V. A Little about Representations

There is still a great deal to uncover about the structure of simple Lie algebras, but it is worthwhile to make a slight detour to discuss something about representations. This will lead to some useful relations for the adjoint representation (c.f. Eqs. (II.4) and (II.5)) and thus for the structure of the Lie algebras themselves.

The study of representations of Lie algebras is based on the simple principles discussed in Chapter I. The reason for this is that the elements e_α, $e_{-\alpha}$, and h_α have commutation relations

$$\begin{aligned}
[h_\alpha, e_\alpha] &= \alpha(h_\alpha)e_\alpha = (h_\alpha, h_\alpha)e_\alpha = \langle\alpha,\alpha\rangle e_\alpha \ , \\
[h_\alpha, e_{-\alpha}] &= -\ \langle\alpha,\alpha\rangle e_\alpha \ , \\
[e_\alpha, e_{-\alpha}] &= (e_\alpha, e_{-\alpha})h_\alpha \ ,
\end{aligned} \tag{V.1}$$

which are just the same as those for t_+, t_-, and t_z, except for normalization.Thus for each pair of roots, α , and -α ,there is an SU(2) we can form. What makes the Lie algebras interesting is that the SU(2)'s are linked together by the commutation relations

$$[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}, \qquad \alpha+\beta \in \Sigma \ . \tag{V.2}$$

We recall that a representation of the Lie algebra is obtained when for each element of the algebra we have a linear transformation (i.e. a matrix) acting on a vector space (i.e. column vectors) in a way which preserves the commutation relations. If we indicate the representation of a, b, and c by A, B, and C, then

$$[a, b] = c \qquad \rightarrow \qquad [A, B] = C \ . \tag{V.3}$$

Let us continue to use this notation so that if $h_1, h_2, \ldots$ is a basis for the Cartan subalgebra H, we will indicate their representatives by $H_1, H_2, \ldots$.Similarly, the representatives of e_α will be E_α. The transformations H_i and E_α act on vectors ϕ^a in a space, V. Since the h's commute, so do the H's. We can choose a basis for the space V in which the H's are diagonal (The representation is in particular a representation for the $SU(2)$ formed by $H_\alpha, E_\alpha, E_{-\alpha}$. We know how to diagonalize H_α. But all the H_α's commute so we can diagonalize them simultaneously.):

$$H_i \phi^a = \lambda_i^a \phi^a \ . \tag{V.4}$$

The eigenvalues λ_i^a depend linearly on the H's.Thus if $h = \sum_i c_i h_i$ so that $H = \sum_i c_i H_i$, then

$$\begin{aligned} H\phi^a &= \left(\sum_i c_i \lambda_i^a\right) \phi^a \\ &\equiv M^a(h)\phi^a \ . \end{aligned} \tag{V.5}$$

We can regard the eigenvalue associated with this vector, ϕ^a, to be a linear function defined on H:

$$M^a\left(\sum_i c_i h_i\right) = \sum_i c_i \lambda_i^a \ . \tag{V.6}$$

The functions M^a are called **weights**. As linear functionals on H, they are members of the dual space, H^*, just as the roots,α, are. We shall see later that the weights can be expressed as real (in fact, rational) linear combinations of the roots. We can use the product $\langle , \rangle$ we defined on the root space also when we deal with the weights.

A simple example of Eq. (V.5) is given by the three dimensional representation of SU(3), Eqs. (II.1) and (II.2).

$$T_z = \begin{bmatrix} \frac{1}{2} & & \\ & -\frac{1}{2} & \\ & & 0 \end{bmatrix} , \qquad Y = \begin{bmatrix} \frac{1}{3} & & \\ & \frac{1}{3} & \\ & & -\frac{2}{3} \end{bmatrix} . \tag{V.7}$$

The weight vectors of the three dimensional representation are

$$\phi^a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \qquad \phi^b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} , \qquad \phi^c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \tag{V.8}$$

We consider the action of $H = aT_z + bY$ on the weight vectors to find the weights:

$$\begin{aligned} H\phi^a &= (\tfrac{1}{2}a + \tfrac{1}{3}b)\phi^a &&= M^a(at_z + by)\phi^a \ , \\ H\phi^b &= (-\tfrac{1}{2}a + \tfrac{1}{3}b)\phi^b &&= M^b(at_z + by)\phi^b \ , \\ H\phi^c &= (-\tfrac{2}{3}b)\phi^c &&= M^c(at_z + by)\phi^c \ . \end{aligned} \tag{V.9}$$

The weights can be expressed in terms of the roots of SU(3), Eq. (III.6). Only two of the roots are linearly independent so we need use only two of them, say $\alpha_1$and α_2.

Then

$$\begin{aligned} M^a &= +\tfrac{2}{3}\alpha_1 + \tfrac{1}{3}\alpha_2 \ , \\ M^b &= -\tfrac{1}{3}\alpha_1 + \tfrac{1}{3}\alpha_2 \ , \\ M^c &= -\tfrac{1}{3}\alpha_1 - \tfrac{2}{3}\alpha_2 \ . \end{aligned} \tag{V.10}$$

In SU(2), $T_+$and T_- act as raising and lowering operators. This concept may be generalized in the following way. Suppose that ϕ^a is a weight vector with weight M^a. Then $E_\alpha \phi^a$ is a weight vector with weight $M^a + \alpha$ unless $E_\alpha \phi^a = 0$:

$$\begin{aligned} H E_\alpha \phi^a &= (E_\alpha H + \alpha(h) E_\alpha)\, \phi^a \\ &= (M^a(h) + \alpha(h))\, E_\alpha \phi^a \ . \end{aligned} \tag{V.11}$$

Thus we can think of the E_α as raising operators and the $E_{-\alpha}$ as lowering operators.

If M is a weight, then it lies in a string of weights M^*, $M^* - \alpha$, ..., M, ..., $M^* - q\alpha$. Let us see how q is determined by M^*. Let ϕ_0 be a weight vector with weight M^*. Then, if it is non-zero, the vector

$$(E_{-\alpha})^j \, \phi_0 = \phi_j \tag{V.12}$$

is a weight vector with weight $M^* - j\alpha$. On the other hand, $E_\alpha \phi_k$ has weight $M^* - (k-1)\alpha$, and is proportional to ϕ_{k-1}. We can find q by using the relation

$$E_{-\alpha}\phi_q = 0 \ . \tag{V.13}$$

The calculation is simplified by choosing a convenient normalization for the generators $e_\alpha$and $e_{-\alpha}$: $(e_\alpha, e_{-\alpha}) = 1$. Thus

$$[e_\alpha, e_{-\alpha}] = h_\alpha \ . \tag{V.14}$$

In terms of the representation, then

$$[E_\alpha, E_{-\alpha}] = H_\alpha \ . \tag{V.15}$$

We shall also need the relation (see Eq. (III.5))

$$M(h_\alpha) = (h_M, h_\alpha) = \langle M, \alpha \rangle \ . \tag{V.16}$$

By analogy with our treatment of SU(2), we define

$$E_\alpha \phi_k = r_k \phi_{k-1} \tag{V.17}$$

and seek a recursion relation. We find

$$\begin{aligned}
E_\alpha \phi_k =& r_k \phi_{k-1} \\
=& E_\alpha E_{-\alpha} \phi_{k-1} \\
=& (E_{-\alpha} E_\alpha + H_\alpha) \phi_{k-1} \\
=& r_{k-1}\phi_{k-1} + [M^*(h_\alpha) - (k-1)\alpha(h_\alpha)]\phi_{k-1} \\
=& [r_{k-1} + \langle M^*, \alpha \rangle - (k-1)\langle \alpha, \alpha \rangle]\phi_{k-1} \ .
\end{aligned} \tag{V.18}$$

The solution to the recursion relation which satisfies $r_0 = 0$ is

$$r_k = k\langle M^*, \alpha \rangle - \tfrac{1}{2}k(k-1)\langle \alpha, \alpha \rangle \ . \tag{V.19}$$

Now from Eq. (V.13) we know that

$$E_\alpha E_{-\alpha} \phi_q = 0 = r_{q+1}\phi_q \tag{V.20}$$

so we have found q in terms of M^* and α :

$$q = \frac{2\langle M^*, \alpha \rangle}{\langle \alpha, \alpha \rangle} \ . \tag{V.21}$$

In practice, we often have a weight, M, which may or may not be the highest weight in the sequence $M+p\alpha, \ldots\ M, \ldots M-m\alpha$. We can obtain an extremely useful formula for $m-p$ by using Eq. (V.21):

$$
\begin{aligned}
m+p = & \frac{2\langle M+p\alpha,\alpha\rangle}{\langle\alpha,\alpha\rangle} \\
m-p = & \frac{2\langle M,\alpha\rangle}{\langle\alpha,\alpha\rangle}\ .
\end{aligned}
\tag{V.22}
$$

As an example, let us consider the three dimensional representation of SU(3). Now suppose we wish to find the string of weights spaced by α_1 containing the weight $M^a = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$. Using the table of scalar products in Eq. (III.12), we compute

$$
m-p = \frac{2\langle\frac{2}{3}\alpha_1+\frac{1}{3}\alpha_2,\alpha_1\rangle}{\langle\alpha_1,\alpha_1\rangle} = 1\ . \tag{V.23}
$$

In fact, $m=1$ and $p=0$.

So important is Eq. (V.22) that it is worthwhile to pause for a geometrical interpretation. Suppose the number of weights in the string is odd. Then there is a central weight, M_0, such that $p=m$ and $\langle M_0,\alpha\rangle = 0$. This suggests the existence of a symmetry, a reflection which acts about the mid-point. If the full string is $M^*, M^*-\alpha, \ldots M^*-q\alpha$, this symmetry would relate the weights $M = M^*-j\alpha$ and $M' = M^*-(q-j)\alpha = M-(q-2j)\alpha$. Using Eq. (V.22) with $p=j$ and $m=q-j$, we see that $q-2j = 2\langle M,\alpha\rangle/\langle\alpha,\alpha\rangle$. Thus the symmetry among the weights can be expressed by

$$
S_\alpha: \quad M \to M - \frac{2\langle M,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha\ . \tag{V.24}
$$

It is clear that this works similarly if the string is even. In either event, S_α maps weights into weights.

This symmetry is called a **Weyl reflection**. If we consider elements made up of any number of reflections, these elements form a group called the **Weyl group**. The Weyl group maps weights into weights. Associated with each weight is a **weight space** consisting of all weight vectors with a given weight. For an irreducible representation of SU(2), each weight space is one dimensional, but in general this is not so.

If we consider a string of weights, $M^*, \ldots M^* - q\alpha$, and the associated weight spaces, we can restrict our consideration to the subalgebra generated by e_α, $e_{-\alpha}$, and h_α. This SU(2) subalgebra is represented by E_α, $E_{-\alpha}$, and H_α. The representation of SU(2) on the weight spaces associated with the string is in general reducible. It contains at least one copy of the $2q+1$ dimensional representation of SU(2). In addition, there may be other representations of lesser dimension. Each of these representations will be arranged symmetrically with respect to the reflection S_α with the result that S_α will map a weight M into a weight M' whose weight space has the same dimension.

The symmetry of some SU(3) representations is apparent in Fig. V.1.

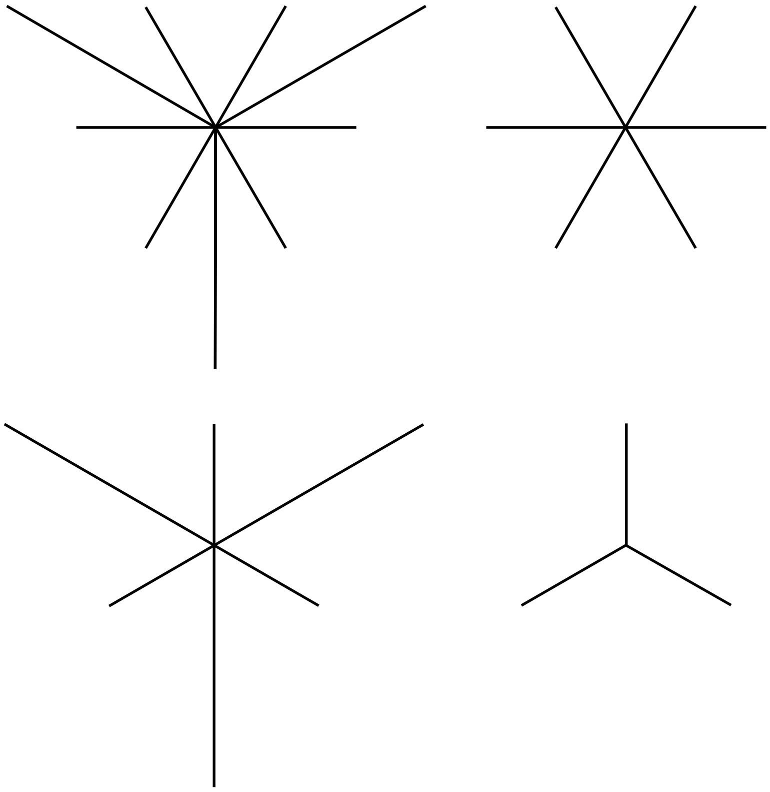

Fig. V.1

References

The material is standard. See, for example, JACOBSON, pp. 112–119.

Exercise

1. Find the elements of the Weyl group for SU(3) and their multiplication table.