III. The Killing Form

A fundamental step in the analysis of Lie algebras is to establish a geometrical picture of the algebra. We shall eventually see how this geometry can be developed in terms of the roots of the algebra. Before turning to the roots, we must first define something resembling a scalar product for elements of the Lie algebra itself. We shall state our definitions for an arbitrary Lie algebra and illustrate them with SU(3).

Let L be a Lie algebra and let $a, b \in L$. The **Killing form** is defined by

$$(a, b) = \mathrm{Tr}\ \mathrm{ad}\ a\ \mathrm{ad}\ b\ . \tag{III.1}$$

Remember that $\mathrm{ad}\, a$ is an operator which acts on elements of L and maps them into new elements of L. Thus the indicated trace can be evaluated by first taking a basis for L, say $x_1, x_2, \ldots$. Then we calculate for each x_j, the quantity $[a, [b, x_j]]$ and express the result in terms of the x_i's. The coefficient of x_j is the contribution to the trace. It is easy to show that the trace is independent of the choice of basis. As an example of the Killing form, consider SU(3). Using Table II.1 we see that

$$(t_z, t_z) = 3. \tag{III.2}$$

This can be calculated simply using the matrix representation of the operator $\mathrm{ad}\, t_z$, Eq. (II.7), or more tediously

$$\begin{aligned}
[t_z,[t_z,t_z]] &= 0\ , & [t_z,[t_z,u_+]] &= \tfrac{1}{4}u_+ \\
[t_z,[t_z,y]] &= 0\ , & [t_z,[t_z,u_-]] &= \tfrac{1}{4}u_- \\
[t_z,[t_z,t_+]] &= t_+\ , & [t_z,[t_z,v_+]] &= \tfrac{1}{4}v_+ \\
[t_z,[t_z,t_-]] &= t_-\ , & [t_z,[t_z,v_-]] &= \tfrac{1}{4}v_-\ .
\end{aligned} \tag{III.3}$$

It is easy to see that a term like (t_z, t_+) must vanish. From Table II.1 we see that $\mathrm{ad}\, t_z\ \mathrm{ad}\, t_+\ (t_z) = -t_+$ and hence gives no contribution to (t_z, t_+), *etc.* If we take the Killing form between two of our basis elements, only a few are non-zero:

$$\begin{aligned}
(t_z,t_z) &= 3\ , & (y,y) &= 4\ , & (t_+,t_-) &= 6\ , \\
(v_+,v_-) &= 6\ , & (u_+,u_-) &= 6\ .
\end{aligned} \tag{III.4}$$

The Killing form is not a scalar product. In particular it is not positive definite. For example, since we are considering complex combinations of the SU(3) generators, we can calculate $(iy, iy) = -4$.

There is a scalar product associated with the Lie algebra, but it is not defined on the Lie algebra itself, but rather on the space containing the roots. We recall that the roots live in a space called H^*, the dual space to the Cartan subalgebra, H. Often we can restrict ourselves to the space H_0^*, the real, linear combinations of the roots.

The Killing form enables us to make a connection between the Cartan subalgebra, H, and its dual H^*.[1] If $\rho \in H^*$, there exists a unique element $h_\rho \in H$ such that for every $k \in H$,

$$\rho(k) = (h_\rho, k) \ . \tag{III.5}$$

This unique connection between H and H^* occurs not for all Lie algebras but only for the class of semi-simple Lie algebras which are the ones we shall be mostly concerned with. For semi-simple Lie algebras the Killing form, as we shall see, is non-degenerate. This means, in particular, that if $(a, b) = 0$ for every $b \in H$, then $a = 0$. More prosaically, non-degeneracy means that if $x_1, x_2 \ldots$ is a basis for H, then the matrix (x_i, x_j) can be inverted. Thus the values of (a, x_j) completely determine a.

This one-to-one relationship between H and H^* can be illustrated with SU(3). Referring to Eqs. (II.9) and (II.10), we designate three non-zero roots by

$$\begin{aligned}
\alpha_1(at_z + by) &= a \\
\alpha_2(at_z + by) &= -\tfrac{1}{2}a + b \\
\alpha_3(at_z + by) &= \tfrac{1}{2}a + b \ .
\end{aligned} \tag{III.6}$$

The other non-zero roots are the negatives of these. Now we determine the elements in H corresponding to α_1, α_2, and α_3. Each of these h's is to lie in H and is thus of the form

$$h_{\alpha_i} = c_i t_z + d_i y \ . \tag{III.7}$$

Using the previously computed values of the Killing form, Eq. (III.4), we see that

$$\begin{aligned}
(h_{\alpha_i}, t_z) &= 3c_i \\
(h_{\alpha_i}, y) &= 4d_i \ .
\end{aligned} \tag{III.8}$$

To determine the coefficients c_i and d_i, we combine the definition of h_α, Eq. (III.5), with the expressions for the roots, Eq. (III.6):

$$\begin{aligned}
\alpha_1(t_z) =& 1 = (h_{\alpha_1}, t_z) = & 3c_1 \ , \\
\alpha_1(y) =& 0 = (h_{\alpha_1}, y) = & 4d_1 \ , \\
\alpha_2(t_z) =& -\tfrac{1}{2} = (h_{\alpha_2}, t_z) = & 3c_2 \ , \\
\alpha_2(y) =& 1 = (h_{\alpha_2}, y) = & 4d_2 \ , \\
\alpha_3(t_z) =& \tfrac{1}{2} = (h_{\alpha_3}, t_z) = & 3c_3 \ , \\
\alpha_3(y) =& 1 = (h_{\alpha_3}, y) = & 4d_3 \ .
\end{aligned} \tag{III.9}$$

Thus we find the elements of H which correspond to the various roots:

$$h_{\alpha_1} = \tfrac{1}{3}t_z; \quad h_{\alpha_2} = -\tfrac{1}{6}t_z + \tfrac{1}{4}y; \quad h_{\alpha_3} = \tfrac{1}{6}t_z + \tfrac{1}{4}y \ . \tag{III.10}$$

Of course, this correspondence is linear. It would have sufficed to determine h_{α_1} and h_{α_2} and then noted that since $\alpha_3 = \alpha_1 + \alpha_2, h_{\alpha_3} = h_{\alpha_1} + h_{\alpha_2}$. Indeed, using Eq. (III.10) we can find the element of H which corresponds to any element of H^* since such elements can be expressed in terms of, say, α_1 and α_2.

We are now in a position to display the previously advertised scalar product. Let α and β be real linear combinations of the roots, that is, $\alpha, \beta \in H_0^*$ and let h_α and h_β be the elements in H associated with them according to Eq. (III.5). Then we define a product on H_0^* by

$$\langle \alpha, \beta \rangle \equiv (h_\alpha, h_\beta) \ . \tag{III.11}$$

For the particular case of SU(3), using Eq. (III.4), we have

$$
\begin{aligned}
\langle\alpha_1,\alpha_1\rangle &= (h_{\alpha_1},h_{\alpha_1}) = (\tfrac{1}{3}t_z,\tfrac{1}{3}t_z) = \tfrac{1}{3}\\
\langle\alpha_2,\alpha_2\rangle &= (h_{\alpha_2},h_{\alpha_2}) = \tfrac{1}{3}\\
\langle\alpha_3,\alpha_3\rangle &= (h_{\alpha_3},h_{\alpha_3}) = \tfrac{1}{3}\\
\langle\alpha_1,\alpha_2\rangle &= (h_{\alpha_1},h_{\alpha_2}) = -\tfrac{1}{6}\\
\langle\alpha_1,\alpha_3\rangle &= (h_{\alpha_1},h_{\alpha_3}) = \tfrac{1}{6}\\
\langle\alpha_2,\alpha_3\rangle &= (h_{\alpha_2},h_{\alpha_3}) = \tfrac{1}{6}\ .
\end{aligned}
\tag{III.12}
$$

From these specific values, we can see that for SU(3), $\langle,\rangle$ provides a scalar product on the root space, H_0^*. Indeed, we can interpret Eq. (III.12) geometrically by representing the roots by vectors of length $1/\sqrt{3}$. The angles between the vectors are such that $\cos\theta = \pm\frac{1}{2}$ as shown in Fig. III.1.

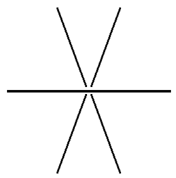

Fig. III.1

It is important to note that $\langle,\rangle$ is quite different from $(\ ,\ )$. There is no "natural" basis for the Cartan subalgebra so some of the symmetry is not apparent. Thus we found $(t_z,t_z)=3$, but $(y,y)=4$. Moreover, we might as well have chosen iy instead of y and found $(iy,iy)=-4$. There are naturally distinguished elements of H^*, namely the roots. As a result, the product on H^* displays more clearly the symmetry of the Lie algebra.

So far we have limited our discussion to the Lie algebras of SU(2) and SU(3) (or, more precisely, their complex extensions). Let us now generalize this and explain the terms "simple" and "semi-simple".

Suppose L is a Lie algebra. A **subalgebra**, M, is simply a subspace of L which is closed under the Lie product. For example, t_z, t_+, and t_- generate a subalgebra of SU(3) which is indeed isomorphic (equivalent) to SU(2). Symbolically, if M is a subalgebra and $x, y \in M$, then $[x, y] \in M$. An **ideal** is a special kind of subalgebra. If J is an ideal, and $x \in J$ and y is any element of L, then $[x, y] \in J$. If J were only a subalgebra instead of an ideal, we would have to restrict y to be in J rather than just in L.

As an example, consider the group U(3), the set of all three-by-three unitary matrices. We can think of its Lie algebra as being the set of all Hermitian three-by-three matrices. This is the same as for SU(3) except that the matrices need not be traceless. Thus we might take for a basis, the eight matrices displayed in Eq. (III.1), plus the three-by-three identity matrix.

Now consider the one-dimensional space spanned by the identity matrix, that is, the space given by multiples of the identity. This space, J, is an ideal because if $x \in J$ and y is any element of the Lie algebra, $[x, y] = 0 \in J$. In fact, if we consider the space of all traceless matrices, J', we see that it too is an ideal. This follows since the trace of a commutator is necessarily traceless. Thus every element in U(3) can be written as a sum of one element from J and one element from J'. The full algebra is the sum of the two ideals.

A Lie algebra which has no ideals (except the trivial ones comprising the full algebra itself or the ideal consisting solely of 0) is called **simple**. A subalgebra in which all members commute is called **abelian**. An algebra with no abelian ideals is called **semi-simple**. Thus the Lie algebra of SU(3) is simple, while that of U(3) is neither simple nor semi-simple.

A semi-simple Lie algebra is the sum of simple ideals. Consider, for example, the five-by-five traceless hermitian matrices which are zero except for two diagonal blocks, one three-by-three and one two-by-two. Suppose we consider only matrices where each of these two blocks is separately traceless. The resulting set is a Lie algebra which can be considered the sum of two ideals, one of which is isomorphic to SU(2) and the other of which is isomorphic to SU(3). If we require only that the sum of the traces of the two diagonal blocks vanish, the resulting algebra is larger, including matrices proportional to one whose diagonal elements are $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{2}, -\frac{1}{2}$. This element and its multiples form an abelian ideal so this larger algebra $(SU(3) \times SU(2) \times U(1))$ is not semi-simple.

Because semi-simple Lie algebras are simply sums of simple ones, most of their properties can be obtained by first considering simple Lie algebras.

There is an intimate relationship between the Killing form and semi-simplicity: the Killing form is non-degenerate if and only if the Lie algebra is semi-simple. It is not hard to prove half of this fundamental theorem [2](which is due to Cartan): if the Killing form is non-degenerate, then L is semi-simple. Suppose L is not semi-simple and let B be an abelian ideal. Let $b_1, b_2, \ldots$ be a basis for B. We can extend this to a basis for the full algebra L by adding $y_1, y_2, \ldots$ where $y_i \notin B$. Now let us calculate (b_1, a) where $a \in L$. First consider $[b_1, [a, b_j]]$. The inner commutator lies in B since B is an ideal. But then the second commutator vanishes since B is abelian. Next consider $[b_1, [a, y_j]]$ The final result must lie in B since $b_1 \in B$ so its expansion has no components along the y_k's and along y_j in particular. Thus there is no contribution to the trace. The trace vanishes and the Killing form is degenerate.

Footnotes

1. We follow here JACOBSON, p. 110.

2. JACOBSON, pp. 69–70.

Exercise

1. Define a bilinear form on SU(3) using the three-dimensional representation as follows. Let x and y be a linear combination of the matrices in Eq. (II.1) and define $((x,y)) = \mathrm{Tr}\, xy$. Compare this with the Killing form, i.e. $(x,y) = \mathrm{Tr}\,\mathrm{ad}\,x\,\mathrm{ad}\,y$. It suffices to consider x and y running over some convenient basis.