VI. More on the Structure of Simple Lie Algebras

In this Chapter we shall use the results on representations just obtained to learn about the algebras themselves by considering the adjoint representation. In the adjoint representation, the Lie algebra itself serves as the vector space on which the E's and H's act. Thus if x is an element of the Lie algebra L, then e_α is represented by E_α where

$$E_\alpha x = \mathrm{ad}\ e_\alpha(x)\ . \tag{VI.1}$$

Before studying the adjoint representation, let us first state a few properties of simple (and semi-simple) Lie algebras which may sound intuitive or obvious, but which require real mathematical proof. As is often the case, it is these innocent sounding statements which are the most difficult to prove and we omit their proofs, which may be found in standard mathematical texts.

First, if α is a root, then $\langle \alpha, \alpha \rangle \neq 0$. While we have asserted that $\langle\ ,\ \rangle$ will become a scalar product on the root space, we have not proved it. In fact, we shall prove it later, based on the assumption that $\langle \alpha, \alpha \rangle \neq 0$.

Second, if α is a root, then the only multiples of α which are roots are α, $-\alpha$, and 0. We can show that $-\alpha$ must be a root because $(e_\alpha, e_\beta) = 0$ unless $\alpha + \beta = 0$, and we cannot have $(e_\alpha, x) = 0$for every x in the Lie algebra, because then the Lie algebra would not be semi-simple (see Chapter III). It might be thought that to show that 2α is not a root would be simple, since $e_{2\alpha}$ might arise from $[e_\alpha, e_\alpha]$ which is certainly zero. However, this proves nothing since $e_{2\alpha}$ might arise from, say,$[e_{\alpha+\beta}, e_{\alpha-\beta}]$. Nevertheless, the result is true. In fact, if α, β, and $\alpha + \beta$ are roots, then $[e_\alpha, e_\beta] \neq 0$.

Third, there is only one linearly independent root vector for each root. This may be stated in terms of the adjoint representation: every weight space (except the root zero space which is the Cartan subalgebra) is one dimensional.

The adjoint representation has other important properties. We know that there is no limit to the length of a string of weights for a representation: even for SU(2) we can have arbitrarily long strings, $j, j-1 \ldots -j$. However, in the adjoint representation, a string can have no more than four weights in it. That is, a string of roots can have no more than four roots in it. We shall see that this has far-reaching consequences.

Suppose to the contrary, there is a string containing five roots which we label without loss of generality $\beta - 2\alpha, \beta - \alpha, \beta, \beta + \alpha$, and $\beta + 2\alpha$. Since α is a root, $2\alpha = (\beta + 2\alpha) - \beta$ is not a root, nor is $2(\beta + \alpha) = (\beta + 2\alpha) + \beta$. Thus $\beta + 2\alpha$ is in a β-string of roots with only one element. Thus from Eq. (V.22)

$$\langle \beta + 2\alpha, \beta \rangle = 0 \ . \tag{VI.2}$$

Similarly,

$$\langle \beta - 2\alpha, \beta \rangle = 0 \ . \tag{VI.3}$$

But then $\langle \beta, \beta \rangle = 0$, which is impossible. Geometrically, we see that β is perpendicular to both $\beta + 2\alpha$ and $\beta - 2\alpha$ which is possible only if $\beta = 0$.

Now if the α-string containing β is four elements long, $\beta + 2\alpha, \beta + \alpha, \beta, \beta - \alpha$, then $m - p$ in Eq. (V.22) can be only ± 3 or ± 1. If the string is three elements long, $m - p$ can be only ± 2 or 0. If it is two elements long, $m - p$ is ± 1, and if it is only one element long, $m - p$ is 0.

We can obtain more information from the ubiquitous Eq. (V.22). Using it twice, we write

$$\frac{\langle\alpha,\beta\rangle\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle}=\tfrac{1}{4}mn \tag{VI.4}$$

where m and n are integers given by the appropriate values of $m-p$. We recognize the left hand side as $\cos^2\theta$ where θ is the angle between the vectors α and β. Anticipating that we really have a scalar product, we use the Schwarz inequality to assert that $mn/4$ must be less than unity unless α and β are proportional. Thus $\cos^2\theta$ can take on only the values $0, \frac{1}{4}, \frac{1}{2}$, and $\frac{3}{4}$.

We shall later see how this restriction of permissible angles limits the possibilities for simple Lie algebras. Indeed, we shall see that every simple Lie algebra falls either into one of four sequences of "classical" algebras or is one of the five "exceptional" Lie algebras first enumerated by Killing. Since every semi-simple Lie algebra is a sum of simple Lie algebras, this will give an exhaustive list of the semi-simple Lie algebras as well.

For the present, we pursue our analysis of the nature of roots of simple Lie algebras. First we show that every root is expressible as a linear combination of a basis set of roots with real, rational coefficients. Suppose $\alpha_1, \alpha_2 \ldots$ is a basis of roots for H^*. (It is not hard to show the roots span H^*.) Let β be a root expressed as $\beta=\sum_i q_i\alpha_i$. Then

$$2\frac{\langle\beta,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}=\sum_i q_i\ 2\frac{\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}\ . \tag{VI.5}$$

This is a set of linear equations for q_i. All the coefficients are rational and indeed integers according to Eq. (V.22). Therefore, when we solve for the q_i, they will all be rational.

We can go further and show that $\langle\alpha,\beta\rangle$ is rational when α and β are roots. Using Eq. (IV.3), we have

$$\begin{aligned}\langle\beta,\beta\rangle &= (h_\beta, h_\beta)\\ &= \sum_{\alpha\in\Sigma}\alpha(h_\beta)\alpha(h_\beta)\\ &= \sum_{\alpha\in\Sigma}\langle\alpha,\beta\rangle^2 \ . \end{aligned} \tag{VI.6}$$

The root α is in some β-string of roots, so $2\langle\alpha,\beta\rangle = (m-p)_\alpha\langle\beta,\beta\rangle$ for some integral $(m-p)_\alpha$. Thus

$$\begin{aligned}\langle\beta,\beta\rangle &= \sum_{\alpha\in\Sigma}\tfrac{1}{4}\left[(m-p)_\alpha\right]^2\langle\beta,\beta\rangle^2,\\ &= \left(\sum_{\alpha\in\Sigma}\tfrac{1}{4}\left[(m-p)_\alpha\right]^2\right)^{-1} . \end{aligned} \tag{VI.7}$$

This shows that $\langle\beta,\beta\rangle$ is rational. Also, $\langle\alpha,\beta\rangle = (m-p)_\alpha\langle\beta,\beta\rangle/2$ is rational. We see, then, from Eq. (VI.7) that $\langle\ ,\ \rangle$ is positive definite on the space of rational linear combinations of roots. In particular, this means that $\langle\ ,\ \rangle$ is a scalar product.

References

This is standard material. See, for example, JACOBSON, pp. 112–119.

Exercise

1. Assuming that for each root α there is only one linearly independent root vector, show that if α, β, and $\alpha+\beta$ are roots, then $[e_\alpha, e_\beta] \neq 0$. *Hint:* consider the adjoint representation and then the $SU(2)$ generated by $e_\alpha, e_{-\alpha}$, and h_α.