XVI. Branching Rules

Having determined, with much help from E. B. Dynkin, the maximal semi-simple subalgebras of the simple Lie algebras, we want to pursue this further to learn how an irreducible representation of an algebra becomes a representation of the subalgebra. To do this we shall have to be more precise about the embedding of the subalgebra in the algebra. Indeed, as we have already seen, the three dimensional representation of $SU(3)$ may become either a reducible representation or an irreducible representation of $SU(2)$ depending on the embedding.

We start with a subalgebra G' embedded in the algebra G by a mapping $f : G' \rightarrow G$, where f is a homomorphism, that is, it preserves the commutation relations:

$$f([x', y']) = [f(x'), f(y')] , \qquad x', y' \in G' . \tag{XVI.1}$$

Moreover, we can arrange it so that the Cartan subalgebra $H' \subset G'$ is mapped by f into the Cartan subalgebra $H \subset G$. Note that if ϕ is a representation of G, then $\phi \circ f$ is a representation of G'.

If we are to make progress, we must deal not only with the algebras but with the root spaces $H_0^*{}'$ and H_0^* as well. Given the mapping f, we define $f^* : H_0^* \rightarrow H_0^*{}'$ by

$$f^* \circ \rho = \rho \circ f \tag{XVI.2}$$

where $\rho \in H_0^*$. That is to say, if $h' \in H'$

$$(f^* \circ \rho)(h') = \rho(f(h')) \ . \qquad \text{(XVI.3)}$$

Instead of thinking of G' as external to G, it is easier to imagine it already within G. Then $f : H' \to H$ simply maps H' onto itself as the identity. We recall that there is a one-to-one correspondence between the elements of the root space, $\rho \in H_0^*$ and the elements h_ρ of the Cartan subalgebra. This one-to-one mapping connects to H' a space which we regard as $H_0^*{}'$. Now let us decompose H_0^* as the sum of $H_0^*{}'$ and a space H^* orthogonal to it. That is, if $\tau \in H_0^*{}'$ and $\kappa \in H^*$, then $\langle\tau, \kappa\rangle = 0$. Then the elements of H^* are functionals which when applied to H' give zero. This is so because if $\kappa \in H^*$ corresponds to $h_\kappa \in H$ and $\tau \in H_0^*{}'$ corresponds to $h_\tau \in H'$, then $\kappa(h_\tau) = (h_\kappa, h_\tau) = \langle\kappa, \tau\rangle = 0$. Now the action of f^* is simply to project elements of H_0^* onto $H_0^*{}'$. This follows because if $\rho = \rho_1 + \rho_2, \rho_1 \in H_0^*{}', \rho_2 \in H^*$, then for $h' \in H'$, $f^* \circ \rho(h') = \rho(f(h')) = (\rho_1 + \rho_2)(h') = \rho_1(h')$. Thus $f^* \circ \rho = \rho_1$.

Let M be a weight of a representation ϕ of G:

$$\phi(h)\xi_M = M(h)\xi_M \ . \qquad \text{(XVI.4)}$$

Then if $h \in H' \subset H$,

$$\begin{aligned} \phi(f(h))\xi_M =& M(f(h))\xi_M \\ =& f^* \circ M(h)\xi_M \ . \end{aligned} \qquad \text{(XVI.5)}$$

It follows that if M is a weight of ϕ, then $f^* \circ M$ is a weight of the representation $\phi \circ f$ of G'. More graphically, the weights of the representation of the subalgebra are obtained by projecting the weights of the algebra from H_0^* onto $H_0^{*\,\prime}$.

A simple but important consequence of this conclusion is that if the rank of G' is the same as the rank of G, then the subalgebra is regular. This is so because if the ranks are the same, then $H_0^{*\,\prime}$ coincides with H_0^* so the projection is simply the identity. Thus if we start with the adjoint representation of G, it will be mapped (by the identity) into a reducible representation of G' which contains the adjoint of G'. But the weights of the adjoint of G' must then have been among the weights of the adjoint of G.

Let us pause to consider an example. Let us take $G = G_2$ and $G' = A_2$. We know this is a regular subalgebra. Indeed, examining the root diagram, we see that the six long roots form the familiar hexagon of the adjoint representation of $SU(3) = A_2$. The projection f^* here is just the identity map since the algebra and the subalgebra have the same rank. The fourteen dimensional adjoint representation becomes the sum of the eight dimensional adjoint representation of $SU(3)$ and two conjugate three dimensional representations.

We state without proof two of Dynkin's theorems.[1] If G' is a regular subalgebra of G and ϕ is representation of G, then $\phi \circ f$ is reducible. An approximate converse is also true. If G is A_n, B_n or C_n, and G' has a reducible representation which makes it a subalgebra of G by being respectively n-dimensional with no bilinear invariant or 2n+1 dimensional with a symmetric bilinear invariant, or 2n dimensional with an anti-symmetric bilinear invariant, then G' is regular. In other words, if A_n, B_n, or C_n has an S-subalgebra, that S-subalgebra must have an irreducible representation of dimension, n, 2n+1, or 2n respectively.

What happened to D_n in this theorem? As we saw in the last chapter, $O(s) \times O(t)$ is an S-subalgebra of $O(s+t)$ if both s and t are odd.

We now proceed to the determination of f^*, the mapping which connects H_0^* to $H_0^{*\,\prime}$. Once we know f^* we can find the weights of $\phi \circ f$ for any representation ϕ of a given algebra. From these weights we can infer the irreducible representations into which $\phi \circ f$ decomposes. In fact, extensive tables of these branching rules have been compiled by computer (see McKay and Patera). Here we seek to develop some intuitive understanding of the procedure.

To be explicit, we shall take an example: B_3, that is O(7). By the methods of the previous chapter we can easily find the maximal regular semi-simple subalgebras: $A_3 = D_3$ and $A_1+A_1+A_1$. In seeking the S-subalgebras, we note that the technique $O(s) \times O(t) \subset O(st)$ is of no avail for $st = 7$. On the other hand, A_1 has a seven dimensional representation which has a symmetric bilinear form. Thus we might anticipate that A_1 is a maximal S-subalgebra of B_3. In fact, as we shall see, $A_1 \subset G_2$ and G_2 is maximal in B_3.

There is a simple and effective way to find the mappings f^* for the regular subalgebras. Remember the procedure for constructing the extended Dynkin diagrams. We added the vector $-\gamma$ to the diagram for the simple algebra, where γ was the highest root. We then had a diagram for the set which was the union of $-\gamma$ and the simple roots. From this set, we struck one root. The remainder furnished a basis of simple roots for the subalgebra. The Dynkin coefficients for the weights relative to the new basis of simple roots are just the Dynkin coefficients with respect to the surviving old simple roots, together with the Dynkin coefficient with respect to $-\gamma$. Thus we simply calculate the Dynkin coefficient of each weight with respect to $-\gamma$ and use it in place of one of the old coefficients. Calculating the coefficient with respect to $-\gamma$ is trivial since we can express $-\gamma$ as a linear combination of the simple roots.

Let us use this technique to analyze the regular subalgebras of B_3. The Dynkin coefficients of γ are $(0,1,0)$. Referring to the Cartan matrix we see that $-\gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3$. Since $\langle\gamma,\gamma\rangle = \langle\alpha_1,\alpha_1\rangle = \langle\alpha_2,\alpha_2\rangle = 2\langle\alpha_3,\alpha_3\rangle$, the coefficient of a weight with respect to $-\gamma$ is $-a_1 - 2a_2 - a_3$ if the Dynkin coefficients with respect to $\alpha_1, \alpha_2,$ and α_3 are $a_1, a_2,$ and a_3 respectively.

In this way we construct the **extended weight scheme** for the seven dimensional representation of B_3:

$$\boxed{1\ \ 0\ \ 0\ \ -1}$$

$$\boxed{-1\ \ 1\ \ 0\ \ -1}$$

$$\boxed{0\ \ -1\ \ 2\ \ 0}$$

$$\boxed{0\ \ 0\ \ 0\ \ 0}$$

$$\boxed{0\ \ 1\ \ -2\ \ 0}$$

$$\boxed{1\ \ -1\ \ 0\ \ 1}$$

$$\boxed{-1\ \ 0\ \ 0\ \ 1}$$

Now the A_3 regular subalgebra is obtained by using the fourth column, the one for $-\gamma$, rather than the third one. Deleting the third column we have

$$\boxed{1\ \ 0\ \ -1}$$

$$\boxed{-1\ \ 1\ \ -1}$$

$$\boxed{0\ \ -1\ \ 0}$$

$$\boxed{0\ \ 0\ \ 0}$$

$$\boxed{0\ \ 1\ \ 0}$$

$$\boxed{1\ \ -1\ \ 1}$$

$$\boxed{-1\ \ 0\ \ 1}$$

This is a representation of A_3. The candidates for highest weights of irreducible components are (0,1,0) and (0,0,0) since these are the only ones with purely non-negative Dynkin coefficients. Indeed, these give a six dimensional representation and a one-dimensional representation. Moreover, we can deduce the projection operator for this subalgebra directly by comparing three weights in the original basis to their values in the basis for the subalgebra A_3. In this way we find

$$
\begin{aligned}
f^*_{A_3}(1,0,0) &= (1,0,-1)\\
f^*_{A_3}(0,1,0) &= (0,1,-2)\\
f^*_{A_3}(0,0,1) &= (0,0,-1)\ .
\end{aligned}
\tag{XVI.6}
$$

Knowing this mapping gives us an alternative method of finding the weights of some irreducible representation of B_3 with respect to the subalgebra A_3. Thus for example, we can map the weights of the adjoint representation (it suffices just to consider the positive roots) of B_3 into weights of A_3:

$$
\begin{array}{ccc}
\boxed{0\ \ 1\ \ 0} & \longrightarrow & \boxed{0\ \ 1\ \ -2}\\[6pt]
\boxed{1\ \ -1\ \ 2} & \longrightarrow & \boxed{1\ \ -1\ \ -1}\\
\boxed{-1\ \ 0\ \ 2} & \longrightarrow & \boxed{-1\ \ 0\ \ -1}\\
\boxed{1\ \ 0\ \ 0} & \longrightarrow & \boxed{1\ \ 0\ \ -1}\\
\boxed{-1\ \ 1\ \ 0} & \longrightarrow & \boxed{-1\ \ 1\ \ -1}\\
\boxed{1\ \ 1\ \ -2} & \longrightarrow & \boxed{1\ \ 1\ \ -1}\\
\boxed{0\ \ -1\ \ 2} & \longrightarrow & \boxed{0\ \ -1\ \ 0}\\
\boxed{-1\ \ 2\ \ -2} & \longrightarrow & \boxed{-1\ \ 2\ \ -1}\\
\boxed{2\ \ -1\ \ 0} & \longrightarrow & \boxed{2\ \ -1\ \ 0}
\end{array}
$$

From these weights and their negatives, it is clear that the candidates for highest weights are (1,0,1) and (0,1,0) which indeed correspond to representations of dimension fifteen and six respectively.

If we consider the subalgebra $A_1 + A_1 + A_1$, deleting the Dynkin coefficients with respect to the second root, we find the seven dimensional representation of B_3 is mapped into

$$\boxed{1\ \ 0\ \ -1}$$

$$\boxed{-1\ \ 0\ \ -1}$$

$$\boxed{0\ \ 2\ \ 0}$$

$$\boxed{0\ \ 0\ \ 0}$$

$$\boxed{0\ \ -2\ \ 0}$$

$$\boxed{1\ \ 0\ \ 1}$$

$$\boxed{-1\ \ 0\ \ 1}$$

This is the reducible representation whose Dynkin expression is [(0)+(2)+(0)] + [(1)+(0)+(1)]. In the notation which indicates dimensionality, it is (1,3,1)+(2,1,2).

It is clear that the regular subalgebras can be dealt with in a very simple fashion. The S-subalgebras require more effort. It is always possible to order the Cartan subalgebras of the initial algebra and of its subalgebra so that if $x > y$, $x, y \in H'$ then $f^*(x) > f^*(y)$. Now we exploit this by writing the weights of the seven dimensional representations of B_3 and G_2 beside each other:

$$
\begin{array}{cc}
\boxed{1\ \ 0\ \ 0} & \boxed{0\ \ 1} \\[6pt]
\boxed{-1\ \ 1\ \ 0} & \boxed{1\ \ -1} \\
\boxed{0\ \ -1\ \ 2} & \boxed{-1\ \ 2} \\
\boxed{0\ \ 0\ \ 0} & \boxed{0\ \ 0} \\
\boxed{0\ \ 1\ \ -2} & \boxed{1\ \ -2} \\
\boxed{1\ \ -1\ \ 0} & \boxed{-1\ \ 1} \\
\boxed{-1\ \ 0\ \ 0} & \boxed{0\ \ -1}
\end{array}
$$

Thus it is clear that we must have

$$
\begin{aligned}
f^*_{G_2}(1,0,0) &= (0,1) \\
f^*_{G_2}(0,1,0) &= (1,0) \\
f^*_{G_2}(0,0,1) &= (0,1)\ .
\end{aligned}
\tag{XVI.7}
$$

Equipped with this mapping, we can project the weights of any representation of B_3 onto the space of weights of G_2 and thus identify the branching rules.

For example, we again consider the positive roots of B_3 to find the branching rule for the adjoint representation:

$$\begin{array}{ccc}
\boxed{0\ \ 1\ \ 0} & \to & \boxed{1\ \ 0} \\
\boxed{1\ \ -1\ \ 2} & \to & \boxed{-1\ \ 3} \\
\boxed{-1\ \ 0\ \ 2} & \to & \boxed{0\ \ 1} \\
\boxed{1\ \ 0\ \ 0} & \to & \boxed{0\ \ 1} \\
\boxed{-1\ \ 1\ \ 0} & \to & \boxed{1\ \ -1} \\
\boxed{1\ \ 1\ \ -2} & \to & \boxed{1\ \ -1} \\
\boxed{0\ \ -1\ \ 2} & \to & \boxed{-1\ \ 2} \\
\boxed{-1\ \ 2\ \ -2} & \to & \boxed{2\ \ -3} \\
\boxed{2\ \ -1\ \ 0} & \to & \boxed{-2\ \ 2} \\
\boxed{0\ \ 0\ \ 0} & \to & \boxed{0\ \ 0}
\end{array}$$

The weights with non-negative Dynkin coefficients are (1,0), (0,1), and (0,1). Now the fourteen dimensional representation has highest weight (1,0) and includes the weight (0,1). Thus we see that the 21 dimensional representation of B_3 becomes the sum of a 14 dimensional representation and a 7 dimensional representation.

Footnote

1. DYNKIN III, pp. 158–159.

References

Again, the entire chapter is due to DYNKIN III. The works of SLANSKY and of MC KAY AND PATERA provide exhaustive tables.

Exercises

1. Find the branching rules for the ten-dimensional representation of SU(5) for the maximal semi-simple subalgebras.

2. Find the branching rules for the 26 and 52 dimensional representations of F_4 into its maximal semi-simple subalgebras.