VIII. The Classical Lie Algebras

The general considerations of the previous chapter can be applied to the most familiar simple Lie algebras, the classical Lie algebras, SU(n), SO(n), and Sp(2n). These algebras are defined in terms of matrices and are simpler to visualize than some of the exceptional Lie algebras we shall encounter soon. The explicit construction of the Cartan subalgebra and the root vectors and roots for the classical algebras should make concrete our earlier results.

The space of all $n \times n$ matrices has a basis of elements e_{ab} where the components of e_{ab} are

$$(e_{ab})_{ij} = \delta_{ai}\delta_{bj} \ . \tag{VIII.1}$$

Thus the multiplication rule for the matrices is

$$e_{ab}e_{cd} = e_{ad}\delta_{bc} \tag{VIII.2}$$

and the commutator is

$$[e_{ab}, e_{cd}] = e_{ad}\delta_{bc} - e_{cb}\delta_{ad} \; . \tag{VIII.3}$$

The matrix $I = \sum_i e_{ii}$ commutes with all the basis elements. It thus forms the basis for a one-dimensional Abelian subalgebra. Consequently, the Lie algebra of all the $n \times n$ matrices is not semi-simple. However, if we restrict ourselves to traceless $n \times n$ matrices, we do obtain a semi-simple (in fact, simple) Lie algebra called A_{n-1}. This is the complex version of SU(n).

The elements of A_{n-1} are linear combinations of the e_{ab}'s for $a \neq b$ and of elements $h = \sum_i \lambda_i e_{ii}$ where $\sum_i \lambda_i = 0$. From Eq. (VIII.3) we find the commutation relation

$$[h, e_{ab}] = (\lambda_a - \lambda_b) e_{ab} \; . \tag{VIII.4}$$

Thus e_{ab} is a root vector corresponding to the root $\sum_i \lambda_i e_{ii} \rightarrow \lambda_a - \lambda_b$.

Let us choose as a basis for the root space

$$\begin{aligned}
\alpha_1 : \qquad & \sum_i \lambda_i e_{ii} \rightarrow \lambda_1 - \lambda_2 \\
\alpha_2 : \qquad & \sum_i \lambda_i e_{ii} \rightarrow \lambda_2 - \lambda_3 \\
& \qquad \ldots \\
\alpha_{n-1} : \qquad & \sum_i \lambda_i e_{ii} \rightarrow \lambda_{n-1} - \lambda_n
\end{aligned} \tag{VIII.5}$$

and declare these positive with $\alpha_1 > \alpha_2 \ldots > \alpha_{n-1}$. It is easy to see that these same roots are the simple roots.

In order to find the scalar product $\langle \; , \; \rangle$, we first determine the Killing form as applied to elements of the Cartan algebra, using Eq. (IV.3) and Eq. (VIII.4):

$$\begin{aligned}\Big(\sum_i \lambda_i e_{ii}, \sum_j \lambda'_j e_{jj}\Big) &= \mathrm{Tr}\,\mathrm{ad}\left(\sum_i \lambda_i e_{ii}\right)\mathrm{ad}\left(\sum_j \lambda'_j e_{jj}\right)\\ &= \sum_{p,q}(\lambda_p-\lambda_q)(\lambda'_p-\lambda'_q)\\ &= 2n\sum_p \lambda_p\lambda'_p\ . \end{aligned} \tag{VIII.6}$$

The Killing form determines the connection between the Cartan subalgebra , H, and the root space H_0^*. That is, it enables us to find h_{α_i}:

$$\begin{aligned}(h_{\alpha_j}, \sum_i \lambda_i e_{ii}) &= \alpha_j(\sum_i \lambda_i e_{ii})\\ &= \lambda_j - \lambda_{j+1}\ . \end{aligned} \tag{VIII.7}$$

Combining this with Eq. (VIII.6), we see that

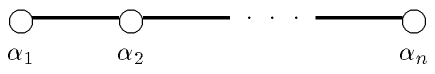
$$h_{\alpha_i} = (e_{ii} - e_{i+1\ i+1})/(2n) \tag{VIII.8}$$

and

$$\langle \alpha_i, \alpha_j\rangle = (2\delta_{ij} - \delta_{i\ j+1} - \delta_{i+1\ j})/(2n)\ . \tag{VIII.9}$$

This agrees in particular with our earlier computation for SU(3). From the value of $\langle\alpha_i,\alpha_j\rangle$ we see that the Cartan matrix and Dynkin diagram are given by

$$A_n : \quad \begin{bmatrix} 2 & -1 & 0 & & & . \\ -1 & 2 & -1 & & & . \\ 0 & -1 & . & & & . \\ & & & . & -1 & 0 \\ & & & -1 & 2 & -1 \\ . & . & . & 0 & -1 & 2 \end{bmatrix}$$



where we have chosen to represent A_n rather than A_{n-1}.

We next consider the **symplectic group** Sp(2m) and its associated Lie algebra. The group consists of the $2m\times 2m$ matrices A with the property $A^tJA = J$ where $(\ )^t$ indicates transpose and J is the $2m \times 2m$ matrix

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \ . \tag{VIII.10}$$

The corresponding requirement for the Lie algebra is obtained by writing $A = \exp(\mathcal{A}) \approx I + \mathcal{A}$. Thus we have $\mathcal{A}^t = J\mathcal{A}J$. In terms of$m \times m$ matrices, we can write

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_3 & \mathcal{A}_4 \end{bmatrix} \tag{VIII.11}$$

and find the restrictions $\mathcal{A}_1^t = -\mathcal{A}_4, \mathcal{A}_2 = \mathcal{A}_2^t, \mathcal{A}_3 = \mathcal{A}_3^t$. In accordance with these, we choose the following basis elements $(j, k \leq m)$:

$$\begin{aligned} e^1_{jk} &= e_{jk} - e_{k+m,j+m} \ , \\ e^2_{jk} &= e_{j,k+m} + e_{k,j+m} \ , \qquad j \leq k \\ e^3_{jk} &= e_{j+m,k} + e_{k+m,j} \ , \qquad j \leq k \ . \end{aligned} \tag{VIII.12}$$

The Cartan subalgebra has a basis $h_j = e^1_{jj}$. By direct computation we find that if $h = \sum_i h_i\lambda_i$,

$$\begin{aligned} \left[h, e^1_{jk}\right] &= +(\lambda_j - \lambda_k)e^1_{jk} \ , \quad j \neq k \\ \left[h, e^2_{jk}\right] &= +(\lambda_j + \lambda_k)e^2_{jk} \ , \quad j \leq k \\ \left[h, e^3_{jk}\right] &= -(\lambda_j + \lambda_k)e^3_{jk} \ , \quad j \leq k \ . \end{aligned} \tag{VIII.13}$$

We take as an ordered basis of roots $\alpha_1(h) = \lambda_1 - \lambda_2$, $\alpha_2(h) = \lambda_2 - \lambda_3$, $\ldots\alpha_{m-1}(h) = \lambda_{m-1} - \lambda_m$, $\alpha_m(h) = 2\lambda_m$. With this ordering, the α_i's are themselves simple roots. For example, the root $\alpha(h) = \lambda_{m-1} + \lambda_m$ is not simple since it is the sum of α_{m-1} and α_m.

We calculate the Killing form on the Cartan subalgebra explicitly by considering in turn the contribution of each root to the trace which defines the form.

$$\begin{aligned}\Big(\sum_i \lambda_i h_i, \sum_j \lambda'_j h_j\Big) &= \sum_{p,q}(\lambda_p - \lambda_q)(\lambda'_p - \lambda'_q) + 2\sum_{p\le q}(\lambda_p + \lambda_q)(\lambda'_p + \lambda'_q)\\ &= \sum_{p,q}[(\lambda_p - \lambda_q)(\lambda'_p - \lambda'_q) + (\lambda_p + \lambda_q)(\lambda'_p + \lambda'_q)] + \sum_p 4\lambda_p\lambda'_p\\ &= 4(m+1)\sum_p \lambda_p\lambda'_p\ . \end{aligned} \tag{VIII.14}$$

We easily see then that

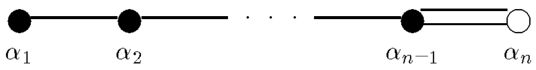
$$\begin{aligned} h_{\alpha_i} &= \frac{(h_i - h_{i+1})}{4(m+1)}\ , \qquad i < m\\ h_{\alpha m} &= \frac{h_m}{2(m+1)}\ . \end{aligned} \tag{VIII.15}$$

Since $(h_i, h_j) = \delta_{ij}4(m+1)$, we can compute directly all the terms we need for the Cartan matrix:

$$\begin{aligned} \langle\alpha_i, \alpha_j\rangle &= \frac{1}{4(m+1)}(2\delta_{ij} - \delta_{i\ j+1} - \delta_{i+1\ j}), \qquad i,j \neq m\\ \langle\alpha_i, \alpha_m\rangle &= -\frac{1}{2(m+1)}\delta_{i+1\ m}, \qquad i \neq m\\ \langle\alpha_m, \alpha_m\rangle &= \frac{1}{(m+1)}\ . \end{aligned} \tag{VIII.16}$$

The Lie algebra which is associated with Sp(2n) is denoted C_n. From Eq. (VIII.16) we derive its Cartan matrix and Dynkin diagram:

$$C_n : \begin{bmatrix} 2 & -1 & 0 & . & . & . \\ -1 & 2 & -1 & & & . \\ 0 & -1 & . & & & . \\ . & & & . & -1 & 0 \\ . & & & -1 & 2 & -1 \\ . & . & . & 0 & -2 & 2 \end{bmatrix}$$



The **orthogonal groups** are given by matrices which satisfy $A^tA = I$. Using the correspondence between elements of the group and elements of the Lie algebra as discussed in Chapter I, $A = \exp \mathcal{A} \approx I + \mathcal{A}$,we see that the requirement is $\mathcal{A} + \mathcal{A}^t = 0$. Clearly these matrices have only off-diagonal elements. As a result, it would be hard to find the Cartan subalgebra as we did for A_n and C_n by using the diagonal matrices. To avoid this problem, we perform a unitary transformation on the matrices A. This will yield an equivalent group of matrices obeying a modified condition. Let us write

$$A = UBU^\dagger \ , \tag{VIII.17}$$

so that

$$A^tA = U^{\dagger t}B^tU^tUBU^\dagger \ = I. \tag{VIII.18}$$

Setting $K = U^tU$, we have $B^tKB = K$. Writing $B \approx I + \mathcal{B}$, we have

$$\mathcal{B}^tK + K\mathcal{B} = 0. \tag{VIII.19}$$

A convenient choice for the even dimensional case, $n = 2m$, is

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -i \\ -1 & -1 \end{bmatrix}, \tag{VIII.20}$$

so that

$$K = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{VIII.21}$$

Representing $\mathcal{B}$ in terms of $m \times m$ matrices,

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{B}_3 & \mathcal{B}_4 \end{bmatrix} \tag{VIII.22}$$

the condition becomes

$$\mathcal{B}_1 = -\mathcal{B}_4^t \; , \quad \mathcal{B}_2 = -\mathcal{B}_2^t \; , \quad \mathcal{B}_3 = -\mathcal{B}_3^t \; . \tag{VIII.23}$$

We can now select a basis of matrices obeying these conditions:

$$\begin{aligned} e^1_{j\,k} &= e_{j,k} - e_{k+m,j+m} , \\ e^2_{j\,k} &= e_{j,k+m} - e_{k,j+m} , \qquad j < k \\ e^3_{j\,k} &= e_{j+m,k} - e_{k+m,j} , \qquad j < k \end{aligned} \tag{VIII.24}$$

and designate the basis for the Cartan subalgebra by

$$h_j = e^1_{jj} \; . \tag{VIII.25}$$

Writing a general element of the Cartan subalgebra as

$$h = \sum_i \lambda_i h_i \; , \tag{VIII.26}$$

we compute the various roots

$$
\begin{aligned}
\left[h, e^1_{jk}\right] &= (\lambda_j - \lambda_k) e^1_{jk} \qquad && j \neq k \\
\left[h, e^2_{jk}\right] &= (\lambda_j + \lambda_k) e^2_{jk} \qquad && j < k \\
\left[h, e^3_{jk}\right] &= -(\lambda_j + \lambda_k) e^3_{jk} \qquad && j < k \ .
\end{aligned} \tag{VIII.27}
$$

Note that for e^2_{jk} and e^3_{jk} we must have $j \neq k$ or else the matrix vanishes. Thus there are no roots corresponding to $\pm 2\lambda_j$. We may take as a basis of simple roots $\alpha_1(h) = \lambda_1 - \lambda_2, \alpha_2(h) = \lambda_2 - \lambda_3, \ldots \alpha_{m-1}(h) = \lambda_{m-1} - \lambda_m, \alpha_m(h) = \lambda_{m-1} + \lambda_m$.

The Killing form restricted to the Cartan subalgebra is given by

$$
\begin{aligned}
(\sum_i \lambda_i h_i, \sum_j \lambda'_j h_j) &= \sum_{i\neq j} (\lambda_i - \lambda_j)(\lambda'_i - \lambda'_j) + 2\sum_{i<j} (\lambda_i + \lambda_j)(\lambda'_i + \lambda'_j) \\
&= \sum_{i,j} [(\lambda_i - \lambda_j)(\lambda'_i - \lambda'_j) + (\lambda_i + \lambda_j)(\lambda'_i + \lambda'_j)] - \sum_i 4\lambda_i \lambda'_i \\
&= 4(m-1) \sum_i \lambda_i \lambda'_i \ .
\end{aligned} \tag{VIII.28}
$$

From this relation we can determine the h_{α_i}'s:

$$
h_{\alpha_i} = \frac{h_i - h_{i+1}}{4(m-1)} \ , \qquad i < m \tag{VIII.29a}
$$

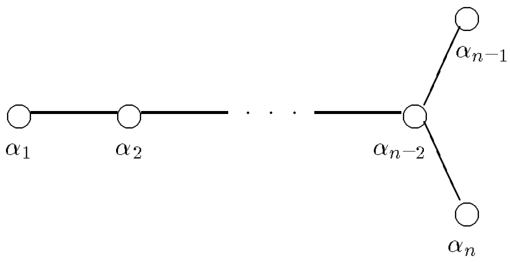
$$
h_{\alpha_m} = \frac{h_{m-1} + h_m}{4(m-1)} \ . \tag{VIII.29b}
$$

The scalar products of the roots are now easily computed:

$$
\begin{aligned}
\langle \alpha_i, \alpha_j \rangle &= [2\delta_{ij} - \delta_{ij+1} - \delta_{i+1j}]/[4(m-1)] \qquad i, j < m \\
\langle \alpha_m, \alpha_m \rangle &= 1/[2(m-1)] \\
\langle \alpha_{m-1}, \alpha_m \rangle &= 0, \\
\langle \alpha_{m-2}, \alpha_m \rangle &= -1/[4(m-1)] \ .
\end{aligned} \tag{VIII.30}
$$

Thus the Cartan matrix and Dynkin diagram are

$$D_n: \quad \begin{bmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & & & & \cdot \\ 0 & -1 & & & & & \cdot \\ \cdot & & & 2 & -1 & 0 & 0 \\ \cdot & & & -1 & 2 & -1 & -1 \\ \cdot & & & 0 & -1 & 2 & 0 \\ \cdot & & & 0 & -1 & 0 & 2 \end{bmatrix}$$



For the odd dimensional case of the orthogonal group, we proceed the same way except that we set

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & i_m & -i_m \\ 0 & -1_m & -1_m \end{bmatrix} \tag{VIII.31}$$

so that

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0_m & 1_m \\ 0 & 1_m & 0_m \end{bmatrix} \tag{VIII.32}$$

where the subscript m indicates an $m \times m$ matrix. The corresponding matrix $\mathcal{B}$ may be parameterized as

$$\mathcal{B} = \begin{bmatrix} b_1 & c_1 & c_2 \\ d_1 & \mathcal{B}_1 & \mathcal{B}_2 \\ d_2 & \mathcal{B}_3 & \mathcal{B}_4 \end{bmatrix} . \tag{VIII.33}$$

For the $2m \times 2m$ pieces of the matrix, the conditions are the same as for the even dimensional orthogonal algebra. The constraints on the new matrices are

$$b_1 = 0 , \qquad c_1 = -d_2^t , \qquad c_2 = -d_1^t . \tag{VIII.34}$$

Thus we must add to our basis for the $2m$ dimensional orthogonal algebra the elements $(1 \le j \le m)$:

$$e_j^4 = e_{0j} - e_{j+m\ 0} ; \qquad e_j^5 = e_{j0} - e_{0\ j+m} . \tag{VIII.35}$$

The corresponding roots are seen to be

$$\left[h, e_j^4\right] = -\lambda_j e_j^4 ; \qquad \left[h, e_j^5\right] = \lambda_j e_j^5 . \tag{VIII.36}$$

Using these new roots, together with those found for the even dimensional case, we compute the Killing form

$$\begin{aligned} &\Big(\sum_i \lambda_i h_i, \sum_j \lambda'_j h_j\Big) \\ &= \sum_{i \neq j} (\lambda_i - \lambda_j)(\lambda'_i - \lambda'_j) + 2\sum_{i<j} (\lambda_i + \lambda_j)(\lambda'_i + \lambda'_j) + 2\sum_i \lambda_i \lambda'_i \\ &= 4(m - \tfrac{1}{2}) \sum_i \lambda_i \lambda'_i . \end{aligned} \tag{VIII.37}$$

From this we can infer the values

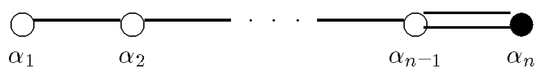
$$
\begin{aligned}
h_{\alpha_i} &= \frac{h_{\alpha_i} - h_{\alpha_{i+1}}}{4(m-\frac{1}{2})} \,, \qquad i < m \\
h_{\alpha_m} &= \frac{h_{\alpha_m}}{4(m-\frac{1}{2})}
\end{aligned} \tag{VIII.38}
$$

where now the simple roots have the values $\alpha_1(h) = \lambda_1 - \lambda_2, \alpha_2(h) = \lambda_2 - \lambda_3, \ldots \alpha_{m-1}(h) = \lambda_{m-1} - \lambda_m, \alpha_m(h) = \lambda_m$. Note that the last of these was not even a root for the even dimensional case. Using the Killing form, it is easy to compute the scalar product on the root space:

$$
\begin{aligned}
\langle \alpha_i, \alpha_j \rangle &= \frac{1}{4(m-\frac{1}{2})}(2\delta_{ij} - \delta_{i\ j+1} - \delta_{i+1\ j}), \qquad i < m \\
\langle \alpha_m, \alpha_i \rangle &= 0, \qquad i < m-1 \\
\langle \alpha_m, \alpha_{m-1} \rangle &= -\frac{1}{4(m-\frac{1}{2})} \,, \\
\langle \alpha_m, \alpha_m \rangle &= \frac{1}{4(m-\frac{1}{2})} \,.
\end{aligned} \tag{VIII.39}
$$

Accordingly, the Cartan matrix and Dynkin diagram are

$$
B_n: \quad \begin{bmatrix}
2 & -1 & 0 & . & . & . \\
-1 & 2 & -1 & & & . \\
0 & -1 & & & & . \\
. & & & 2 & -1 & 0 \\
. & & & -1 & 2 & -2 \\
. & . & . & 0 & -1 & 2
\end{bmatrix}
$$



Notice the similarity between B_n and C_n. In the Cartan matrix they differ only by the interchange of the last off-diagonal elements. The corresponding change in the Dynkin diagrams is to reverse the shading of the dots.

References

This material is discussed in DYNKIN I, JACOBSON, pp. 135-141, and MILLER, pp. 351–354.

Exercise

1. Starting with the Dynkin diagrams, construct drawings of the roots of B_2, D_2, A_3, B_3, and C_3.