XV. Subalgebras

Some examples may be illuminating as an introduction to the topic of subalgebras. Suppose we start with the algebra $G = A_5$, i.e. SU(6), the traceless 6×6 matrices. Now one subalgebra is obtained by considering only those matrices with non-zero 4×4 and 2×2 diagonal pieces, and zeros in the 4×2 off- diagonal pieces. If the two diagonal blocks are required to be traceless separately, then the restricted set is the subalgebra $G' = A_3 + A_1 \subset A_5 = G$. It is clear that we can take as the Cartan subalgebra $H' \subset G'$ the diagonal matrices, so $H' \subset H$. The dimension of H' is one fewer than that of H since there is a one dimensional subspace of H proportional to the diagonal element which is $+1$ for the first four components and -2 on the last two.

The root vectors of G' are just the root vectors of G which have non-zero components only in the two diagonal blocks. If the space proportional to e_α is denoted G_α, we have

$$G = H + \sum_{\alpha \in \Sigma} G_\alpha \tag{XV.1}$$

while for some set $\Sigma' \subset \Sigma$

$$G' = H' + \sum_{\alpha \in \Sigma'} G_\alpha \ . \tag{XV.2}$$

A subalgebra with this property is called **regular**. In addition to this SU(6) example, the subalgebras we employed in the method of parts – which were obtained by deleting dots from Dynkin diagrams – were regular. Not all subalgebras, however, are regular.

Let us consider again A_5 and a particular embedding of $G' = A_2 + A_1 (SU(3) \times SU(2))$. We know that every matrix in the Lie algebra of SU(2) is a linear combination of σ_1, σ_2, and σ_3 and every matrix in the Lie algebra of SU(3) is a linear combination of $\lambda_1, \lambda_2, \ldots \lambda_8$. Let us add to these $\sigma_0$and λ_0 which are the 3×3 and 2×2 identity matrices. Now every 6×6 matrix can be written in terms of

$$\begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{bmatrix} , \quad \begin{bmatrix} \lambda_i & 0 \\ 0 & -\lambda_i \end{bmatrix} , \quad \begin{bmatrix} 0 & \lambda_i \\ \lambda_i & 0 \end{bmatrix} , \quad \begin{bmatrix} 0 & -i\lambda_i \\ i\lambda_i & 0 \end{bmatrix} ,$$

i.e. $\sigma_0 \otimes \lambda_i, \sigma_3 \otimes \lambda_i, \sigma_1 \otimes \lambda_i, \sigma_2 \otimes \lambda_i, \quad i = 0, 1, \ldots 8$. Now this is equivalent to regarding the six dimensional vectors in the carrier space as having two indices, with the σ acting on the first and the λ on the second.

Now suppose we consider only elements of the forms $\sigma_0 \otimes \lambda_i$ and $\sigma_i \otimes \lambda_0$. Then an element of one form commutes with an element of the other. Thus these elements form a subalgebra which is $A_2 + A_1$. The Cartan subalgebra of the $A_2 + A_1$ has a basis $\sigma_3 \otimes \lambda_0, \sigma_0 \otimes \lambda_3, \sigma_0 \otimes \lambda_8$. The root vectors are $\sigma_+ \otimes \lambda_0, \sigma_- \otimes \lambda_0, \sigma_0 \otimes t_+, \sigma_0 \otimes t_-, \sigma_0 \otimes u_+, \sigma_0 \otimes u_-, \sigma_0 \otimes v_+$, and $\sigma_0 \otimes v_-$. We see that $H' \subset H$. However, the root vectors of G' are not among those of G. Thus, for example,

$$\sigma_+ \otimes \lambda_0 = \begin{bmatrix} 0 & \lambda_0 \\ 0 & 0 \end{bmatrix} \ . \tag{XV.3}$$

We cannot write G' in the form Eq. (XV.2), so the subalgebra is not regular.

The six dimensional representation of A_5 gave a reducible representation of the regular subalgebra, $A_3 + A_1$: $6 \rightarrow (4,1) + (1,2)$. The non-regular subalgebra, $A_2 + A_1$, gave an irreducible representation: $6 \rightarrow (3,2)$. As we shall see, this is typical.

As a further example of regular and non-regular subalgebras, consider SU(2) as a subalgebra of SU(3). If the SU(2) is generated by t_+, t_-, and t_z, the SU(2) is a regular subalgebra. On the other hand, there is a three dimensional representation of SU(2). The 3×3 matrices of this representation are elements of SU(3) so this provides a second embedding of SU(2) in SU(3), which is not regular. Under the regular embedding, the 3 dimensional representation of SU(3) becomes a reducible 2 + 1 dimensional representation of SU(2), while under the second embedding, it becomes an irreducible representation of SU(2).

It is clear that a moderate sized algebra may have an enormous number of subalgebras. To organize the task of finding them we introduce the concept of a **maximal subalgebra**. G' is a maximal subalgebra of G if there is no larger subalgebra containing it except G itself. Now we can proceed in stages finding the maximal subalgebras, then their maximal subalgebras, and so on.

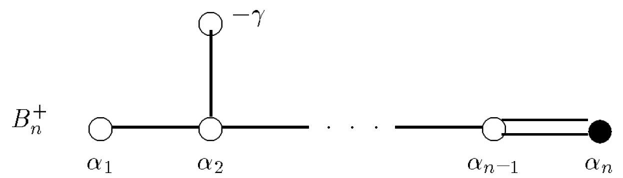
There is a slight flaw in this approach. A maximal subalgebra of a semi-simple algebra need not be semi-simple. Consider, for example, SU(2) and the subalgebra generated by t_+ and t_z. It is certainly maximal, since if we enlarge it we shall have all of SU(2). However, the subalgebra is not simple because t_+ generates an ideal in it. We shall generally restrict ourselves to the consideration of maximal semi-simple subalgebras, that is, semi-simple algebras contained in no other semi-simple subalgebras except the full algebra.

Dynkin introduced the notions of an **R-subalgebra** and an **S-subalgebra**. An R-subalgebra is a subalgebra which is contained in some regular subalgebra. An S-subalgebra is one which is not. The task then is to find the regular maximal subalgebras and the maximal S-subalgebras. The regular subalgebras are more easily dealt with.

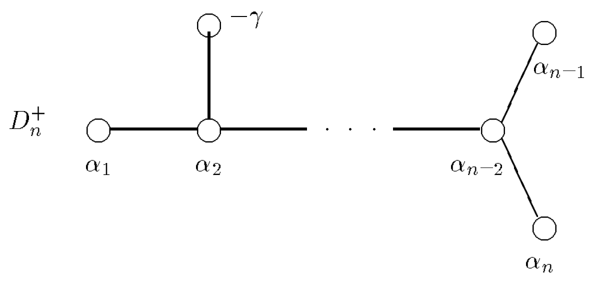
Suppose G' is a regular subalgebra of a simple algebra G, $\Sigma' \subset \Sigma$ is the set of its roots, and $\Pi' \subset \Sigma'$ is a basis of simple roots for G'. Now if $\alpha', \beta' \in \Pi'$, then $\alpha' - \beta' \notin \Sigma'$. In fact $\alpha' - \beta' \notin \Sigma$, since if $\alpha' - \beta' \in \Sigma, [e_{\alpha'}, e_{-\beta'}] \sim e_{\alpha'-\beta'}$ so then $\alpha' - \beta' \in \Sigma'$. Thus to find regular subalgebras of G we seek sets $\Pi' \subset \Sigma$ such that $\alpha', \beta' \in \Pi' \Rightarrow \alpha' - \beta' \notin \Sigma$. Then we take as G' the subalgebra generated by the $e_{\alpha'}, e_{-\alpha'}, h_{\alpha'} \in \Pi'$.

An algorithm for this has been provided by Dynkin. Start with Π, the simple roots of G. Enlarge it to $\overline{\Pi}$ by adding the most negative root in Σ. Now $\overline{\Pi}$ has the property that if $\alpha', \beta' \in \overline{\Pi}$, then $\alpha' - \beta' \notin \Sigma$. However, $\overline{\Pi}$ is a linearly dependent set. Thus, if we eliminate one or more vectors from $\overline{\Pi}$ to form Π', it will have the required properties. In general, Π' will generate a semi-simple algebra, not a simple one.

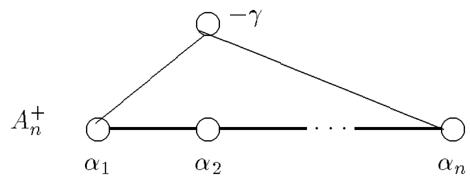
This procedure is easy to follow using Dynkin diagrams. We form the **extended Dynkin diagram** associated with $\overline{\Pi}$ by noting that the vector added to Π is the negative of the highest weight ,γ, of the adjoint representation. Since we know the Dynkin coefficients of this weight, it is easy to add the appropriate dot. For B_n the adjoint representation has highest weight $(0, 1, 0, \ldots 0)$, so the extended diagram is

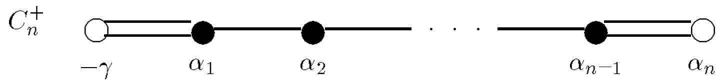


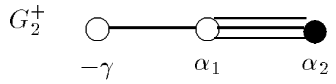
Similarly, for D_n, the highest weight of the adjoint is $(0, 1, 0, \ldots 0)$ so the extended diagram is



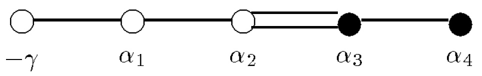
In an analogous fashion, we find the remaining extended diagrams:



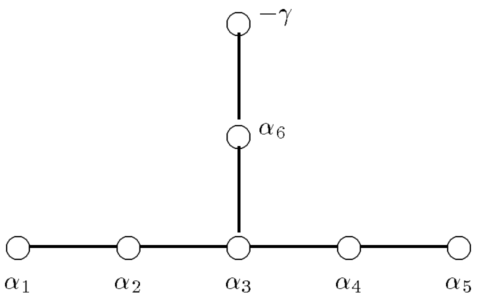




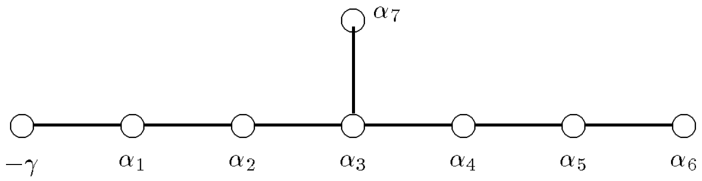
F_4^+


$-\gamma$
α_1
α_2
α_3
α_4

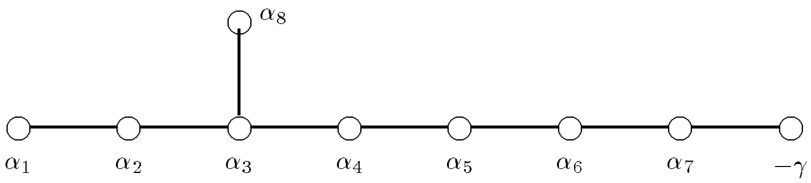
E_6^+


$-\gamma$
α_6
α_1
α_2
α_3
α_4
α_5

E_7^+


α_7
$-\gamma$
α_1
α_2
α_3
α_4
α_5
α_6

E_8^+


α_8
α_1
α_2
α_3
α_4
α_5
α_6
α_7
$-\gamma$

A few examples will make clear the application of these diagrams. By deleting a single dot from the extended diagram for G_2 we obtain in addition to the diagram for G_2 itself

Thus we see that A_2 and $A_1 + A_1$ are regular subalgebras of G_2. Starting with B_6 we find among others, the subalgebra $B_3 + D_3$. In other words, we have $O(13) \supset O(7) \times O(6)$.

The A_n algebras are somewhat pathological. If we remove a single dot from the extended diagram, the result is simply the original diagram. If we remove two dots we obtain a regular subalgebra, but one that is maximal only among the semi-simple subalgebras, not maximal among all the subalgebras. This is actually familiar: from SU(5) one obtains not just SU(3) x SU(2), but also SU(3) x SU(2) x U(1), which itself lies in a larger, non-semi-simple subalgebra of SU(5).

Dynkin's rule[1] for finding the maximal regular subalgebras is this: the regular semi-simple maximal subalgebras are obtained by removing one dot from the extended Dynkin diagram. The non-semi-simple maximal regular subalgebras are obtained by selecting one of the simple roots, $\alpha \in \Pi$, and finding the subalgebra generated by e_α and h_α, together with $e_\beta, e_{-\beta}$, and h_β for all the simple roots β other than α. Such a non-semi-simple algebra contains a semi-simple subalgebra generated by excluding e_α and h_α as well. This may be maximal among the semi-simple subalgebras, or it may be contained in an S-subalgebra.

Dynkin's rule has been shown to be not entirely correct.[2] In particular, it would have $A_3 + A_1$ maximal in F_4 while in fact $A_3 + A_1 \subset B_4 \subset F_4$. Similarly, $A_3 + A_3 + A_1 \subset D_6 + A_1 \subset E_7$ and in E_8, $A_3 + A_5 \subset D_8$, $A_1 + A_2 + A_5 \subset A_2 + E_6$, and $A_7 + A_1 \subset E_7 + A_1$.

In an analogy with the example $SU(4) \times SU(2) \subset SU(6)$, we see that $SU(s) \times SU(t) \subset SU(s+t), O(s) \times O(t) \subset 0(s+t)$, and $Sp(2s) \times Sp(2t) \subset Sp(2s+2t)$, where the embedding is the obvious one. Each of these gives a regular subalgebra except for $O(s) \times O(t)$ when s and t are odd, as is easily verified from the extended diagrams. The last embedding is thus an S-subalgebra.

We have already seen two examples of maximal S-subalgebras. One was the embedding $A_2 + A_1 \subset A_5$ which produced the decomposition of the fundamental representation $6 \rightarrow (3,2)$. We can simplify the notation in this section by passing from the Lie algebras to the associated groups. Thus we have $SU(6) \supset SU(3) \times SU(2)$. More generally, we have $SU(st) \supset SU(s) \times SU(t)$ as a (non-simple) maximal S-subalgebra.

For the orthogonal groups we follow the path used in Chapter VIII. Rather than require $A^t A = I$, we consider the more general relation $B^t K B = K$ where K is a symmetric $n \times n$ matrix. This is the same as taking all $n \times n$ matrices, B, which preserve a symmetric bilinear form $(\phi, \eta) = \sum_{i,j} \phi_i K_{ij} \eta_j, K_{ij} = K_{ji}$. Now consider the groups $O(s_1)$and $O(s_2)$ preserving $(\phi_1, \eta_1)_1$and $(\phi_2, \eta_2)_2$. If we consider the $s_1 s_2$ dimensional space spanned by vectors like $\phi_1 \otimes \phi_2$, we have a symmetric bilinear form defined by $(\phi_1 \otimes \phi_2, \eta_1 \otimes \eta_2) = (\phi_1, \eta_1)_1 (\phi_2, \eta_2)_2$. The subgroup $O(s_1) \times O(s_2)$ acts as $B_1 \otimes B_2(\phi_1 \otimes \phi_2) = B_1 \phi_1 \otimes B_2 \phi_2$ for $B_1 \in O(s_1)$ and $B_2 \in O(s_2)$. It is clear that this subgroup indeed leaves the symmetric bilinear form invariant and thus $O(s_1) \times O(s_2) \subset O(s_1 s_2)$. Indeed, it is a maximal S-subgroup.

In a similar fashion, we can consider $Sp(2n)$ to be the set of $2n \times 2n$ matrices preserving an anti-symmetric bilinear form: $(\phi, \eta) = -(\eta, \phi)$. Now if we take $Sp(2n_1) \times Sp(2n_2)$ we will act on a space of dimension $4n_1 n_2$. With $(\phi_1 \otimes \phi_2, \eta_1 \otimes \eta_2) = (\phi_1, \eta_1)_1 (\phi_2, \eta_2)_2$ we see that the form is symmetric and $Sp(2n_1) \times Sp(2n_2) \subset O(4n_1 n_2)$. Analogously, $Sp(2n) \times O(s) \subset Sp(2ns)$ as a maximal S-subgroup.

The other maximal S-subalgebra we have encountered is in the embedding $SU(2) \subset SU(3)$ whereby the three dimensional representation of $SU(3)$ becomes the three dimensional representation of SU(2). Since $SU(2)$ has a three dimensional representation by 3×3 matrices, it is bound to be a subalgebra of $SU(3)$. More generally, if G is a simple algebra with an n-dimensional representation, $G \subset SU(n)$. Is G then maximal in $SU(n)$? For the most part, the answer is this: if the n-dimensional representation of G has an invariant symmetric bilinear form, G is maximal in $SO(n)$, if it has an invariant anti-symmetric bilinear form it is maximal in $Sp(n)$ (n must be even). If it has no invariant bilinear form, it is maximal in $SU(n)$. The exceptions to this are few in number and have been detailed by Dynkin.[3]

Let us consider an example with $SU(3)$, which has an eight dimensional adjoint representation. Rather than think of the vectors in the eight dimensional space as columns, it is convenient to think of them as 3×3 traceless matrices:

$$\phi = \sum_i \phi_i \lambda_i \tag{XV.4}$$

where the λ_i are those of Eq. (II.1). Now we remember that the adjoint representation is given by

$$ad\, x \lambda_i = [x, \lambda_i] \ . \tag{XV.5}$$

The invariance of a symmetric form (ϕ, η) under an infinitesimal transformation $\exp \mathcal{B} \approx I + \mathcal{B}$ yields

$$(\mathcal{B}\phi, \eta) + (\phi, \mathcal{B}\eta) = 0 \ . \tag{XV.6}$$

For the adjoint representation, the linear tranformation $\mathcal{B}$ corresponding to an element x of the Lie algebra is simply $\mathcal{B}\phi = [\mathcal{B}, \phi]$. Now if we define

$$(\phi, \eta) = Tr\ \phi\eta \tag{XV.7}$$

the invariance of the form follows from the identity

$$Tr\ [x, \phi]\, \eta + Tr\ \phi\, [x, \eta] = 0 \ . \tag{XV.8}$$

Thus we see there is a symmetric invariant bilinear form for the adjoint representation and thus $SU(3) \subset SO(8)$. Of course the demonstration is more general: $SU(n) \subset SO(n^2-1)$.

It is clear that we must learn how to determine when an n-dimensional representation of a simple Lie algebra admits a bilinear form and whether the form is symmetric or anti-symmetric. As a beginning, let us consider $SU(2)$ and in particular the $2j+1$ dimensional representation. We shall construct explicitly the bilinear invariant. Let

$$\xi = \sum_i c_i \phi_i$$

$$\eta = \sum_i d_i \phi_i \tag{XV.9}$$

where the ϕ_i are a basis for the representation space and

$$\begin{aligned} T_z \phi_m &= m \phi_m \\ T_+ \phi_m &= \sqrt{j(j+1) - m(m+1)}\,\phi_{m+1} \\ T_- \phi_m &= \sqrt{j(j+1) - m(m-1)}\,\phi_{m-1} \ . \end{aligned} \tag{XV.10}$$

Suppose the bilinear form is

$$(\xi, \eta) = \sum_{m,n} a_{mn} c_m d_n \ . \tag{XV.11}$$

Now the invariance of the form requires in particular

$$(T_z \xi, \eta) + (\xi, T_z \eta) = 0 \ . \tag{XV.12}$$

Thus in the sum we must have $m + n = 0$ so

$$(\xi, \eta) = \sum_m a_m c_m d_{-m} \ . \tag{XV.13}$$

If we next consider the requirement

$$(T_{-}\xi, \eta) + (\xi, T_{-}\eta) = 0 \ , \tag{XV.14}$$

we find that up to a multiplicative constant the bilinear form must be

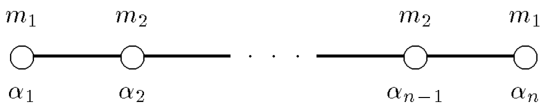
$$(\xi, \eta) = \sum_m (-1)^m c_m d_{-m} \ . \tag{XV.15}$$

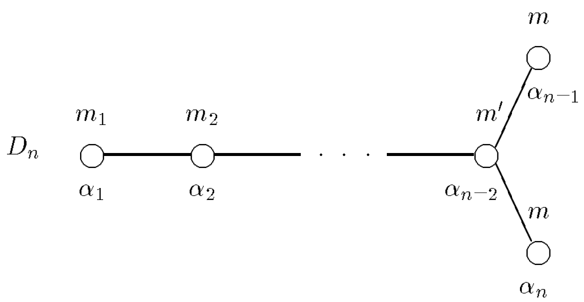
We see that we have been able to construct a bilinear invariant and moreover, if j is integral, the form is symmetric under interchange of ξ and η, while if j is half-integral it is anti-symmetric.

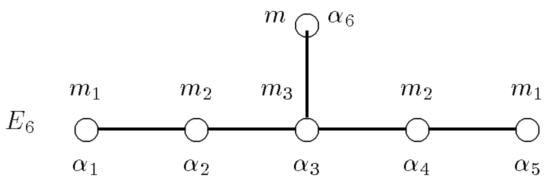
The generalization to all the simple Lie algebras is only slightly more complicated. From the analogs of Eq. (XV.12), we conclude that we can form a bilinear invariant only if for every weight M in the representation, $-M$ is a weight also. To determine which irreducible representations have this property it suffices to consider the representations from which we can build all others. For example, for A_n, the representation with highest weight $(1, 0, \ldots 0)$ does not contain the weight $(-1, 0, \ldots 0)$ but instead $(0, 0, \ldots, 0, -1)$. On the other hand, the adjoint representation whose highest weight is $(1, 0, \ldots 0, 1)$ does contain the weight $(-1, 0, \ldots 0, -1)$. More generally for A_n, if the highest weight has the symmetric configuration $(n_1, n_2, \ldots n_2, n_1)$, the weights do occur in opposite pairs, M and $-M$.

The general result is the following. Representations of $B_n, C_n, D_{2n}, G_2, F_4$, E_7, and E_8 always have an invariant bilinear form. The algebras A_n, D_{2n+1}, and E_6 have invariant bilinear forms for representations of the forms:

A_n







Now as for the symmetry or anti-symmetry of the bilinear form it turns out to be determined by the number of levels in the weight diagram. Dynkin calls the number of levels less one the **height of the representation**. Thus for $SU(2)$, the $2j+1$ dimensional representation has a height $2j$. For all the simple Lie algebras, just as for SU(2), if the height is even, the form is symmetric, and if the height is odd, the form is anti-symmetric. Now Dynkin has determined the heights of the irreducible representations of the simple Lie algebras in terms of their Dynkin coefficients.[4] The results are summarized in the following Table:

A_n	$(n, (n-1)2, \ldots, n)$
B_n	$(1 \cdot 2n, 2 \cdot (2n-1), \ldots, (n-1)(n+2), n(n+1)/2)$
C_n	$\big(1 \cdot (2n-2), 2(2n-2), \ldots, (n-1)(n+1), n^2\big)$
D_n	$(1 \cdot (2n-2), 2 \cdot (2n-3), \ldots, (n-2)(n+1), n(n-1)/2, n(n-1)/2)$
G_2	$(10, 6)$
F_4	$(22, 42, 30, 16)$
E_6	$(16, 30, 42, 30, 16, 22)$
E_7	$(34, 66, 96, 75, 52, 27, 49)$
E_8	$(92, 182, 270, 220, 168, 114, 58, 136)$

The height of the representation is calculated by multiplying each Dynkin coefficient by the number from the table corresponding to its location. Thus, for example, the adjoint representation of A_n, with Dynkin coefficients $(1,0,\ldots 0,1)$ has height $2n$. It is clear that to determine whether a bilinear form is symmetric or anti-symmetric we need only consider those entries in the table and those Dynkin coefficients which are odd. It is apparent that since $SU(3)$ representations have a bilinear form only if the Dynkin coefficients of the highest weight are of the form (n,n), all such bilinear forms are symmetric. On the other hand, we see that $SU(6)$ has a representation with highest weight $(0,0,1,0,0)$ and dimension 20 which has an anti-symmetric bilinear invariant. Thus $SU(6) \subset Sp(20)$.

There are a few instances in which the procedure described above does not identify a maximal subalgebra. These exceptions have been listed by Dynkin[5] and we shall not dwell on them here. One example will suffice to indicate the nature of these exceptions. There is a 15 dimensional representation of A_2, which has a highest weight $(2,1)$. We would expect this to be maximal in A_{14} since it has no bilinear invariant. In fact, there is an embedding of A_2 in A_5 under which the 15 dimensional representation of A_5, $(0,1,0,0,0)$, becomes the fifteen dimensional representation of A_2. Thus we have the chain $A_2 \subset A_5 \subset A_{14}$. We can understand the embedding of A_2 in A_5 as follows. Since A_2 has a six dimensional representation, it is maximal in A_5 , i.e. $SU(6)$. The anti-symmetric product of the six with itself, in both A_2 and A_5 is an irreducible fifteen dimensional representation. Thus the embedding which maps the six into the six, also maps the fifteen into the fifteen.

It is clear that the approach above will not help us find the S-subalgebras of the exceptional Lie algebras. Fortunately, this problem has been solved, again by Dynkin. In order to display his results we must introduce some additional notation. In particular, we need a means of specifying a particular embedding of a subalgebra G' in an algebra, G. This is done with the **index of the embedding**. In Chapter XI, we introduced the concept of the index of a representation, which is simply the ratio of the bilinear form obtained from the trace of the product of two representation matrices to the bilinear form which is the Killing form normalized in a particular way. Here we define the index of an embedding to be the ratio of the bilinear form on G' obtained by lifting the value of the Killing form on G to the Killing form on G' itself:

$$j_f(x', y')_2' = (f(x'), f(y'))_2 \tag{XV.16}$$

where j_f is the index of the embedding and $f : G' \to G$ is the embedding. As we have seen, on a simple Lie algebra all invariant bilinear forms are proportional, so this definition makes sense. Now suppose ϕ is a representation of G, that is, a mapping of elements of G onto a space of linear transformations. Then $\phi \circ f$ is a representation of G'. Moreover, for $x', y' \in G'$:

$$\begin{aligned} Tr\ \phi(f(x'))\phi(f(y')) =& l_{\phi\circ f}(x', y')_2' \\ =& l_\phi(f(x')), f(y'))_2 \\ =& l_\phi j_f(x', y')_2' \end{aligned} \tag{XV.17}$$

so we see that the index of the embedding is determined by the ratio of the indices of the representations ϕ and $\phi \circ f$:

$$j_f = \frac{l_{\phi\circ f}}{l_\phi} \ . \tag{XV.18}$$

Consider G_2, which we know has a 7 dimensional representation. Thus we might hope to embed A_1, which has a 7 dimensional representation, in G_2. This is in fact possible. Now we compute the index of the seven dimensional representation of A_1 according to the methods of Chapter XI: $l = 6 \times 7 \times 8/6 = 56$. For the seven dimensional representation of G_2 we have $l = 7\langle \Lambda, \Lambda + 2\delta\rangle/14 = 2$. See the Problems following Chapter XI. Thus the index of the embedding is 28. Dynkin indicates this subalgebra by A_1^{28}. If there is more than one subalgebra with the same index, we can use primes to indicate this.

Having established this notation, we list the results of Dynkin [3]for the maximal S-subalgebras of the exceptional Lie algebras:

Maximal S-subalgebras of Exceptional Lie Algebras

G_2	A_1^{28}
F_4	$A_1^{156}, G_2^1 + A_1^8$
E_6	$A_1^9, G_2^3, C_4^1, G_2^1 + A_2^{2''}, F_4^1$
E_7	$A_1^{399}, A_1^{231}, A_2^{21}, G_2^1 + C_3^{1''}, F_4^1 + A_2^{3''}, G_2^2 + A_1^7, A_1^{24} + A_1^{15}$
E_8	$A_1^{1240}, A_1^{760}, A_1^{520}, G_2^1 + F_4^1, A_2^{6'} + A_1^{16}, B_2^{12}$

We summarize here the results on the maximal semi-simple subalgebras of the simple Lie algebras:

1. Regular subalgebras are found using the algorithm of extended Dynkin diagrams.

   a. Dropping a dot from an extended diagram yields a regular subalgebra which is semi-simple and maximal unless it is one of the exceptions mentioned on pp. 134 - 135.

   b. Dropping a dot from a basic diagram yields a subalgebra which may be maximal among the semi-simple subalgebras.

2. Non-Regular (S-subalgebras)

   a. Of classical algebras:

   i. Non-simple: $SU(s) \times SU(t) \subset SU(st)$, $SO(s) \times SO(t) \subset SO(st)$, $Sp(s) \times Sp(t) \subset SO(st)$, $Sp(s) \times SO(t) \subset Sp(st)$, and $O(s) \times O(t) \subset O(s+t)$ for s and t odd.

   ii. Simple: If G has an n dimensional representation it is maximal in $SU(n)$, $SO(n)$, or $Sp(n)$, unless it is one of the few exceptions listed by Dynkin. If the representation has a symmetric bilinear form the subalgebra is maximal in $SO(n)$. If it has an anti-symmetric bilinear form, it is maximal in $Sp(n)$. If it has no bilinear form, it is maximal in $SU(n)$.

   b. Of exceptional Lie algebras: the maximal S-subalgebras are listed above.

Footnotes

1. DYNKIN II, p. 148.
2. GOLUBITSKY and ROTHSCHILD.
3. DYNKIN III.
4. DYNKIN III, p. 365.
5. DYNKIN II, p. 231.

References

This material is comes from DYNKIN II, III.

Very useful tables are provided in SLANSKY.

Incredibly extensive tables are given in MC KAY and PATERA.

Exercises

1. Find the maximal semi-simple subalgebras of A_4.
2. Find the maximal semi-simple subalgebras of D_5. Note that it is necessary to consider some subalgebras which are only maximal among the semi-simple subalgebras. *ans.* $A_3 + A_1 + A_1, A_4, B_4, D_4, A_1 + B_2, B_2 + B_2$.
3. Find the maximal semi-simple subalgebras of F_4.
4. Find the maximal semi-simple subalgebras of B_2.