II. SU(3)

The preceding review of SU(2) will be central to the understanding of Lie algebras in general. As an illustrative example, however, SU(2) is not really adequate. The Lie algebra of SU(3) is familiar to particle physicists and exhibits most of the features of the larger Lie algebras that we will encounter later.

The group SU(3) consists of the unitary, three-by-three matrices with determinant equal to unity. The elements of the group are obtained by exponentiating iM, where M is a traceless, three-by-three, hermitian pmatrix. There are eight linearly independent matrices with these properties.

One choice for these is the λ matrices of Gell-Mann:

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{II.1}$$

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

The first three are just the Pauli matrices with an extra row and column added. The next four also have a similarity to σ_x and σ_y. To exploit this similarity we define

$$\begin{aligned} T_x &= \tfrac{1}{2}\lambda_1 , \qquad T_y = \tfrac{1}{2}\lambda_2 , \qquad T_z = \tfrac{1}{2}\lambda_3 , \\ V_x &= \tfrac{1}{2}\lambda_4 , \qquad V_y = \tfrac{1}{2}\lambda_5 , \\ U_x &= \tfrac{1}{2}\lambda_6 , \qquad U_y = \tfrac{1}{2}\lambda_7 , \qquad Y = \frac{1}{\sqrt{3}}\lambda_8 . \end{aligned} \tag{II.2}$$

There is no U_z or V_z because there are only two linearly independent diagonal generators. By historical tradition, they are chosen to be T_z and Y. Just as with SU(2), it is convenient to work with the complex combinations

$$T_\pm = T_x \pm iT_y, \quad V_\pm = V_x \pm iV_y, \quad U_\pm = U_x \pm iU_y . \tag{II.3}$$

It is straightforward to compute all the commutation relations between the eight generators. See Table II.1. We can consider these commutation relations to define the abstract Lie algebra of SU(3). That is, a representation of SU(3) is a correspondence $t_z \to T_z$, $t_+ \to T_+$, $t_- \to T_-$, $u_+ \to U_+$, *etc.* which preserves the commutation relations given in Table II.1. The three-by-three matrices given above form one representation, but as is well-known, there are six dimensional, eight dimensional, ten dimensional representations, *etc.*

Table II.1

The SU(3) commutation relations. The label on the row gives the first entry in the commutator and the column label gives the second.

	t_+	t_-	t_z	u_+	u_-	v_+	v_-	y
t_+	0	$2t_z$	$-t_+$	v_+	0	0	$-u_-$	0
t_-	$-2t_z$	0	t_-	0	$-v_-$	u_+	0	0
t_z	t_+	$-t_-$	0	$-\frac{1}{2}u_+$	$\frac{1}{2}u_-$	$\frac{1}{2}v_+$	$-\frac{1}{2}v_-$	0
u_+	$-v_+$	0	$\frac{1}{2}u_+$	0	$\frac{3}{2}y - t_z$	0	t_-	$-u_+$
u_-	0	v_-	$-\frac{1}{2}u_-$	$-\frac{3}{2}y + t_z$	0	$-t_-$	0	u_-
v_+	0	$-u_+$	$-\frac{1}{2}v_+$	0	t_-	0	$\frac{3}{2}y + t_z$	$-v_+$
v_-	u_-	0	$\frac{1}{2}v_-$	$-t_-$	0	$-\frac{3}{2}y - t_z$	0	v_-
y	0	0	0	u_+	$-u_-$	v_+	$-v_-$	0

The eight dimensional representation is especially interesting. It can be obtained in the following fashion. We seek a mapping of the generators $t_+,\ t_-,\ t_z,\ u_+$, *etc.* into linear operators which act on some vector space. Let that vector space be the Lie algebra, L, itself, that is, the space of all linear combinations of $t's,\ u's$, *etc.* With t_z we associate the following linear transformation. Let $x \in L$ and take

$$x \to [t_z, x] \ . \tag{II.4}$$

We call this linear transformation $\mathrm{ad}\, t_z$. More generally, if $x,\ y \in L, \mathrm{ad}\, y(x) = [y, x]$.

Now the mapping $y \to \mathrm{ad}\, y$ is just what we want. It associates with each element y in the Lie algebra, a linear tranformation, namely $\mathrm{ad}\, y$. To see that this is a representation, we must show it preserves the commutation relations, that is, if $[x, y] = z$ it must follow that $[\mathrm{ad}\, x, \mathrm{ad}\, y] = \mathrm{ad}\, z$. (It is worth noting here that the brackets in the first relation stand for some abstract operation, while those in the second indicate ordinary commutation.) This is easy to show:

$$\begin{aligned}
[\mathrm{ad}\, x, \mathrm{ad}\, y]\, w &= [x, [y, w]] - [y, [x, w]] \\
&= [x, [y, w]] + [y, [w, x]] \\
&= - [w, [x, y]] \\
&= [[x, y]\, , w] = [z, w] \\
&= \mathrm{ad}\, z(w) \ .
\end{aligned} \tag{II.5}$$

In the third line the Jacobi identity was used.

This representation is eight dimensional since L is eight dimensional. The operators $\mathrm{ad}\, x$ can be written as eight-by-eight matrices if we select a particular basis. For example, if the basis for L is $t_+,\ t_-,\ t_z,\ u_+,\ u_-,\ v_+,\ v_-$, and y (in that order), the pmatrix for $\mathrm{ad}\, t_+$ is found to be (using Table II.1)

$$\operatorname{ad} t_+ = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{II.6}$$

while that for $\operatorname{ad} t_z$ is

$$\operatorname{ad} t_z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{II.7}$$

Both $\operatorname{ad} t_z$ and $\operatorname{ad} y$ are diagonal. Thus if $x = at_z + b\ y$, then $\operatorname{ad} x$ is diagonal. Explicitly,we find

$$\operatorname{ad} x = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}a+b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}a-b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}a+b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}a-b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{II.8}$$

In other words, $t_+,\ t_-,\ t_z,\ u_+,\ u_-,\ v_+,\ v_-$, and y are all eigenvectors of $\operatorname{ad} x$ with eigenvalues $a, -a, 0, -\frac{1}{2}a+b, \frac{1}{2}a-b, \frac{1}{2}a+b, -\frac{1}{2}a-b$, and 0 respectively.

The procedure we have followed is central to the analysis of larger Lie algebras. We have found a two dimensional subalgebra (all linear combinations of t_z and y) which is **abelian** (that is, if x and y are in the subalgebra, $[x, y] = 0$). We have chosen a basis for the rest of the Lie algebra so that each element of the basis is an eigenvector of $\operatorname{ad} x$ if x is in the subalgebra (called the **Cartan subalgebra**). It is for this reason that we chose to work with t_+ and t_- rather than t_x and t_y, *etc.*

Once we have selected the Cartan subalgebra, H, the determination of the eigenvectors of $\operatorname{ad} x$ for $x \in H$ does not depend on a specific choice of basis for H. That is, we could choose any two linearly independent combinations of t_z and y as the basis for H. Of course, the eigenvectors are not uniquely determined, but are determined only up to a multiplicative constant: if u_+ is an eigenvector of $\operatorname{ad} x$, then so is cu_+, where c is a number. The eigenvalues, however, are completely determined, since, for example, u_+ and cu_+ have the same eigenvalue.

These eigenvalues depend, of course, on what x is. Specifically, we have

$$\begin{aligned}
\operatorname{ad}(at_z + by)t_+ &= \quad at_+ \\
\operatorname{ad}(at_z + by)t_- &= \quad -at_- \\
\operatorname{ad}(at_z + by)t_z &= \quad 0t_z \\
\operatorname{ad}(at_z + by)u_+ &= \quad (-\tfrac{1}{2}a + b)u_+ \\
\operatorname{ad}(at_z + by)u_- &= \quad (\tfrac{1}{2}a - b)u_- \\
\operatorname{ad}(at_z + by)v_+ &= \quad (\tfrac{1}{2}a + b)v_+ \\
\operatorname{ad}(at_z + by)v_- &= \quad (-\tfrac{1}{2}a - b)v_- \\
\operatorname{ad}(at_z + by)y &= \quad 0y \ .
\end{aligned} \tag{II.9}$$

The eigenvalues depend on x in a special way: they are linear functions of x. We may write

$$\alpha_{u_+}(at_z + by) = -\tfrac{1}{2}a + b \ , \tag{II.10}$$

etc. The functions α_{u_+} are linear functions which act on elements of the Cartan subalgebra, H, and have values in the complex numbers. The mathematical term for a linear function which takes a vector space, V (here V is H, the Cartan subalgebra) into the complex numbers is a **functional**. The linear functionals on V comprise a vector space called the **dual space** , denoted V^*. Thus we say that the functionals α lie in H^*. These functionals, α, are called **roots** and the corresponding generators like u_+ are called **root vectors**.

The concept of a dual space may seem excessively mathematical, but it is really familiar to physicists in a variety of ways. If we consider a geometrical vector space, say the three-dimensional vectors, there is a well-defined scalar (dot) product. Thus if $\mathbf{a}$ and $\mathbf{b}$ are vectors, $\mathbf{a} \cdot \mathbf{b}$ is a number. We can think of $\mathbf{a} \cdot$ as an element in the dual space. It acts on vectors to give numbers. Moreover, it is linear: $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$. A less trivial example is the bra-ket notation of Dirac: $|\ \rangle$ represents vectors in the space V, $\langle|$ represents vectors in the dual space, V^*.

References

SU(3) is familiar to particle physicists and is presented in many texts. Particularly notable presentations are found in GASIOROWICZ and in CARRUTHERS.

Exercises

1. Show that $\lambda_1, \lambda_2,$ and λ_3 close among themselves under the commutation relations, that is that they generate an SU(2) subalgebra. Show the same is true for $\lambda_2, \lambda_5,$ and λ_7.

2. Show that

$$\sum_i \lambda^i_{ab}\lambda^i_{cd} = -\frac{1}{3}\sum_i \lambda^i_{ad}\lambda^i_{cb} + \frac{16}{9}\delta_{ad}\delta_{cb} \; .$$