matroid is planar if and only if it is graphic and cographic.*

Proof. Apply Corollary (10.2.13), or compare the excluded minors of Theorems (10.2.11) and (10.4.2) to obtain the conclusion. □

For the second corollary, we need the following auxiliary result.

(10.4.4) Lemma. *Every 1-element reduction of R_{10} or R_{12} produces a matroid with an $M(K_{3,3})$ or $M(K_{3,3})^*$ minor.*

Proof. For R_{10}, the proof is easy. One first shows that R_{10} is highly symmetric as follows. Consider a binary matrix D with five rows. Each column of D has exactly three 1s, with each possible case occurring. Thus, D has $\binom{5}{3} = 10$ columns. Index the columns of D by a 10-element set E. We claim that E and the subsets of E indexing GF(2)-independent columns of D define a matroid isomorphic to R_{10}. For a proof, we perform row operations in D until a 5×10 matrix $[I \mid B]$ results. Simple checking confirms that the matrix B is either up to indices the matrix of (10.2.8) for R^{10}, or is the matrix of (10.4.5) below. A pivot on the 1 in the $(1, 2)$ position of the latter matrix converts it to the former one.

$$\text{(10.4.5)} \qquad \begin{array}{|ccccc|}\hline 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ \hline \end{array}$$

Alternate matrix for R_{10}

By (10.2.8), the minor $R_{10} \backslash z$ of R_{10} is isomorphic to $M(K_{3,3})$. By the just-proved symmetry and duality, every 1-element deletion (resp. contraction) in R_{10} produces a matroid isomorphic to $M(K_{3,3})$ (resp. $M(K_{3,3})^*$).

The proof for R_{12} is about as easy. That matroid is also isomorphic to its dual. Thus, we can confine ourselves to the contraction case. With the aid of (10.2.9), one can prove presence of an $M(K_{3,3})$ or $M(K_{3,3})^*$ minor after each 1-element contraction. □

The proof of Lemma (10.4.4) contains the following result about R_{10}.

(10.4.6) Lemma. *Up to indices, the matrices of* (10.2.8) *and* (10.4.5) *are the only binary representation matrices for* R_{10}.

We use Theorem (10.4.1) as the main ingredient in the proof of the following corollary.

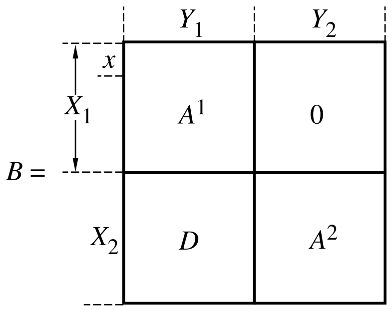
(10.4.7) Corollary. *Every regular* 1*-element extension of a connected planar matroid is graphic or cographic.*

Proof. Let M be a 1-element regular extension of a connected planar matroid N. The proof is by induction on the number of elements of N. The result clearly holds for small M, say with at most eight elements. Planarity and the conclusions of the corollary are preserved under dualizing. Thus, we may assume that for some x, $N = M/x$.

If M is 1-separable, then x is a loop or coloop of M. Since N is planar, M is planar as well. Hence, we may suppose M to be connected.

Assume M to be 2-separable. We may select a representation matrix B of M as given by (8.1.1) with GF(2)-rank $D = 1$ and $x \in X_1$. Below, (10.4.8) displays that matrix. Since $x \in X_1$, B displays N. If $|X_1 \cup Y_1| = 2$, then M is a series extension of N, and thus is planar. If $|X_1 \cup Y_1| \geq 3$, then $((X_1 \cup Y_1) - \{x\}, X_2 \cup Y_2)$ is a 2-separation of N. Let N_1 and N_2 be the components of the related 2-sum decomposition of N.

(10.4.8)



Matrix B for M with 2-separation

By Lemma (8.2.6), N_1 and N_2 are connected. Then for some 1-element expansion of N_1 by x, say N_1', the matroid M is a 2-sum with N_1' and N_2 as components. Since N_1 is connected, by induction N_1' is graphic or cographic. We know that N_2 is planar. By Lemma (8.2.7), M is graphic or cographic.

As final case, consider M to be 3-connected. If M is not graphic or cographic, then by Theorem (10.4.1), M has an R_{10} or R_{12} minor. But by Lemma (10.4.4), every 1-element reduction of either minor, and thus of M, is not planar, a contradiction. $\square$

Corollary (10.4.7) is false when the given planar matroid is not assumed to be connected. For example, for any element z of K_5, the 1-sum of $M(K_5/z)$ and $M(K_5\backslash z)^*$ is planar. But there is a regular 1-element extension that has $M(K_5)$ and $M(K_5)^*$ as a minor. Thus, that extension is nongraphic and noncographic.

In the next section, we introduce a major switch of topic. We examine a simple yet powerful idea that produces interesting graph decomposition theorems. In Chapters 11–13, we rely on the matroid generalization of this idea to prove profound matroid decomposition theorems.

10.5 Decomposition Theorems for Graphs

In subsequent chapters, we make repeated use of a recursive construction of matroid decomposition theorems. In this section, we use a graph example to motivate and explain that construction. In the process, we deduce by rather elementary checking a famous decomposition theorem of Wagner about the class of graphs without K_5 minors.

The basic idea of the construction is as follows. Given is a class $\mathcal{G}$ of connected graphs. The class is closed under isomorphism and under the taking of minors. On hand are also two subclasses $\mathcal{L}$ and $\mathcal{H}$ of $\mathcal{G}$. The two subclasses are so selected that each graph $G \in \mathcal{G}$ is in $\mathcal{L}$ or has, for some $H \in \mathcal{H}$, an H minor.

One iteration of the construction is as follows. We select a graph H of $\mathcal{H}$ and use one of the decomposition theorems of Chapter 6 or 7 to establish a result of the following type: If a graph $G \in \mathcal{G}$ has an H minor, then G has a certain decomposition caused by H, or G is a member of a certain collection of graphs, say $\{L_1, L_2, \ldots, L_m\}$, or G has a minor that is a member of a certain second collection of graphs, say $\{H_1, H_2, \ldots, H_n\}$. We now derive from $\mathcal{L}$ the set $\mathcal{L}' = \mathcal{L} \cup \{L_1, L_2, \ldots, L_m\}$, and from $\mathcal{H}$ the set $\mathcal{H}' = (\mathcal{H} - H) \cup \{H_1, H_2, \ldots, H_n\}$. By the derivation of $\mathcal{L}'$ and $\mathcal{H}'$, we have established the following theorem: Each graph $G \in \mathcal{G}$ can be decomposed, or belongs to $\mathcal{L}'$, or has an H' minor for some $H' \in \mathcal{H}'$.

At this point, we have completed one iteration of the construction. If $\mathcal{H}'$ is empty, we stop; most likely the cited theorem is an interesting decomposition result for the graphs of $\mathcal{G}$. If $\mathcal{H}'$ is nonempty, we have two choices: We may stop, or we may carry out another iteration by declaring $\mathcal{L}'$ to be $\mathcal{L}$ and $\mathcal{H}'$ to be $\mathcal{H}$. Evidently, the recursive construction is nothing but a concatenation of decomposition results, each of which is deduced from suitably selected theorems of Chapters 6 and 7.

An example will help to clarify the construction. We want a decomposition theorem for the graphs without K_5 minors. That class of graphs turns out to be important for a number of combinatorial problems. Details are included in Section 10.7. We ignore the applications for the time being, and concentrate on the construction of a decomposition theorem for that class. In agreement with the preceding outline, we define $\mathcal{G}$ to be the set of connected graphs without K_5 minors. The subclass $\mathcal{L}$ is the collection of connected planar graphs, and $\mathcal{H}$ is the set $\{K_{3,3}\}$. By Theorem (7.4.1) and the exclusion of K_5 minors from the graphs of $\mathcal{G}$, each graph $G \in \mathcal{G}$ is planar or has a $K_{3,3}$ minor. Thus, $\mathcal{L}$ and $\mathcal{H}$ do satisfy the condition stated earlier, i.e., each graph $G \in \mathcal{G}$ is in $\mathcal{L}$ or has an H minor for the single graph $H = K_{3,3}$ of $\mathcal{H}$.

We are ready for the first iteration of the construction. We will rely on the splitter theorem for graphs of Section 7.2, listed there as Corollary (7.2.10). We repeat that result below.

(10.5.1) Theorem. *Let $\mathcal{G}$ be a class of connected graphs that is closed under isomorphism and under the taking of minors. Let H be a 3-connected graph of $\mathcal{G}$ with at least six edges.*

(a) *If H is not a wheel, then H is a splitter of $\mathcal{G}$ if and only if $\mathcal{G}$ does not contain any graph derived from H by one of the following two extension steps:*

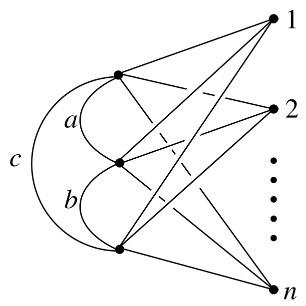
 (1) *Connect two nonadjacent nodes of N by a new edge.*

 (2) *Partition a vertex of degree at least 4 into two vertices, each of degree at least 2, and connect these two vertices by a new edge.*

(b) *If H is a wheel, then H is a splitter of $\mathcal{G}$ if and only if $\mathcal{G}$ does not contain any of the extensions of H described under* (a) *and does not contain the next larger wheel.*

The application of the splitter theorem involves several graphs, which we define next. The graphs are $K_{3,n}$, $n \geq 3$, as well as certain variants of $K_{3,n}$, called $K^1_{3,n}$, $K^2_{3,n}$, and $K^3_{3,n}$. The graph $K^3_{3,n}$ is given by (10.5.2) below. The other graphs are obtained from $K^3_{3,n}$ by deletion of some edges. In the notation of (10.5.2), $K^2_{3,n}$ is the graph $K^3_{3,n} \backslash c$, and $K^1_{3,n}$ is the graph $K^3_{3,n} \backslash \{b, c\}$. At times, we refer to $K_{3,n}$ as $K^0_{3,n}$. We collect the graphs just defined in a set $\mathcal{K} = \{K^i_{3,n} \mid 0 \leq i \leq 3, n \geq 3\}$.

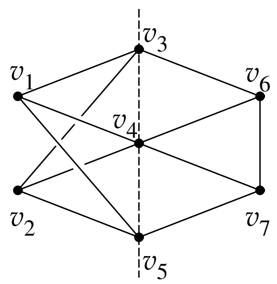
(10.5.2)



Graph $K^3_{3,n}$

We need one additional graph V defined by

(10.5.3)



Graph V

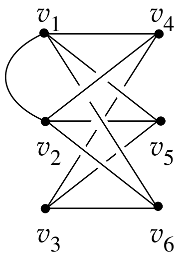
For the moment, the reader should ignore the dashed line in the drawing of V. It indicates a 3-separation that we will utilize later. Note that V has no K_5 minors. Indeed, V has eleven edges and seven vertices, and if V has a K_5 minor, then such a minor must be produced by a single contraction or deletion. But any such reduction results in a graph with at least six vertices.

The splitter Theorem (10.5.1) involves 3-connected 1-edge extensions of graphs. The next lemma supplies information about such extensions for the graphs of $\mathcal{K}$.

(10.5.4) Lemma. *For any graph G of $\mathcal{K}$, any 3-connected 1-edge extension of G is isomorphic to another graph of $\mathcal{K}$, or has a K_5 minor, or has a V minor.*

Proof. The proof involves a simple checking of cases for the graphs $K^i_{3,n}$, $0 \leq i \leq 3$, with n fixed, plus induction. As examples, we cover the cases of $K_{3,3}$ and $K^1_{3,3}$. Since every vertex of $K_{3,3}$ has degree 3, any 3-connected 1-edge extension must be an addition. Indeed, just one addition case exists, the graph $K^1_{3,3}$. We depict that graph below.

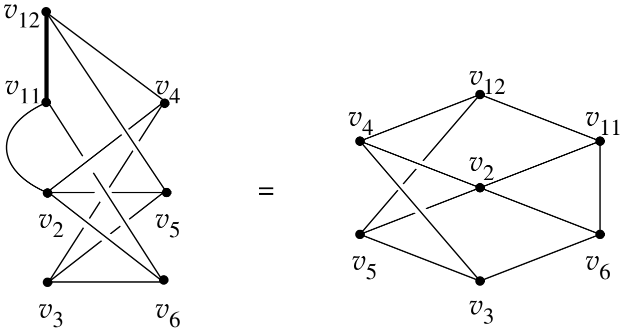
(10.5.5)



Graph $K^1_{3,3}$

The 3-connected 1-edge extensions of $K^1_{3,3}$ are as follows. We start with the expansion cases. By symmetry, all cases are isomorphic to the following one, where we split the vertex v_1. The resulting graph is isomorphic to V, as is evident from the next drawing.

(10.5.6)



3-connected 1-edge expansion of $K^1_{3,3}$

We turn to the addition cases. All such instances are isomorphic to two graphs, one of which is $K^2_{3,3}$, while the second one becomes K_5 upon contraction of one edge. We leave the verification of this claim to the reader. □

With these preparations, we carry out the first iteration of the construction process. We must select the single graph $K_{3,3}$ of $\mathcal{H}$ as H. From the splitter Theorem (10.5.1) and Lemma (10.5.4), we deduce the following decomposition result.

(10.5.7) Theorem. *Every connected graph without K_5 minors is 1- or 2-separable, or is planar, or is isomorphic to one of the graphs $K^i_{3,n}$, $0 \leq i \leq 3$, $n \geq 3$, or has a V minor.*

Proof. Let G be any graph without K_5 minors. In the nontrivial case, G is nonplanar. By Theorem (7.4.1), we know that G has a $K_{3,3}$ minor, and thus a $K^i_{3,n}$ minor with an edge set of maximum cardinality. If G itself is that minor, we are done. Otherwise, we apply the splitter Theorem (10.5.1) to the class of graphs consisting of all minors of G and their isomorphic versions. The graph $K^i_{3,n}$ plays the role of the splitter. Thus, G is 2-separable, or is 3-connected and has a 3-connected 1-edge extension of a $K^i_{3,n}$ minor. In the first case, we are done. In the second case, we know by Lemma (10.5.4) and the assumed maximality of the edge set of $K^i_{3,n}$ that G has a V minor. $\square$

We have reached the end of the first iteration. In the notation of the general construction process, the current set $\mathcal{L}'$ contains the planar graphs, and all graphs $K^i_{3,n}$ plus their isomorphic versions. The set $\mathcal{H}'$ contains just one graph, V.

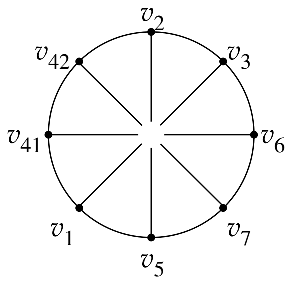
We begin the second iteration. The current $\mathcal{H}$, i.e., the set $\mathcal{H}'$ of iteration 1, contains just V. Thus, that graph is selected as H. We intend to invoke the induced separation result for graphs of Section 6.3, listed there as Corollary (6.3.26). We include that result below.

(10.5.8) Theorem. *Let $\mathcal{G}$ be a class of connected graphs that is closed under isomorphism and under the taking of minors. Let a 3-connected graph $H \in \mathcal{G}$ have a 3-separation (F_1, F_2) with $|F_1|, |F_2| \geq 4$. Assume that H/F_2 has no loops and $H \backslash F_2$ has no coloops. Furthermore, assume that for every 3-connected 1-edge extension of H in $\mathcal{G}$, say by edge z, the pair $(F_1, F_2 \cup \{z\})$ is a 3-separation of that extension. Then for any 3-connected graph $G \in \mathcal{G}$ with an H minor, the following holds. Any 3-separation of any such minor that corresponds to (F_1, F_2) of H under one of the isomorphisms induces a 3-separation of G.*

We need a 3-separation (F_1, F_2) of V of (10.5.3) for the application of Theorem (10.5.8). Thus, we define F_1 to be the set of edges of V incident at node v_1 or v_2, and declare F_2 to be the set of the remaining edges. The 3-separation (F_1, F_2) of V is informally indicated in (10.5.3) by the dashed line.

We will encounter one additional graph G_8 given by (10.5.9) below. For the moment, the unusual indexing of the node labels of G_8 should be ignored. It will make sense shortly. The graph G_8 has twelve edges, and every vertex has degree 3. We claim that G_8 does not have K_5 minors; otherwise, the contraction of two suitably selected edges could produce a graph where at least five vertices have degree of at least 4. But that is not possible.

(10.5.9)



Graph G_8

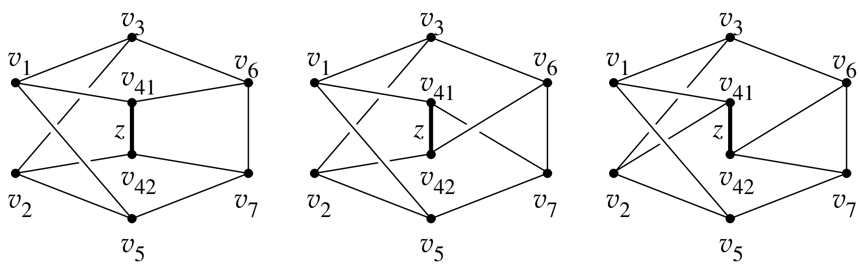
The decomposition result for the second iteration is as follows.

(10.5.10) Theorem. *Let G be a 3-connected graph without K_5 minors, but with a V minor. Then the 3-separation of that minor defined from (F_1, F_2) of V induces a 3-separation of G, or G has a G_8 minor.*

Proof. We apply Theorem (10.5.8) with the class of connected graphs without K_5 minors as $\mathcal{G}$, and with the graph V of (10.5.3) as H. We readily verify that V/F_2 has no loops and that $V\backslash F_2$ has no coloops. Thus, by Theorem (10.5.8), the claimed induced 3-separation exists, or V can be extended by one edge z to a 3-connected graph for which $(F_1, F_2 \cup \{z\})$ is not a 3-separation. We consider all such 3-connected extensions of V by an edge z.

We start with the expansion case. Since $V\&z$ is to be 3-connected, we must split the single degree 4 vertex v_4 of V and insert z. There are three ways to do this. The corresponding graphs V_1, V_2, V_3 are given below.

(10.5.11)



Graph V_1 Graph V_2 Graph V_3

Evidently, $(F_1, F_2 \cup \{z\})$ is not a 3-separation for V_1 or V_2, but this is so for V_3. Thus, V_1 and V_2 are the graphs of interest to us. The graphs V_1 and V_2 are isomorphic. Indeed, one isomorphism from the vertices of V_1 to those of V_2 is an identity except that it takes v_3, v_5, v_6, v_7 of V_1 to v_5, v_3, v_7, v_6 of V_2, respectively. Furthermore, a comparison of V_1 with G_8 of (10.5.9) proves these two graphs to be identical.

We turn to the addition case. Since $V+z$ is to be 3-connected, the added edge z cannot be parallel to another edge. Also, $(F_1, F_2 \cup \{z\})$ is not to be a 3-separation of $V+z$. Thus, one endpoint of z must be v_1 or v_2. The second endpoint must be v_1, v_2, v_6, or v_7. Suppose v_1 is one endpoint and v_2 the second one. Contract in $V+z$ the edges (v_3, v_6) and (v_5, v_7). A K_5 minor results, which is a contradiction. Suppose v_1 is one endpoint and v_6 is the second one. Contract in $V+z$ the edges (v_2, v_3) and (v_5, v_7). Again a K_5 minor results. The remaining cases are isomorphic to the latter one.

We conclude that in each case G has a G_8 minor as desired. □

We combine Theorems (10.5.7) and (10.5.10) to the following result.

(10.5.12) Theorem. *Every connected graph without K_5 minors is* 1- *or* 2-*separable, or has a* 3-*separation with at least four edges on each side, or is planar, or is isomorphic to a graph $K^i_{3,n}$, $0 \le i \le 3$, $n \ge 3$, or has a G_8 minor.*

Proof. By Theorem (10.5.7), we may assume G to have a V minor. Then Theorem (10.5.10) establishes presence of the 3-separation or of a G_8 minor. □

We have reached the end of the second iteration. In terms of the general description of the construction, the current set $\mathcal{L}'$ contains the planar graphs, and all graphs $K^i_{3,n}$ plus their isomorphic versions. The set $\mathcal{H}'$ contains just one graph, G_8.

The third iteration involves an application of the splitter Theorem (10.5.1). The result so produced is as follows.

(10.5.13) Theorem. *G_8 is a splitter for the graphs without K_5 minors.*

Proof. We only need to show that every 3-connected 1-edge extension of G_8 has K_5 minors. Since every vertex of G_8 has degree 3, a 3-connected 1-edge expansion is not possible. Up to isomorphism, just two addition cases are possible. Both cases have K_5 minors. We leave the easy verification of this claim to the reader. □

We combine Theorems (10.5.12) and (10.5.13) with the results for 1-, 2-, and 3-sums of Chapter 8 to obtain Wagner's famous decomposition theorem for the graphs without K_5 minors.

(10.5.14) Theorem. *Every connected graph without K_5 minors is a* 1-, 2-, *or* 3-*sum, or is planar, or is isomorphic to $K_{3,3}$ or G_8.*

Proof. Assume G to be a connected graph without K_5 minors. Theorems (10.5.12) and (10.5.13) imply that G is 1- or 2-separable, or has a 3-separation with at least four edges on each side, or is planar, or is isomorphic to G_8 or to a graph $K^i_{3,n}$, $0 \le i \le 3$, $n \ge 3$. If G is isomorphic to a graph $K^i_{3,n}$ different from $K_{3,3}$, then by (10.5.2), G has a 3-separation with at least four edges on each side. We apply some results of Chapter 8. If G is 1- or 2-separable, then according to Section 8.2, G is a 1- or 2-sum. If G has a 3-separation with at least four edges on each side, then by Lemma (8.3.12) and the discussion following that lemma, G is a 3-sum. Thus, we may conclude that G is a 1-, 2-, or 3-sum, or is planar, or is isomorphic to $K_{3,3}$ or G_8, as claimed in the theorem. □

Note that at the end of the third iteration, the set $\mathcal{H}'$ is empty. Thus, the construction process stops.

Recall the Δ-sum decomposition of Section 8.5. In a connected graph G, such a decomposition is carried out as follows. Given is a 3-separation of G with at least four edges on each side. Let H_1 and H_2 be the corresponding subgraphs of G. Thus, both H_1 and H_2 are connected subgraphs of G. Identification of three connecting nodes of H_1 with three connecting nodes of H_2 produces G. For $i = 1, 2$, we enlarge H_i by attaching a triangle to the three connecting nodes. Let G_i be the resulting graph. Then G is a Δ-sum of G_1 and G_2, denoted by $G = G_1 \oplus_\Delta G_2$. The components G_1 and G_2 of a 3-connected Δ-sum $G_1 \oplus_\Delta G_2$ are 3-connected, except possibly for edges parallel to the edges of the connecting triangle in G_1 or G_2.

Also recall the 2-sum decomposition of Section 8.2. The graphs H_1 and H_2 have two connecting nodes each. For $i = 1, 2$, we enlarge H_i by joining the connecting nodes by an edge. Let G_i be the resulting graph. Then G is a 2-sum of G_1 and G_2, denoted by $G = G_1 \oplus_2 G_2$. The components G_1 and G_2 are 2-connected if G is 2-connected.

By inverting the above operations, we obtain the Δ-sum and 2-sum compositions. At times, one may desire to construct a graph recursively by these operations. Initially, one combines two graphs G_1 and G_2 in a 2- or Δ-sum. Then one composes the resulting graph in a 2- or Δ-sum with a graph G_3. Continuing in this fashion, one recursively enlarges the graph on hand by G_4, G_5, etc. We call the G_i, $i \geq 1$, the *building blocks* of this process.

We may use Theorem (10.5.14) to establish such a construction process for the graphs without K_5 minors. The details are specified in the next theorem.

(10.5.15) Theorem. *Any 2-connected graph without K_5 minors is planar, or isomorphic to $K_{3,3}$ or G_8, or may be constructed recursively by 2-sums and Δ-sums. The building blocks of that construction are as follows.*

2-sums: planar graphs, and graphs isomorphic to $K_{3,3}$ or G_8.
Δ-sums: planar graphs.

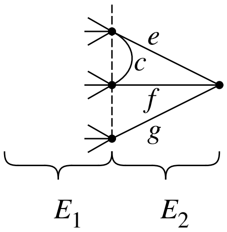
The proof of Theorem (10.5.15) utilizes the following two lemmas.

(10.5.16) Lemma. *Let G be a 3-connected nonplanar graph without K_5 minors and not isomorphic to $K_{3,3}$ or G_8. Assume G to have a triangle C.*

Then G has a 3-separation (E_1, E_2) where $|E_1|, |E_2| \geq 4$ and where one of E_1, E_2 contains C.

Proof. We use induction. If G has ten edges, the smallest case, then direct checking proves the lemma. Otherwise, G is by Theorem (10.5.14) a 3-sum, and thus has a 3-separation (E_1, E_2) where $|E_1|, |E_2| \geq 4$. If one of E_1, E_2 contains C, we are done. Thus, we may assume that just one edge of C, say c, is in E_2. If $|E_2| \geq 5$, we shift the edge c from E_2 to E_1 and have the case where one side of a 3-separation contains C. Thus, we assume that $|E_2| = 4$. It is easy to see that G must be of the form

(10.5.17)



3-separation (E_1, E_2) of G with $|E_2| = 4$ and $c \in E_2$

If G has an edge that forms a triangle with the explicitly shown edges e and g, or with f and g, then we exchange c and such an edge between E_1 and E_2, and again have the desired 3-separation. Otherwise, the minor G/g has no parallel edges. Note that $\{e, f, c\}$ is a triangle of G/g. Indeed, G/g is isomorphic to one of the components of the Δ-sum induced by (E_1, E_2), and thus is 3-connected. We consider two cases, depending on whether G/g is planar.

If G/g is planar, draw it in the plane. If the triangle $\{e, f, c\}$ lies on one face, then it is easily seen that G itself is planar, a contradiction. Thus, $\{e, f, c\}$ partitions the plane into two regions, both of which contain at least one vertex of G/g. Then we readily confirm that G/g, and hence G, has a 3-separation of the form claimed in the lemma.

For the second case, we assume G/g to be nonplanar. Since G/g has a triangle while $K_{3,3}$ and G_8 do not, G/g cannot be isomorphic to either one of the latter graphs. We apply induction and see that G/g as well as G have the desired 3-separation as well. □

(10.5.18) Lemma. *Let G be a 3-connected nonplanar graph without K_5 minors and not isomorphic to $K_{3,3}$ or G_8. Assume G to have either a designated triangle C or a designated edge e. Then G is a Δ-sum $G_1 \oplus_\Delta G_2$, where G_1 contains C or e, whichever applies, and where G_2 is planar.*

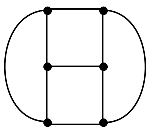
Proof. We prove the case for the triangle C and leave the easier situation with the edge e to the reader. We use induction. The smallest case, which has ten edges, is handled by direct checking. For larger G, we apply Lemma (10.5.16). Thus, G is a Δ-sum $G_1 \oplus_\Delta G_2$ where C is part of the component

G_1. If the second component G_2 is planar, we are done. Otherwise, we may assume G_2 to be 3-connected. We define C' to be the triangle of G_2 involved in the Δ-sum. Due to the presence of the triangle C', the graph G_2 cannot be isomorphic to $K_{3,3}$ or G_8. By induction, G_2 has a Δ-sum decomposition $G_{21} \oplus_\Delta G_{22}$ where C' is in G_{21}, and where G_{22} is planar. Evidently, $G_1 \oplus_\Delta G_{21}$ and G_{22} are the components of a Δ-sum decomposition of G of the desired form. □

Proof of Theorem (10.5.15). Let G be any 2-connected graph without K_5 minors and not isomorphic to $K_{3,3}$ or G_8. If G is 3-connected, the result follows from Lemma (10.5.18). Otherwise, G is a 2-sum. Choose the 2-sum decomposition, say $G_1 \oplus_2 G_2$, so that G_2 has a minimal number of edges. Evidently, any 2-separation of G_2 contradicts the minimality assumption, so G_2 is 3-connected. If G_2 is planar or isomorphic to $K_{3,3}$ or G_8, we are done. Otherwise, let e be the edge of G_2 that is identified with an edge of G_1 in the 2-sum composition creating G. By Lemma (10.5.18), G_2 is a Δ-sum $G_{21} \oplus_\Delta G_{22}$, where G_{21} contains e and where G_{22} is planar. Clearly, G is a Δ-sum where one component is $G_1 \oplus_2 G_{21}$, and where the second component is the planar G_{22}, as demanded in the theorem. □

The recursive construction scheme described at the beginning of this section produces a number of additional decomposition theorems. We include two example theorems that may be obtained that way. The first theorem refers to the graph G_9 of (10.5.19) below, and to $K_5\backslash y$, which is K_5 minus an arbitrarily selected edge y.

(10.5.19)



Graph G_9

(10.5.20) Theorem. *Every 3-connected graph G with at least six edges and without $K_5\backslash y$ minors is, for some $k \geq 3$, isomorphic to the wheel W_k, or is isomorphic to G_9 or $K_{3,3}$.*

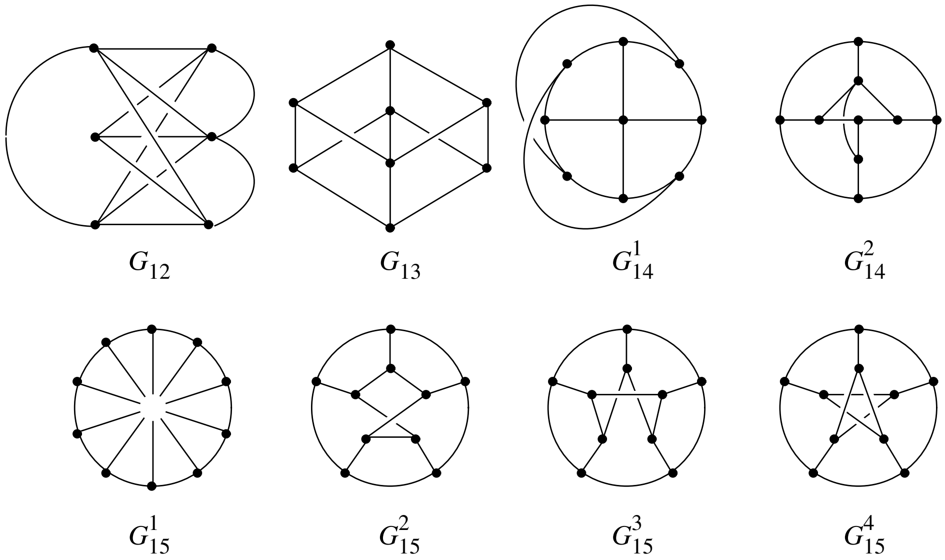
Proof. One first shows that each 3-connected 1-edge extension of any wheel graph with at least four spokes has a $K_{3,3}$ or G_9 minor. Then one suitable application of the splitter Theorem (10.5.1) proves the result. □

The next theorem is much more complicated. We omit the proof, since it involves rather tedious calculations. The theorem refers to a number of graphs, which are listed subsequently under (10.5.22).

(10.5.21) Theorem. *Every connected graph without G_{12} minors is a 1-, 2-, or 3-sum, or is planar, or is isomorphic to K_5, $K_{3,3}$, G_8, G_{13}, G^1_{14}, G^2_{14}, G^1_{15} G^2_{15}, G^3_{15}, or G^4_{15}.*

Here are the graphs mentioned in Theorem (10.5.21).

(10.5.22)



Graphs of Theorem (10.5.21)

The graph G_{15}^4 is the well-known Petersen graph.

From Theorems (10.5.20) and (10.5.21), one may deduce the following 2- and Δ-sum construction results.

(10.5.23) Theorem. *Any 2-connected graph on at least two edges and without $K_5\backslash y$ minors may be constructed recursively by 2-sums. Each building block is a cycle with two or three edges, or is isomorphic to W_k, $k \geq 3$, G_9, or $K_{3,3}$.*

Proof. Let G be a 2-connected graph on at least two edges and without $K_5\backslash y$ minors. If G is not isomorphic to one of the listed graphs, then G has by Theorem (10.5.20) a 2-sum decomposition $G_1 \oplus_2 G_2$. Choose the decomposition so that G_2 has a minimum number of edges. Then G_2 must be 3-connected, and by Theorem (10.5.20) must be one of the prescribed building blocks. $\Box$

(10.5.24) Theorem. *Any 2-connected graph without G_{12} minors is planar, or isomorphic to K_5, $K_{3,3}$, G_8, G_{13}, G_{14}^1, G_{14}^2, G_{15}^1, G_{15}^2, G_{15}^3, or G_{15}^4, or may be constructed recursively by 2-sums and Δ-sums. The building blocks are as follows.*

2-sums: planar graphs, and graphs isomorphic to K_5, $K_{3,3}$, G_8, G_{13}, G_{14}^1, G_{14}^2, G_{15}^1, G_{15}^2, G_{15}^3, or G_{15}^4.
Δ-sums: planar graphs and graphs isomorphic to K_5.

Proof. We use appropriately modified Lemmas (10.5.16) and (10.5.18), and the proof of Theorem (10.5.15). Below, we indicate the necessary adjustments. First, the graphs $K_{3,3}$ and G_8 must throughout be replaced by K_5, $K_{3,3}$, G_8, $G_{13}, \ldots, G_{15}^4$. We note that no graph of that new list has a triangle except for K_5. Second, the proof of the modified Lemma (10.5.16) must be adjusted as follows. In the case of a nonplanar graph G/g, that graph cannot be isomorphic to K_5, $K_{3,3}$, G_8, $G_{13}, \ldots, G_{15}^4$, except for K_5. For the exceptional case, one directly shows G to have a 3-separation of the type demanded by the modified Lemma (10.5.16). Third, in the modified Lemma (10.5.18) and its proof, one now permits the graph G_2 to be isomorphic to K_5. The analogous change applies to the graph G_{22} of the proof of Theorem (10.5.15). □

The decomposition tools provided in this section plus those cited in the references should enable the reader to construct additional decomposition theorems as they are needed. We sketch representative applications for the preceding decomposition theorems in Section 10.7.

Once more we switch topics, and turn to the problem of deciding graphicness of a binary matroid.

10.6 Testing Graphicness of Binary Matroids

Chapters 3, 5, 7, and 8 implicitly contain a quite efficient algorithm for testing graphicness of binary matroids, the topic of this section. Thus, this section is mainly a synthesis of material gleaned from those chapters. We also cover the related problem of deciding whether or not a real $\{0, \pm 1\}$ matrix is the coefficient matrix of a network flow problem.

We start with the graphicness test. Let B be a binary representation matrix of the matroid M to be tested. Small instances are easily decided, so assume that M has at least six elements. If B is not connected, then it clearly suffices that we test each connected component of B for graphicness. Indeed, for some $m \geq 2$, let $G_1, G_2, \ldots, G_m$ be the graphs corresponding to the connected components of B. From each G_i, we select some vertex v_i. Then we combine $G_1, G_2, \ldots, G_m$ to a connected graph G for M and B by identifying the vertices $v_1, v_2, \ldots, v_m$ to one vertex.

Suppose that B is connected, and that we know of a 2-separation of B. By Lemma (8.2.6), M is a 2-sum of two matroids M_1 and M_2. Let B^1 and B^2 be the respective submatrices of B representing the latter matroids. By Lemma (8.2.7), M is graphic if and only if B^1 and B^2 are graphic. Thus, we may analyze B^1 and B^2 instead of B. Indeed, let G_1 and G_2 be graphs for B^1 and B^2, respectively. Then the 2-sum composition of G_1 and G_2 displayed in (8.2.8) produces a graph G for B. In general, G is not unique. But by Theorem (3.2.36), we know that any other graph for B can be obtained from G by a sequence of switchings.

The case remains where B is 3-connected. By Corollary (5.2.15), M has an $M(W_3)$ minor. Furthermore, by Theorem (7.3.4), M has a 3-connected 1-element expansion of an $M(W_3)$ minor, or has a sequence of nested 3-connected minors $M_0, M_1, \ldots, M_t = M$, where M_0 is an $M(W_3)$ minor, and where each M_{i+1} is obtained from M_i by some series expansions, possibly none, followed by a 1-element addition.

By the census of Section 3.3, every 3-connected 1-element expansion of an $M(W_3)$ minor must be an F_7^* minor, which is not regular, and hence not graphic. Thus, evidence of such a minor proves M to be nongraphic. On the other hand, graphicness of each matroid of the sequence $M_0, M_1, \ldots, M_t$ can be efficiently decided as follows. Clearly, the $M(W_3)$ minor M_0 is graphic, and a graph for it is readily found. Suppose for given $0 \leq i < t$, we know M_i to be graphic. By Theorem (3.2.36), the 3-connectedness of M_i implies that just one graph exists for M, say G_i. We assume that we have that 3-connected graph on hand. Now M_{i+1} is obtained from M_i by some series expansions and a 1-element addition. The expansions steps can also be done in G_i, so they preserve graphicness. By Lemma (3.2.49), there is a polynomial, indeed very simple, subroutine for deciding whether the addition step preserves graphicness as well. In the affirmative case, the subroutine also produces the graph for M_{i+1}.

So far, we have assumed that we can locate 2-separations and $M(W_3)$ minors, as well as a 3-connected 1-element expansion of a given $M(W_3)$ minor or the sequence $M_0, M_1, \ldots, M_t$. But these tasks can be efficiently accomplished by the path shortening technique of Chapter 5 and by the efficient method sketched in the proof of Theorem (7.3.6). These two methods plus the simple graphicness testing subroutine of Lemma (3.2.49) constitute an efficient way of deciding whether a given binary matroid is graphic.

We turn to the second testing problem covered in this section. We are to decide whether or not a real $\{0, \pm 1\}$ matrix is the coefficient matrix of a network flow problem. Recall from Section 9.2 that a node/arc incidence matrix of a directed graph is a real $\{0, \pm 1\}$ matrix where each column has only 0s or exactly one $+1$ and one -1. Furthermore, recall that a real matrix is defined to be the coefficient matrix of a network flow problem if it is the node/arc incidence matrix $\tilde{A}$ of a directed graph, or is derived from such a matrix by pivots and deletion of rows and columns. The support matrix $\tilde{B}$ of $\tilde{A}$ is the node/edge incidence matrix of the undirected version of that graph. View $\tilde{B}$ to be over GF(2).

Lemmas (9.2.1), (9.2.2), (9.2.6), (9.2.8), and Corollary (9.2.7) permit the following conclusions about $\tilde{A}$ and $\tilde{B}$. The matrix $\tilde{A}$ is totally unimodular, and $\tilde{B}$ is regular. Any matrix A deduced from $\tilde{A}$ by real pivots and deletion of rows and columns is totally unimodular. The support matrix B of A can be deduced from $\tilde{B}$ by the corresponding GF(2)-pivots. Thus,

B is graphic. Finally, an elementary signing process exists by which one may determine from B the signs of the entries of A, up to a scaling of some rows and columns by -1.

Because of these results, we may test a given real $\{0, \pm 1\}$ matrix A for the network flow property as follows. View the support matrix B of A to be over GF(2). Test B for graphicness with the efficient method described earlier in this section. If B is not graphic, then A cannot have the network flow property, and we stop. Otherwise, sign B to convert it to a totally unimodular matrix A'. By a simple variation of that signing process, which effectively is the proof procedure of Lemma (9.2.6), determine whether or not A' can be converted to A by scaling. Then A is a network flow problem if and only if the answer is affirmative. This procedure can be improved by use of the undirected graph on hand once B has been determined to be graphic. We leave the details to the reader.

A variation of the preceding problem is as follows. We are given a real matrix $\tilde{A}$. We are to settle whether by row scaling, or by row and column scaling, that matrix can be converted to a $\{0, \pm 1\}$ matrix with the network flow property. The question can be reduced to the above situation as follows. If $\mathrm{BG}(\tilde{A})$ is not connected, we apply the test given below to the submatrices of $\tilde{A}$ that correspond to the connected components of $\mathrm{BG}(\tilde{A})$. Hence, assume $\tilde{A}$ to be connected. First, we determine whether by scaling under the assumed restrictions (i.e., scaling of rows only, or scaling of rows and columns), the given $\tilde{A}$ can be converted to a $\{0, \pm 1\}$ matrix. This is readily accomplished by a scaling of the entries of $\tilde{A}$ corresponding to an arbitrarily selected tree of $\mathrm{BG}(\tilde{A})$. We leave it to the reader to fill in the simple details. If a $\{0, \pm 1\}$ matrix cannot be produced, then the network flow property cannot be attained by scaling under the assumed restrictions. If a $\{0, \pm 1\}$ matrix is obtained, we test that matrix for the network flow property. The answer for the original question is affirmative if and only if the scaled matrix has the network flow property. The conclusion is valid since the scaled matrix is unique up to scaling of some rows and columns by -1, as is readily confirmed via the proof of Lemma (9.2.6).

We summarize the above discussion in the following theorem.

(10.6.1) Theorem. *There are polynomial algorithms for each one of the problems* (a), (b), *and* (c) *below.*

(a) *Given is any binary matrix B. Let M be the binary matroid represented by B. It must be decided whether M is graphic. In the affirmative case, an undirected graph G must be produced so that $M(G) = M$.*

(b) *Given is any $\{0, \pm 1\}$ real matrix A. It must be decided whether A has the network flow property. In the affirmative case, a directed graph G must be produced so that the node/arc incidence matrix of G can by row operations be transformed to the matrix $[I \mid A]$.*

(c) *Given is any real matrix A. It must be decided whether A can by row scaling, or by row and column scaling, be transformed to a $\{0, \pm 1\}$ matrix. In the affirmative case, the $\{0, \pm 1\}$ matrix must be produced, and for that matrix, problem* (b) *must be solved.*

In the final section, we sketch applications and extensions and cite relevant references.

10.7 Applications, Extensions, and References

We link the material of this chapter to prior work, and point out applications and extensions.

The representation of a binary 1-element addition of a graphic matroid in Section 10.2 by a graph plus a node subset T is taken from Seymour (1980b). It is one of the many innovative concepts and ideas of that reference. The planarity characterization of Theorem (10.2.11) is properly implied by Tutte's characterization of the graphic matroids (Tutte (1958), (1959), (1965)). The same applies to Corollary (10.2.13), which originally was proved in Whitney (1932), (1933b).

Lemma (10.3.1) is taken from Seymour (1980b). Theorem (10.3.9) has been generalized to nongraphic 3-connected matroids with triangles in Asano, Nishizeki, and Seymour (1984). Seymour (1980b) contains the difficult Theorem (10.3.11). The proof given here is a considerably shortened version of the one of that reference. The key difference lies in the repeated application of Theorem (10.3.9).

The profound characterizations of Section 10.4 in historical order are due to Tutte and Seymour. Theorem (10.4.2) is due to Tutte (1958), (1959), (1965), and Theorem (10.4.1) is due to Seymour (1980b). The latter theorem is one of two main ingredients in the proof of the decomposition theorem of regular matroids, which is covered in the next chapter. Reasonably short proofs of Theorem (10.4.2) are given in Seymour (1980a), Wagner (1985a), and Gerards (1995).

The material of Section 10.5 is based on Truemper (1988). That reference contains details about induced graph decompositions. The famous Theorem (10.5.14) of Wagner was originally proved by quite different arguments (Wagner (1937a), (1970)). Short proofs are given in Halin (1964), (1967), (1981), Ore (1967), and Young (1971). When it was first proved, Theorem (10.5.14) established the equivalence of Hadwiger's conjecture about graph coloring and the four-color conjecture for planar graphs (now a theorem). The details are as follows. A graph G is *colorable with n colors* if the vertices can be colored with n colors so that any two vertices with same color are not connected by an edge. The *chromatic number* of a graph is the least number of colors that permit a coloring of the graph. Hadwiger's conjecture (Hadwiger (1943)) says that any graph with chromatic number n has a K_n minor.

The conjecture is readily seen to be correct for $n = 1, 2$, and 3. For $n = 4$, it was proved in Dirac (1952). Indeed, by Theorem (4.2.6), a 2-connected graph is a series-parallel graph if and only if it has no K_4 minor. It is an elementary exercise to show that any graph whose 2-connected components are series-parallel graphs is colorable with three colors. Thus, a graph with chromatic number equal to 4 must have a K_4 minor.

The case $n = 5$ is more complicated. The first step toward a proof was accomplished in Wagner (1937a) with Theorem (10.5.14). That result allows one to prove that the graphs without K_5 minors are colorable with four colors if this is so for all planar graphs, as follows. Suppose the graph, say G, is 3-connected. By Theorem (10.5.14), G is planar, or is isomorphic to $K_{3,3}$ or G_8, or has a 3-separation with at least four edges on each side. The planar case is handled by the assumption. The graphs $K_{3,3}$ and G_8 are colorable with three colors. In the 3-separation case, G is a Δ-sum, say $G_1 \oplus_\Delta G_2$. By Lemma (8.5.6) or by direct checking, G_1 and G_2 are minors of G. By induction, they have a coloring with at most four colors. In G_1, the colors of the nodes of the connecting triangle must be distinct. The same holds for G_2. By a suitable renaming of the colors of G_2, the graph G can thus be colored with four colors. The case of a 2-separable or 1-separable graph G is even simpler.

The second step in the proof of the case $n = 5$ involves showing that all planar graphs can be colored with four colors. The conjecture of that result was open for about one hundred years. It finally was proved in Appel and Haken (1977), and Appel, Haken, and Koch (1977). An exposition of the proof is included in Saaty and Kainen (1977). Thus, Hadwiger's conjecture is correct for $n \leq 5$. For $n \geq 6$, the conjecture is still open.

Theorem (10.5.15) is part of Wagner (1937a). Theorem (10.5.20) is proved in Wagner (1960). Related decomposition results are described in Halin (1981). Theorem (10.5.21) is taken from Truemper (1988). These theorems are useful for the solution of combinatorial problems via decomposition. An interesting instance is the *max cut problem*. Given is a graph G with nonnegative edge weights. One must find a disjoint union C of cocycles such that the sum of the weights of the edges of C is maximum. This problem is solved in Barahona (1983) for graphs without K_5 minors using Theorem (10.5.15). The same approach applies to graphs without G_{12} minors when Theorem (10.5.24) is substituted for Theorem (10.5.15).

There are numerous other ways in which graphs may be decomposed. The number of results is so large that we cannot even sketch the many ideas, theorems, and applications. Thus, we cite some representative references, but omit details. For example, the *ear decomposition* reduces a given

2-connected graph to a cycle by removing one path at a time while maintaining 2-connectedness. This decomposition is simple, but with its aid significant results have been proved (e.g., in Lovász and Plummer (1975), Kelmans (1987), Lovász (1983), and Frank (1993a)).

For some classes of graphs or for particular graph problems, special decompositions have been developed. Examples are decompositions for perfect graphs (e.g., in Burlet and Uhry (1982), Bixby (1984b), Burlet and Fonlupt (1984), Whitesides (1984), Chvátal (1985), (1987), Chvátal and Hoang (1985), Hoang (1985), Cornuéjols and Cunningham (1985), Hsu (1986), (1987a), (1987b), (1988), and Chvátal, Lenhart, and Sbihi (1990)) and circle graphs (e.g., in Gabor, Supowit, and Hsu (1989)), and for optimization problems (e.g., in Uhry (1979), Boulala and Uhry (1979), Cunningham (1982c), Edmonds, Lovász, and Pulleyblank (1982), Ratliff and Rosenthal (1983), Cornuéjols, Naddef, and Pulleyblank (1983), (1985), Bern, Lawler, and Wong (1987), Lovász (1987), Mahjoub (1988), Coullard and Pulleyblank (1989), Fonlupt and Naddef (1992), and Barahona and Mahjoub (1994a)–(1994d)). Diverse graph decomposition ideas are contained in Lovász and Plummer (1986). An axiomatic treatment of graph decompositions, indeed of decompositions of general combinatorial structures, is given in Cunningham and Edmonds (1980).

The 3-connected components of a graph can be found in linear time by an algorithm of Hopcroft and Tarjan (1973). A decomposition of minimally 3-connected graphs is given in Coullard, Gardner, and Wagner (1993).

The first polynomial test for graphicness of binary matroids was given by Tutte (1960). The graphicness test of Section 10.6 is a simplified version of a scheme of Truemper (1990). Other relevant references have already been cited in Section 3.6. The problem of deciding presence of the network flow property was treated completely and for the first time in Iri (1968). That material appears also in Bixby and Cunningham (1980). An efficient method for deciding graphicness of a matroid not known *a priori* to be binary was first proposed in Seymour (1981c). The matroid is assumed to be specified by a black box for deciding the independence of subsets of the groundset. Related material is included in Bixby (1982a), and Truemper (1982a). The easier case where all circuits are explicitly given is covered in Inukai and Weinberg (1979). The recognition problem of generalized networks, which constitute an extension of directed graphs, is treated in Chandru, Coullard, and Wagner (1985).

Last but by no means least, we should mention a truly astounding proof by Robertson and Seymour of the following daring conjecture due to Wagner: If a given graph property is maintained under minor-taking, then the number of nonisomorphic minor-minimal graphs that do not have the property is finite. The proof of that result involves a powerful graph decomposition concept called tree decomposition that we cannot treat here. The length of the proof is extraordinary. Together with numerous applications, it is being published in a long series of papers (Robertson and Seymour (1983), (1984), (1985), (1986a)–(1986c), (1988), (1990a)–(1990c), (1991), (1994), (1995a)–(1995c), (1996), (1998a)–(1998h)). Related to this work on graphs are the matroid results of Kahn and Kung (1982), and Kung (1986a), (1986b), (1987), (1988), (1990a), (1990b). A survey of applications is made in Fellows (1989). There is some evidence supporting the following conjecture for the matroids that are representable over a given finite field: For any property that is maintained under minor-taking, the number of nonisomorphic minor-minimal matroids not having the property is finite. Truemper (1986) contains an encouraging related result about induced decompositions.