Chapter 1

Introduction

1.1 Summary

A matroid may be specified by a finite set E and a nonempty set of so-called independent subsets of E. The independent subsets must observe two simple axioms. We will introduce them in Chapter 3. Matroids generalize the concept of linear independence of vectors, of determinants, and of the rank of matrices. In fact, any matrix over any field generates a matroid. But there are also matroids that cannot be produced this way.

With matroids, one may formulate rather compactly and solve a large number of interesting problems in diverse fields such as civil, electrical, and mechanical engineering, computer science, and mathematics. Frequently, the matroid formulation of a problem strips away many aspects that at the outset may have seemed important, but that in reality are quite irrelevant. Thus, the matroid formulation affords an uncluttered view of essential problem features. At the same time, the matroid formulation often permits solution of the entire problem, or at least of some subproblems, by powerful matroid techniques.

In this book, we are largely concerned with the binary matroids, which are produced by the matrices over the binary field GF(2). That field has just two elements, 0 and 1, and addition and multiplication obey very simple rules. Any undirected graph may be represented by a certain binary matrix. The graphic matroid produced by such a matrix is an abstraction of the related graph. Thus, binary matroids generalize undirected graphs.

During the past forty years or so, a large number of profound matroid theory results have been produced. A significant portion of these results concerns properties of binary matroids and related matroid decompositions and compositions. Unfortunately, much of the latter material is not easily accessible. This fact, and our own interest in combinatorial decomposition and composition, motivated us to assemble this book.

As we started the writing of the book, we faced a basic conflict. On one hand, we were tempted to prove all matroid results with as much generality as possible. On the other hand, we were also tempted to restrict ourselves to binary matroids, since the proofs would become less abstract. A major argument in favor of the second viewpoint was that the matroid classes analyzed here are binary anyway. Thus, that viewpoint won. Nevertheless, we have mentioned extensions of results to general matroids whenever such extensions are possible.

We proceed as follows. Chapter 2 contains basic definitions concerning graphs and matrices. In Chapter 3, we motivate and define binary matroids. We also prove a number of basic results. In particular, we classify whether the elements of a matroid are loosely or tightly bound together, using the idea of matroid separations and of matroid connectivity. In addition, we learn to shrink matroids to smaller ones by two operations called deletion and contraction. Any such reduced matroid is called a minor of the matroid producing it. Finally, we derive from any matroid another matroid by a certain dualizing operation. Appropriately, the latter matroid is called the dual matroid of the given one.

Chapters 4–6 contain fundamental matroid constructions, tools, and theorems. Chapter 4 is concerned with some elementary constructions of graphs and binary matroids. The constructions rely on replacement rules called series-parallel steps and delta-wye exchanges. In Chapter 5, we introduce a simple yet effective method called the path shortening technique. With its aid, we establish basic connectivity relationships and certain results about the intersection and partitioning of matroids. Chapter 6 contains another elementary matroid tool called the separation algorithm, which identifies certain matroid separations.

The techniques and results of Chapters 4–6 are put to a first use in Chapters 7 and 8. In Chapter 7, we prove the so-called splitter theorem, which links connectivity of a given matroid with the presence of certain minors. With that theorem, we show that a sufficiently connected matroid always contains minors that form a sequence with special properties. In Chapter 8, we establish fundamental notions and theorems about matroid decomposition and composition.

With Chapter 9, we begin the second half of the book. That chapter provides fundamental facts about a very important property of real matrices called total unimodularity. Several translations of the total unimodularity property into matroid language are possible. In one such translation, total unimodularity becomes a property of binary matroids called regularity. Establishing a real matrix to be totally unimodular then becomes equivalent to proving that a certain binary matroid is regular.

In Chapters 10–13, we prove a number of decomposition and composition results about the class of regular matroids and about other, closely related matroid classes. In Chapter 10, we begin with an analysis of the graphic matroids, which are regular. In Chapter 11, we examine the remaining regular matroids, i.e., the nongraphic ones. In Chapter 12, we explore nonregular matroids that, loosely speaking, have many regular minors. The matroids with that property are called almost regular. Finally in Chapter 13, we investigate flows in matroids by borrowing ideas from flows in graphs. A well-known result about the behavior of flows in graphs is the max-flow min-cut theorem. The matroids whose flows exhibit the nice behavior described in that theorem are called the max-flow min-cut matroids. The investigation of Chapter 13 focuses on these matroids.

For each of the classes of matroids mentioned so far (graphic, regular, almost regular, max-flow min-cut), Chapters 9–13 provide polynomial testing algorithms, representative applications, and, except for the almost-regular case, characterizations in terms of excluded minors. In addition, excluded minor characterizations of the binary matroids and of the ternary matroids are given in Chapters 3 and 9, respectively. The ternary matroids are the matroids produced by the matrices over GF(3).

The book may be read as follows. First, one should cover Chapters 2 through 9. During a first reading, one may skip the proofs of the chapters. Chapters 10–13 are largely independent. Thus, one may read them in any order, provided one is willing to occasionally interrupt the reading of a chapter for a quick glance at some auxiliary result of an earlier chapter. In the first section of each chapter, we list relevant earlier chapters, if any.

1.2 Historical Notes

In 1935, H. Whitney realized the mathematical importance of an abstraction of linear dependence. His pioneering paper (Whitney (1935)) contains a number of equivalent axiomatic systems for matroids, and thus laid the foundation for matroid theory. In the 1950s and 1960s, W. T. Tutte built upon H. Whitney's foundation a remarkable body of theory about the structural properties of matroids. In the 1960s, J. Edmonds connected matroids with combinatorial optimization. Within a few years, he produced several key results. In the process, he popularized matroid theory.

From 1965 on, an ever growing number of researchers became interested in matroids. In 1976, D. J. A. Welsh published a book (Welsh (1976)) that contained essentially all results known at that time. As these notes are written, a comprehensive treatment of matroid theory in one book is no longer possible. Selected topics are covered in Crapo and Rota (1970), Lawler (1976), Aigner (1979), Lovász and Plummer (1986), White (1986), (1987), (1991), Kung (1986c), Schrijver (1986), Murota (1987), Recski (1989), Fujishige (1991), Oxley (1992), Bachem and Kern (1992), and Björner, Las Vergnas, Sturmfels, White, and Ziegler (1993). Kung (1986c) includes an excellent historical survey.

Central to this book is the work of P. D. Seymour, W. T. Tutte, and K. Wagner. In historical order, the key results are as follows: K. Wagner's decomposition of the graphs without minors isomorphic to the complete graph on five vertices (Wagner (1937a)); W. T. Tutte's characterization of the regular and graphic matroids (Tutte (1958)) and his efficient test of graphicness (Tutte (1960)); P. D. Seymour's characterization of the max-flow min-cut matroids (Seymour (1977a)), his decomposition of the regular matroids (Seymour (1980b)), and his results on matroid flows (Seymour (1981a)).

Some years ago, these results motivated us to start a systematic investigation using graph and matroid decomposition and composition as main tools (Truemper (1985a), (1985b), (1986), (1987a), (1987b), (1988), (1990), (1992a), (1992b), Tseng and Truemper (1986)). Except for some minor modifications and simplifications, this book comprises a large portion of that effort.

Due to space constraints, the book does not include details of several important matroid results that are related to the material covered here. In particular, we have omitted the principal partitioning results and related earlier material by M. Iri, N. Tomizawa, and others (Kishi and Kajitani (1967), Tsuchiya, Ohtsuki, Ishizaki, Watanabe, Kajitani, and Kishi (1967), Iri (1969), Bruno and Weinberg (1971), Ozawa (1971), Tomizawa (1976b), Iri (1979), Nakamura and Iri (1979), Narayanan and Vartak (1981), Tomizawa and Fujishige (1982), Murota and Iri (1985), and Murota, Iri, and Nakamura (1987)). Tomizawa and Fujishige (1982) provide a detailed historical survey of the work on principal partitions. We should also mention L. Lovász' matroid matching results (Lovász (1980), see also Lovász and Plummer (1986)). That work is not really related to the contents of this book. We are compelled to mention it here since it is one of the very profound achievements in matroid theory. We also have not included, but should mention here, work on oriented matroids. These matroids were independently defined by Bland and Las Vergnas (1978) and Folkman and Lawrence (1978). Two recent books cover most of the known results for oriented matroids (Bachem and Kern (1992), and Björner, Las Vergnas, Sturmfels, White and Ziegler (1993). Finally, many important matroid applications are described in Iri and Fujishige (1981), Iri (1983), Murota (1987), and Recski (1989).