# IX. The Exceptional Lie Algebras

We have displayed the four series of classical Lie algebras and their Dynkin diagrams. How many more simple Lie algebras are there? Surprisingly, there are only five. We may prove this by considering a set of vectors (candidates for simple roots) $\gamma_i \subset H_0^*$ and defining a matrix (analogous to the Cartan matrix)[1]:

$$M_{ij} = 2\frac{\langle\gamma_i, \gamma_j\rangle}{\langle\gamma_j, \gamma_j\rangle} \qquad \text{(IX.1)}$$

and an associated diagram (analogous to the Dynkin diagram), where the $\mathrm{i}^{\mathbf{th}}$ and $\mathrm{j}^{\mathbf{th}}$ points are joined by $M_{ij}M_{ji}$ lines. The set $\gamma_i$ is called **allowable** , (in Jacobson's usage) if

*i.* The $\gamma_i$ are linearly independent, that is, if det M $\neq 0$.

*ii.* $M_{ij} \leq 0$ for $i \neq j$.

*iii.* $M_{ij}M_{ji} = 0, 1, 2, \text{or } 3$.

With these definitions, we can prove a series of lemmas:

1. Any subset of an allowable set is allowable. *Proof:* Since a subset of a linearly independent set is linearly independent, (i) is easy. Equally obvious are (ii) and (iii).

2. An allowable set has more points than joined pairs. *Proof:* Let $\gamma = \sum_i \gamma_i \langle\gamma_i,\gamma_i\rangle^{-\frac{1}{2}}$. Since the set is linearly independent, $\gamma \neq 0$ so $\langle\gamma,\gamma\rangle > 0$. Thus

$$0 < \langle\gamma,\gamma\rangle = \sum_{i<j} 2\,\frac{\langle\gamma_i,\gamma_j\rangle}{\langle\gamma_i,\gamma_i\rangle^{\frac{1}{2}}\langle\gamma_j,\gamma_j\rangle^{\frac{1}{2}}} + \text{no. of points}$$

$$0 < -\sum_{i<j}[M_{ij}M_{ji}]^{\frac{1}{2}} + \text{no. of points}\ . \tag{IX.2}$$

For each pair of joined points, $M_{ij}M_{ji}$ is at least unity, so

$$\text{no. of joined pairs} < \text{no. of points.}$$

3. An allowable set's diagram has no loops. *Proof:* If it did, there would be a subset with at least as many joined pairs as points.

4. If an allowable set has a diagram with a chain of points joined only to successive points by single lines, there is an allowable set whose diagram is the same except that the chain is shrunk to a point. *Proof:* Let the chain be $\beta_1, \beta_2, \ldots \beta_m$ and let $\beta = \sum_i \beta_i$. Now

$$\begin{aligned}\langle\beta,\beta\rangle &= \sum_i \langle\beta_i,\beta_i\rangle + 2\sum_{i<j}\langle\beta_i,\beta_j\rangle \\ &= m\langle\beta_1,\beta_1\rangle - (m-1)\langle\beta_1,\beta_1\rangle \\ &= \langle\beta_1,\beta_1\rangle\end{aligned} \tag{IX.3}$$

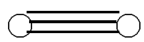
so $\beta$ is the same size as the individual points in the chain. Moreover, if $\gamma$ is joined to the chain at the end,say to $\beta_1$, then $\langle\gamma,\beta_1\rangle = \langle\gamma,\beta\rangle$, since $\langle\gamma,\beta_j\rangle = 0$ for all $j \neq 1$.

5. No more than three lines emanate from a vertex of an allowable diagram. *Proof:* Suppose $\gamma_1, \gamma_2, \ldots \gamma_m$ are connected to $\gamma_0$. Then $\langle\gamma_i, \gamma_j\rangle = 0$ , for $i, j \neq 0$ since there are no loops . Since $\gamma_0$ is linearly independent of the $\gamma_i$, its magnitude squared is greater than the sum of the squares of its components along the orthogonal directions $\gamma_i\langle\gamma_i, \gamma_i\rangle^{-\frac{1}{2}}$:

$$\langle\gamma_0, \gamma_0\rangle \ > \sum_i \langle\gamma_0, \gamma_i\rangle^2 \langle\gamma_i, \gamma_i\rangle^{-1} \ . \tag{IX.4}$$

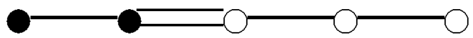
Thus $4 > \sum_i M_{0i}M_{i0}$. But $M_{0i}M_{i0}$ is the number of segments joining $\gamma_0$ and $\gamma_i$.

6. The only allowable configuration with a triple line is



7. An allowable diagram may have one vertex with three segments meeting at a point, but not more. It may have one double line segment, but not more. It may not have both. *Proof:* In each of these instances, it would be possible to take a subset of the diagram and shrink a chain into a point so that the resulting diagram would have a point with more than three line emanating from it. Note that this means that a connected diagram can have roots of at most two sizes, and we henceforth darken the dots for the smaller roots.

8. The diagrams



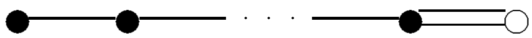
and

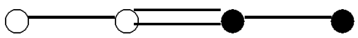
are not allowable. *Proof:* Consider the determinant of $M$ for the first diagram:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

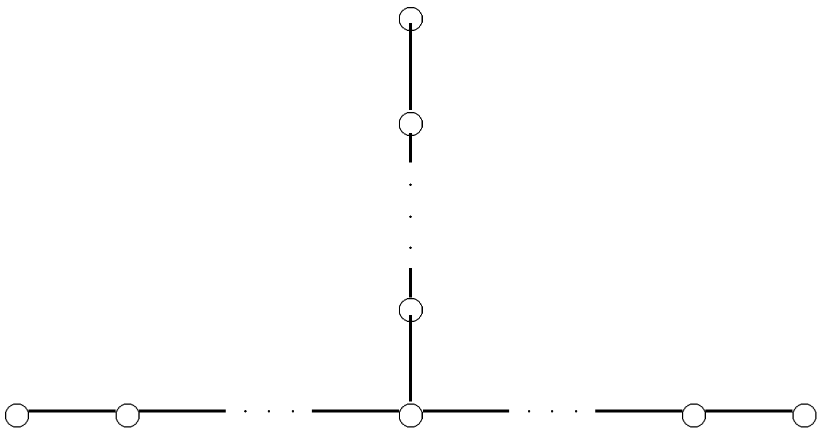
We see that if we add the first and last columns, plus twice the second and fourth, plus three times the third, we get all zeros. Thus the determinant vanishes. The matrix for the second diagram is just the transpose of the first.

9. The only diagrams with a double line segment which may be allowable are of the form:

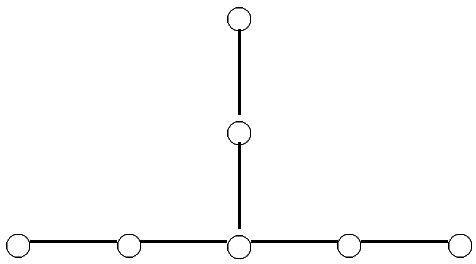




10. By (7) above, the only diagrams with a branch in them are of the form:



11. The diagram below is not allowable

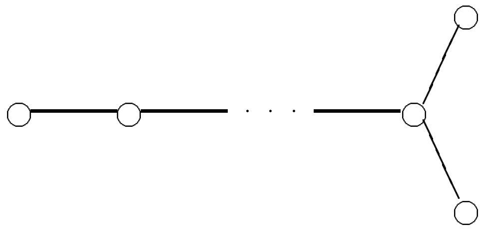


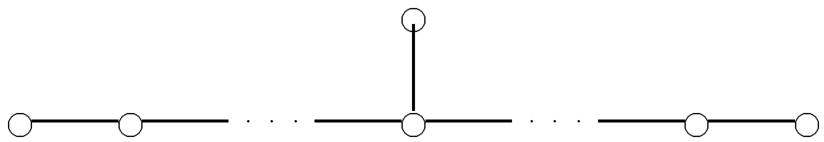
*Proof:* The matrix for the diagram is:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

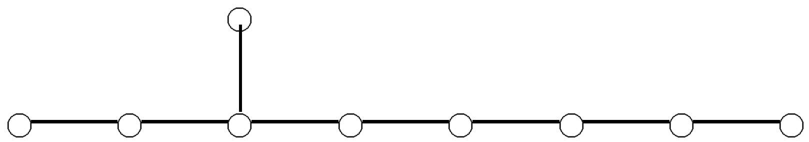
Straightforward manipulation like that above shows that the determinant vanishes.

12. The only allowable diagrams with a branch in them are of the form:



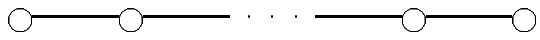


13. The diagram below is not allowable. This is proved simply by evaluating the associated determinant and showing it vanishes.

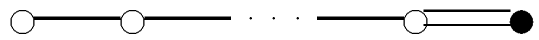


14. The complete list of allowable configurations is

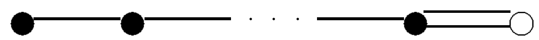
$A_n$



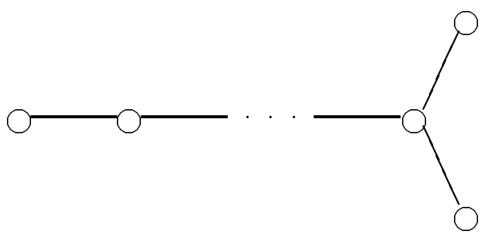
$B_n$



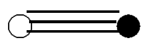
$C_n$



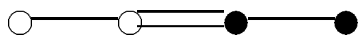
$D_n$



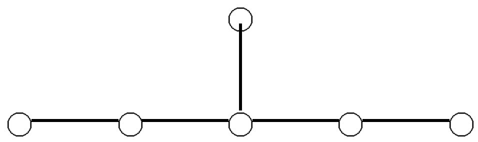
$G_2$



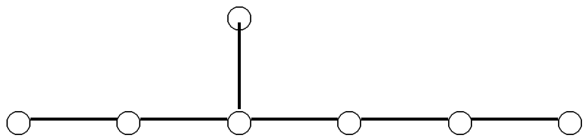
$F_4$



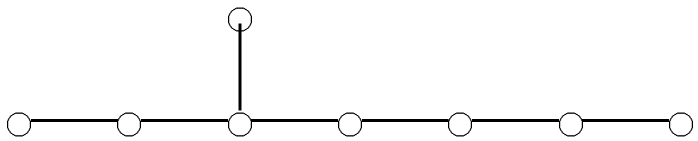
$E_6$



$E_7$



$E_8$



Above are given the names use to designate the five **exceptional Lie algebras**. So far we have only excluded all other possibilities. In fact, these five diagrams do correspond to simple Lie algebras.

## Footnote

1. Throughout the chapter we follow the approach of JACOBSON, pp. 128–135.

## Exercise

1. Prove #11 and #13 above.