VII. Simple Roots and the Cartan Matrix

The next step in analyzing the simple Lie algebras is to define an ordering among the elements in the root space, the space H_0^* of real linear combinations of roots. This ordering is necessarily somewhat arbitrary: there is no natural ordering in the root space. Nevertheless, we shall see that even an arbitrarily chosen ordering can provide much useful information. Let $\alpha_1, \alpha_2 \ldots \alpha_n$ be a fixed basis of roots so every element of H_0^* can be written $\rho = \sum_i c_i \alpha_i$. We shall call ρ **positive** ($\rho > 0$) if $c_1 > 0$, or if $c_1 = 0$, we call ρ positive if $c_2 > 0$, *etc.* If the first non-zero c_i is negative we call ρ negative. Clearly this ordering is possible only because we consider only real linear combinations of roots rather than the full dual space, H^*. We shall write $\rho > \sigma$ if $\rho - \sigma > 0$.

Given the choice of an ordered basis, we can determine which roots are positive and which are negative. A **simple root** is a positive root which cannot be written as the sum of two positive roots. Let us consider SU(3) as an example.

According to Eq. (III.6), the roots are

$$\begin{aligned}
\alpha_1(at_z + by) &= a \\
\alpha_2(at_z + by) &= -\tfrac{1}{2}a + b \\
\alpha_3(at_z + by) &= \tfrac{1}{2}a + b
\end{aligned} \tag{III.6}$$

and the negatives of these roots. Suppose we select as a basis for H_0^* the roots α_1 and α_3, in that order. Now since $\alpha_2 = \alpha_3 - \alpha_1$, α_2 is negative. What are the simple roots? The positive roots are $\alpha_1, -\alpha_2$, and α_3. Now $\alpha_1 = \alpha_3 + (-\alpha_2)$ so α_1 is the sum of two positive roots and is thus not simple. The simple roots are $-\alpha_2$ and α_3, and $-\alpha_2 > \alpha_3$. Of course, this depends on our original ordering of the basis.

We denote the set of simple roots by Π and the set of all roots by Σ. One very important property of the simple roots is that the difference of two simple roots is not a root at all: $\alpha, \beta \in \Pi \Rightarrow \alpha - \beta \notin \Sigma$. To see this, suppose that to the contrary $\alpha - \beta$ is a root. Then either $\alpha - \beta$ or $\beta - \alpha$ is positive. Thus either $\alpha = (\alpha - \beta) + \beta$ or $\beta = (\beta - \alpha) + \alpha$ can be written as the sum of two positive roots which is impossible for simple roots.

If α and β are simple roots, then $\langle \alpha, \beta \rangle \leq 0$. This follows from Eq. (V.22) because β is a root, but $\beta - \alpha$ is not a root. Thus in Eq. (V.22), $m = 0$, so $m - p \leq 0$.

From this result it is easy to show that the simple roots are linearly independent. If the simple roots are not linearly independent we can write an equality

$$\sum_{\alpha_i \in \Pi} a_i \alpha_i = \sum_{\alpha_j \in \Pi} b_j \alpha_j \ , \tag{VII.1}$$

where all the a_i and b_j are non-negative, and no simple root appears on both sides of the equation. (If there were a relation $\sum_i c_i \alpha_i = 0$ with all positive coefficients, the roots α_i could not all be positive.) Now multiplying both sides of Eq. (VII.1) by $\sum_i a_i \alpha_i$,

$$\langle\sum_i a_i\alpha_i,\sum_j a_j\alpha_j\rangle = \langle\sum_i a_i\alpha_i,\sum_j b_j\alpha_j\rangle \ . \tag{VII.2}$$

The left hand side is positive since it is a square, but the right hand side is a sum of negative terms. This contradiction establishes the linear independence of the simple roots. Thus we can take as a basis for the root space the simple roots, since it is not hard to show they span the space.

We now demonstrate a most important property of the simple roots: every positive root can be written as a positive sum of simple roots. This is certainly true for the positive roots which happen to be simple. Consider the smallest positive root for which it is not true. Since this root is not simple, it can be written as the sum of two positive roots. But these are smaller than their sum and so each can, by hypothesis, be written as a positive sum of simple roots. Hence, so can their sum.

From the simple roots, we form the **Cartan matrix**, which summarizes all the properties of the simple Lie algebra to which it corresponds. As we have seen, the dimension of the Cartan subalgebra, H, is the same as that of H_0^*, the root space. This dimension, which is the same as the number of simple roots, is called the **rank** of the algebra. For a rank n algebra, the Cartan matrix is the $n\times n$ matrix

$$A_{ij} = 2\frac{\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} \tag{VII.3}$$

where $\alpha_i, i=1,\ldots n$ are the simple roots.

Clearly, the diagonal elements of the matrix are all equal to two. The matrix is not necessarily symmetric, but if $A_{ij}\neq 0$, then $A_{ji}\neq 0$. In fact, we have shown (see the discussion preceeding Eq. (VI.4)) that the only possible values for the off-diagonal matrix elements are $0,\pm1,\pm2,$ and ± 3. Indeed, since the scalar product of two different simple roots is non-positive, the off-diagonal elements can be only $0,-1,-2,$ and -3.

We have seen that $\langle\ ,\ \rangle$ is a scalar product on the root space. The Schwarz inequality tells us that

$$\langle \alpha_i, \alpha_j \rangle^2 \le \langle \alpha_i, \alpha_i \rangle \langle \alpha_j, \alpha_j \rangle \ , \tag{VII.4}$$

where the inequality is strict unless α_i and α_j are proportional. This cannot happen for $i \neq j$ since the simple roots are linearly independent. Thus we can write

$$A_{ij} A_{ji} < 4, \qquad i \neq j \ . \tag{VII.5}$$

It follows that if $A_{ij} = -2$ or -3, then $A_{ji} = -1$.

Consider again the SU(3) example. For simplicity, (and contrary to our choice above) take the positive basis to be $\alpha_1$and α_2. Then since $\alpha_3 = \alpha_1 + \alpha_2$, the simple roots are also $\alpha_1$and α_2. We computed the relevant scalar products in Eq. (III.12):

$$\begin{aligned} \langle \alpha_1, \alpha_1 \rangle &= \tfrac{1}{3} \\ \langle \alpha_1, \alpha_2 \rangle &= -\tfrac{1}{6} \\ \langle \alpha_2, \alpha_2 \rangle &= \tfrac{1}{3} \ . \end{aligned} \tag{VII.6}$$

From this we compute the Cartan matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \ . \tag{VII.7}$$

The Cartan matrix, together with the ubiquitous Eq. (V.22), suffices to determine all the roots of a given simple Lie algebra. It is enough to determine all the positive roots, each of which can be written as a positive sum of simple roots: $\beta = \sum_i k_i \alpha_i$. We call $\sum_i k_i$ the **level** of the root β. Thus the simple roots are at the first level. Suppose we have determined all the roots up to the nthlevel and wish to determine those at the level n+1. For each root β at the nthlevel, we must determine whether or not $\beta + \alpha_i$ is a root.

Since all the roots through the nthlevel are known, it is known how far back the string of roots extends: $\beta, \beta-\alpha_i, \ldots \beta-m\alpha_i$. From this, we can compute how far forward the string extends:$\beta, \beta+\alpha_i, \ldots \beta+p\alpha_i$. We just put our values into Eq. (V.22):

$$\begin{aligned} m-p &= 2\frac{\langle\beta,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} \\ &= \sum_j 2k_j\frac{\langle\alpha_j,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} \\ &= \sum_j k_j A_{ji}\ . \end{aligned} \tag{VII.8}$$

In particular, $\beta+\alpha_i$ is a root if $p = m - \sum_j k_j A_{ji} > 0$.

It is thus convenient to have an algorithm which keeps track of the n quantities $\sum_j k_j A_{ji}$ for each root as it is determined. It is clear that this is accomplished by adding to the n quantities the jthrow of the Cartan matrix whenever the jthsimple root is added to a root to form a new root.

Let us carry out this construction for SU(3). We begin by writing down the Cartan matrix, then copying its rows to represent the simple roots:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\boxed{2\ \ -1} \qquad \boxed{-1\ \ 2}$$

$$\boxed{1\ \ 1}$$

Beginning with the root α_1 we ask whether the addition of α_2 produces a root in the second level. (Remember that $2\alpha_1$ cannot be a root, nor can $\alpha_1-\alpha_2$). Since the second entry in the box for the first root is negative, the corresponding value of p in Eq. (VII.8) must be positive, so $\alpha_1+\alpha_2$ is a root. The same conclusion would be reached beginning with α_2. Is there a root at level three? Looking back in the α_1 direction, $m=1$. Since the first entry in the box for $\alpha_1+\alpha_2$ is one, we have $p=0$ so we cannot add another α_1. The same applies for α_2. There are no roots at the third level.

As a slightly more complex example, we display the result for the exceptional algebra G_2, which we shall discuss at greater length later:

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\boxed{2 \quad -3} \qquad \boxed{-1 \quad 2}$$

$$\boxed{1 \quad -1}$$

$$\boxed{0 \quad 1}$$

$$\boxed{-1 \quad 3}$$

$$\boxed{1 \quad 0}$$

Not only does the Cartan matrix determine all of the roots, it determines the full commutation relations for the algebra. To see this, let us introduce the notation of Jacobson[1]. Start with any choice of normalization for $e_\alpha$and $e_{-\alpha}$. We have shown that $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha$. Now for every simple root, α_i, define

$$\begin{aligned} e_i &= e_{\alpha_i} \\ f_i &= e_{-\alpha_i} \cdot 2\left[(e_{\alpha_i}, e_{-\alpha_i})\langle\alpha_i, \alpha_i\rangle\right]^{-1} \\ h_i &= h_{\alpha_i} \cdot \frac{2}{\langle\alpha_i, \alpha_i\rangle} \ . \end{aligned} \tag{VII.9}$$

By direct computation we find

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_j \\ [h_i, e_j] &= A_{ji} e_j \\ [h_i, f_j] &= -A_{ji} f_j \ . \end{aligned} \tag{VII.10}$$

The commutator $[e_i, f_j]$ vanishes unless $i = j$ since it would be proportional to $e_{\alpha_i - \alpha_j}$ and $\alpha_i - \alpha_j$ is not a root since α_i and α_j are simple.

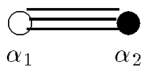
A full basis for the Lie algebra can be obtained from the e_i's, f_i's and h_i's. All of the raising operators can be written in the form $e_{i_1}, [e_{i_1}, e_{i_2}], [e_{i_1}, [e_{i_2}, e_{i_3}]]$, *etc.*, and similarly for the lowering operators constructed from the f's. Two elements obtained from commuting in this way the same set of e's, but in different orders, are proportional with constant of proportionality being determined by the Cartan matrix through the commutation relations in Eq. (VII.10). Among the various orderings we choose one as a basis element. Following the same procedure for the f's and adjoining the h's we obtain a complete basis. The commutation relations among them can be shown to be determined by the simple commutation relations in Eq. (VII.10), that is, by the Cartan matrix.

The Cartan matrix thus contains all the information necessary to determine entirely the corresponding Lie algebra. Its contents can be summarized in an elegant diagrammatic form due to Dynkin. The **Dynkin diagram** of a semi-simple Lie algebra is constructed as follows. For every simple root, place a dot. As we shall show later, for a simple Lie algebra, the simple roots are at most of two sizes. Darken the dots corresponding to the smaller roots. Connect the ithand jthdots by a number of straight lines equal to $A_{ij}A_{ji}$. For a semi-simple algebra which is not simple, the diagram will have disjoint pieces, each of which corresponds to a simple algebra.

For SU(3) and G_2, we have the Cartan matrices and Dynkin diagrams shown below:

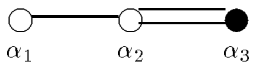
$$SU(3) = A_2 \qquad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$G_2 \qquad \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$



The darkened dot for G_2 corresponds to the second root, since the presence of the (-3) in the second row indicates that the second root is the smaller.

In subsequent sections we will determine the full set of Dynkin diagrams which represent simple Lie algebras. Here we anticipate the result somewhat in order to demonstrate how the Cartan matrix and Dynkin diagrams determine each other. Consider the Dynkin diagram:



The Cartan matrix is determined by noting that $A_{13} = A_{31} = 0$, since the first and third dots are not connected. Since one line connects the first and second points, we must have $A_{12} = A_{21} = -1$. The second and third points are connected by two lines so $A_{23}A_{32} = 2$. Since the third root is smaller than the second, it must be that $A_{23} = -2$ while $A_{32} = -1$. Thus we have

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

Footnote

1. JACOBSON, p. 121.

References

Dynkin diagrams were first introduced in DYNKIN I. An excellent review of much of the material presented in this and other chapters is found in the Appendix to DYNKIN III.

Exercises

1. Find the Dynkin diagram for

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} .$$

2. Find all the roots of B_2 whose Cartan matrix is

$$\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} .$$

Draw a picture of the roots of B_2 like that in Fig. III.1.

3. Draw a picture of the roots of G_2 and compare with Fig. III.1.