Semi-Simple Lie Algebras and Their Representations

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Preface

Particle physics has been revolutionized by the development of a new "paradigm", that of gauge theories. The SU(2) x U(1) theory of electroweak interactions and the color SU(3) theory of strong interactions provide the present explanation of three of the four previously distinct forces. For nearly ten years physicists have sought to unify the SU(3) x SU(2) x U(1) theory into a single group. This has led to studies of the representations of SU(5), O(10), and E_6. Efforts to understand the replication of fermions in generations have prompted discussions of even larger groups.

The present volume is intended to meet the need of particle physicists for a book which is accessible to non-mathematicians. The focus is on the semi-simple Lie algebras, and especially on their representations since it is they, and not just the algebras themselves, which are of greatest interest to the physicist. If the gauge theory paradigm is eventually successful in describing the fundamental particles, then some representation will encompass all those particles.

The sources of this book are the classical exposition of Jacobson in his *Lie Algebras* and three great papers of E.B. Dynkin. A listing of the references is given in the Bibliography. In addition, at the end of each chapter, references are given, with the authors' names in capital letters corresponding to the listing in the bibliography.

The reader is expected to be familiar with the rotation group as it arises in quantum mechanics. A review of this material begins the book. A familiarity with SU(3) is extremely useful and this is reviewed as well. The structure of semi-simple Lie algebras is developed, mostly heuristically, in Chapters III - VII, culminating with the introduction of Dynkin diagrams. The classical Lie algebras are presented in Chapter VIII and the exceptional ones in Chapter IX. Properties of representations are explored in the next two chapters. The Weyl group is developed in Chapter XIII and exploited in Chapter XIV in the proof of Weyl's dimension formula. The final three chapters present techniques for three practical tasks: finding the decomposition of product representations, determining the subalgebras of a simple algebra, and establishing branching rules for representations. Although this is a book intended for physicists, it contains almost none of the particle physics to which it is germane. An elementary account of some of this physics is given in H. Georgi's title in this same series.

This book was developed in seminars at the University of Michigan and the University of California, Berkeley. I benefited from the students in those seminars, especially H. Haber and D. Peterson in Ann Arbor and S. Sharpe in Berkeley. Sharpe, and H.F. Smith, also at Berkeley, are responsible for many improvements in the text. Their assistance is gratefully acknowledged.

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