IV. The Structure of Simple Lie Algebras

Our study of the Lie algebra of SU(3) revealed that the eight generators could be divided up in an illuminating fashion. Two generators, t_z and y, commuted with each other. They formed a basis for the two dimensional Cartan subalgebra. The remaining generators, u_+, u_-, v_+, v_-, t_+, and t_- were all eigenvectors of ad t_z and ad y, that is, $[t_z, u_+]$ was proportional to u_+, *etc.* More generally, each of the six was an eigenvector of ad h for every $h \in H$. The corresponding eigenvalue depended linearly on h. These linear functions on H were elements of H^*, the dual space of H. The functions which gave the eigenvalues of ad h were called roots and the real linear combinations of these roots formed a real vector space, H_0^*.

The SU(3) results generalize in the following way. Every semi-simple Lie algebra is a sum of simple ideals, each of which can be treated as a separate simple Lie algebra. The generators of the simple Lie algebra may be chosen so that one subset of them generates a commutative Cartan subalgebra, H. The remaining generators are eigenvectors of ad h for every $h \in H$. Associated with each of these latter generators is a linear function which gives the eigenvalue. We write

$$(\mathrm{ad}\ h)e_\alpha = \alpha(h)e_\alpha\ . \tag{IV.1}$$

This is the generalization of Eq. (II.9) where we have indicated generators like u_+, u_-, *etc.*, generically by e_α.

The roots of SU(3) exemplify a number of characteristics of semi-simple Lie algebras in general. First, if α is a root, so is $-\alpha$. This is made explicit in Eq. (II.9), where we see that the root corresponding to t_- is the negative of that corresponding to t_+, and so on. Second, for each root, there is only one linearly independent generator with that root. Third, if α is a root, 2α is not a root.

How is the Cartan subalgebra determined in general? It turns out that the following procedure is required. An element $h \in L$ is said to be **regular** if $\mathrm{ad}\, h$ has as few zero eigenvalues as possible, that is, the multiplicity of the zero eigenvalue is minimal. In the SU(3) example, from Eq. (II.8) we see that $\mathrm{ad}\, t_z$ has a two dimensional space with eigenvalue zero, while $\mathrm{ad}\, y$ has a four dimensional space of this sort. The element t_z is regular while y is not. A Cartan subalgebra is obtained by finding a maximal commutative subalgebra containing a regular element. The subalgebra generated by t_z and y is commutative and it is maximal since there is no other element we can add to it which would not destroy the commutativity.

If we take as our basis for the algebra the **root vectors**, $e_{\alpha_1}, e_{\alpha_2} \ldots$ plus some basis for the Cartan subalgebra, say $h_1, h_2 \ldots$, then we can write a matrix representation for $\mathrm{ad}\, h$:

$$\mathrm{ad}\, h = \begin{bmatrix} 0 & & & & & & & & \\ & 0 & & & & & & & \\ & & \cdot & & & & & & \\ & & & 0 & & & & & \\ & & & & \alpha_1(h) & & & & \\ & & & & & \alpha_2(h) & & & \\ & & & & & & \cdot & & \\ & & & & & & & \cdot & \\ & & & & & & & & \alpha_n(h) \end{bmatrix} . \tag{IV. 2}$$

From this we can see that the Killing form, when acting on the Cartan subalgebra can be computed by

$$(h_1, h_2) = \sum_{\alpha \in \Sigma} \alpha(h_1)\alpha(h_2) \ , \tag{IV.3}$$

where Σ is the set of all the roots.

We know the commutation relations between the root vectors and the members of the Cartan subalgebra, namely Eq. (IV.1). What are the commutation relations between the root vectors? We have not yet specified the normalization of the e_α's , so we can only answer this question up to an overall constant.

Let us use the Jacobi identity on $[e_\alpha, e_\beta]$:

$$\begin{aligned} [h, [e_\alpha, e_\beta]] &= -\,[e_\alpha, [e_\beta, h]] - [e_\beta, [h, e_\alpha]] \\ &= \beta(h)\,[e_\alpha, e_\beta] + \alpha(h)\,[e_\alpha, e_\beta] \\ &= (\alpha(h) + \beta(h))\,[e_\alpha, e_\beta] \ . \end{aligned} \tag{IV.4}$$

This means that either $[e_\alpha, e_\beta]$ is zero, or it is a root vector with root $\alpha + \beta$, or $\alpha + \beta = 0$, in which case $[e_\alpha, e_\beta]$ commutes with every $h$and is thus an element of the Cartan subalgebra.

It is easy to show that $(e_\alpha, e_\beta) = 0$ unless $\alpha+\beta=0$. This is simply a generalization of the considerations surrounding Eq. (III.3). We examine $[e_\alpha,[e_\beta,x]]$ where x is some basis element of L, either a root vector or an element of the Cartan subalgebra. If $x \in H$, the double commutator is either zero or proportional to a root vector $e_{\alpha+\beta}$. In either case, there is no contribution to the trace. If x is a root vector, say $x = e_\gamma$, the double commutator is either zero or of the form $e_{\alpha+\beta+\gamma}$, and thus does not contribute to the trace unless $\alpha+\beta=0$.

We have seen that $[e_\alpha, e_{-\alpha}]$ must be an element of the Cartan subalgebra. We can make this more explicit with a little calculation. First we prove an important property, **invariance**, of the Killing form:

$$(a,[b,c]) = ([a,b],c) \ , \tag{IV.5}$$

where a, b, and c are elements of the Lie algebra. The proof is straightforward:

$$\begin{aligned}(a,[b,c]) =& \mathrm{Tr}\ \mathrm{ad}\, a \mathrm{ad}\,[b,c] \\ =& \mathrm{Tr}\ \mathrm{ad}\, a\,[\mathrm{ad}\, b, \mathrm{ad}\, c] \\ =& \mathrm{Tr}\ [\mathrm{ad}\, a, \mathrm{ad}\, b]\, \mathrm{ad}\, c \\ =& \mathrm{Tr}\ \mathrm{ad}\,[a,b] \mathrm{ad}\, c \\ =& ([a,b],c) \ .\end{aligned} \tag{IV.6}$$

Now we use this identity to evaluate $([e_\alpha, e_{-\alpha}], h)$ where h is some element of the Cartan subalgebra.

$$\begin{aligned}([e_\alpha,e_{-\alpha}],h) =& (e_\alpha,[e_{-\alpha},h]) \\ =& \alpha(h)(e_\alpha,e_{-\alpha}) \ .\end{aligned} \tag{IV.7}$$

Both sides are linear functions of h. Referring to Eq. (III.5), we see that $[e_\alpha, e_{-\alpha}]$ is proportional to h_α, where h_α has the property

$$(h_\alpha, k) = \alpha(k), \qquad h_\alpha, k \in H \ . \tag{IV.8}$$

More precisely, we have

$$[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha}) h_\alpha \ . \tag{IV.9}$$

This is, of course, in accord with our results for SU(3). As an example, let $e_\alpha = u_+, e_{-\alpha} = u_-$. From Table II.1, we see that $[u_+, u_-] = 3y/2 - t_z$. From Eq. (III.4), $(u_+, u_-) = 6$, while from Eqs. (III.6), (III.10), and (II.9), we find that $h_{u_+} = y/4 - t_z/6$. Thus indeed, $[u_+, u_-] = (u_+, u_-)h_{u_+}$.

The Killing form is the only invariant bilinear form on a simple Lie algebra, up to trivial modification by multiplication by a constant. To demonstrate this, suppose that $((\ ,\ ))$ is another such form. Then

$$\begin{aligned}((h_\beta, [e_\alpha, e_{-\alpha}])) &= ((h_\beta, (e_\alpha, e_{-\alpha})h_\alpha)) \\ &= (e_\alpha, e_{-\alpha})((h_\beta, h_\alpha)) \\ &= (([h_\beta, e_\alpha], e_{-\alpha})) \\ &= (h_\beta, h_\alpha)((e_\alpha, e_{-\alpha})) \ .\end{aligned} \tag{IV.10}$$

Thus $((h_\beta, h_\alpha))/(h_\beta, h_\alpha) = ((e_\alpha, e_{-\alpha}))/(e_\alpha, e_{-\alpha})$ and this ratio is independent of α as well. Thus we can write

$$\frac{((h_\beta, h_\alpha))}{(h_\beta, h_\alpha)} = k = \frac{((e_\alpha, e_{-\alpha}))}{(e_\alpha, e_{-\alpha})} \ . \tag{IV.11}$$

In a simple algebra, we can start with a single root, α, and proceed to another root, β such that $(h_\beta, h_\alpha) \neq 0$ and continue until we have exhausted the full set of roots, so a single value of k holds for the whole algebra. Separate simple factors of a semi-simple algebra may have different values of k however.

We can summarize what we have thus far learned about the structure of semi-simple Lie algebras by writing the commutation relations. We indicate the set of roots by Σ and the Cartan subalgebra by H:

$$
\begin{aligned}
[h_1, h_2] &= 0 \ , && h_1, h_2 \in H \\
[h, e_\alpha] &= \alpha(h) e_\alpha \ , && \alpha \in \Sigma \\
[e_\alpha, e_\beta] &= N_{\alpha\beta} e_{\alpha+\beta} \ , && \alpha + \beta \in \Sigma \\
&= (e_\alpha, e_{-\alpha}) h_\alpha \ , && \alpha + \beta = 0 \\
&= 0 \ , \quad \alpha + \beta \neq 0 \ , && \alpha + \beta \notin \Sigma \ .
\end{aligned}
\tag{IV.12}
$$

Here $N_{\alpha\beta}$ is some number depending on the roots α and β which is not yet determined since we have not specified the normalization of e_α.

References

A rigorous treatment of these matters is given by JACOBSON.

Exercise

1. Show that $at_z + by$ is almost always regular by finding the conditions on a and b such that it is not regular.

2. Show that invariant bilinear symmetric forms are really invariants of the Lie group associated with the Lie algebra.