XIV. Reducing Product Representations

In Chapter X we began the consideration of finding the irreducible components of a product representation. The procedure for SU(2) is familiar and trivial. The product of the representations characterized by j_1 and j_2, the maximal values of T_z, contains the irreducible representations for j such that $|j_1 - j_2| \leq j \leq j_1 + j_2$ once each (of course we take only integral j if $j_1 + j_2$ is integral and j half integral otherwise). For SU(n), the reduction is most easily obtained by the method of Young Tableaux. The general solution to the problem of finding the irreducible components of the product of two irreducible representations of a simple Lie algebra can be obtained from the Weyl character formula, but the result (Kostant's formula) involves a double sum over the Weyl group and is not particularly practical. In this chapter, we introduce some techniques which are sufficient for solving the problem in most cases of moderate dimensions.

Of course we already have sufficient tools to solve the problem by brute force. We can calculate all the weights of the representations associated with highest weights Λ_1 and Λ_2, and using the Freudenthal recursion relations we can determine the multiplicity of each weight. Then the multiplicity of the weight M in the product representation is found by writing $M = M_1 + M_2$, where M_1 and M_2 are weights of the two irreducible representations. The multiplicity of the weight M in the product representation is

$$n_M = \sum_{M=M_1+M_2} n_{M_1} n_{M_2} \tag{XIV.1}$$

where n_{M_1} and n_{M_2} are the multiplicities of the weights M_1 and M_2. Now we know $\Lambda_1 + \Lambda_2$ is the highest weight of one irreducible component so we can subtract its weights (with proper multiplicities) from the list. Now the highest remaining weight must be the highest weight of some irreducible component which again we find and eliminate from the list. Continuing this way, we exhaust the list of weights, and most likely, ourselves.

A more practical approach is to use some relations which restrict the possible choices of irreducible components. The first such relation is the obvious one: $\Lambda_1 + \Lambda_2$ is the highest weight of one irreducible component. The second relation is a generalization of the rule demonstrated before for the anti-symmetric product of a representation with itself.

Dynkin's Theorem for the Second Highest Representation provides a simple way to find one or more irreducible representations beyond the highest. Suppose we have two irreducible representations with highest weights Λ_1 and Λ_2, with highest weight vectors ξ_0 and η_0. We shall say that two elements of the root space, β and α are joined if $\langle\alpha,\beta\rangle \neq 0$. Moreover, we shall say that a chain of simple roots, $\alpha_1, \alpha_2, \ldots \alpha_k$, connects Λ_1 and Λ_2 if Λ_1 is joined to α_1 but no other of the α's, Λ_2 is joined to α_k and no other α in the chain, and each α_i is joined only to the succeeding and preceding α's. We can represent this with a Dynkin diagram by adding a dot for each of Λ_1 and Λ_2 and connecting them by segments to the simple roots with which they are joined. Then a chain is the shortest path between the weights Λ_1 and Λ_2. For example, consider SU(8) and the representations (1,1,0,0,0,0,0) and (0,0,0,0,2,1,0).

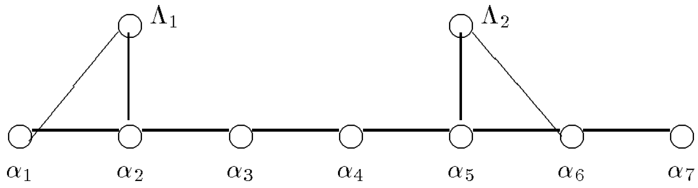

Fig. V.1

Here $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ is a chain joining Λ_1 and Λ_2, but $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ is not.

Dynkin's theorem tells us that if $\alpha_1, \ldots \alpha_k$ is a chain joining Λ_1 and Λ_2, then $\Lambda_1 + \Lambda_2 - \alpha_1 - \alpha_2 - \ldots - \alpha_k$ is the highest weight of an irreducible representation in the product representation formed from the irreducible representations with highest weights Λ_1 and Λ_2. In the above example, the product representation contains $(1,1,0,0,2,1,0) - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 = (2,0,0,0,1,2,0)$.

The theorem is proved by establishing that there is a weight vector with the weight described by the theorem which is annihilated by all the raising operators associated with the simple roots. Thus this weight vector generates, through the lowering operators, a separate irreducible representation. Of course, there are other weight vectors with the same weight which do not share this property.

We begin by constructing a sequence of weight vectors starting with ξ_0. Since $\langle \Lambda_1, \alpha_1 \rangle \neq 0$ and must be non-negative, it is positive and thus $\Lambda_1 - \alpha_1$ is a weight and has a weight vector

$$\xi_1 = E_{-\alpha_1} \xi_0 \ . \tag{XIV.2}$$

Also, since $\langle \Lambda_1, \alpha_2 \rangle = 0$, $\Lambda_1 - \alpha_2$ is not a weight. However, $\Lambda_1 - \alpha_1 - \alpha_2$ is a weight since $\langle \Lambda_1 - \alpha_1, \alpha_2 \rangle = \langle -\alpha_1, \alpha_2 \rangle > 0$. Proceeding in this way we construct

$$\begin{aligned} \xi_j &= E_{-\alpha_j} \xi_{j-1} \\ &= E_{-\alpha_j} \ldots E_{-\alpha_1} \xi_0 \ . \end{aligned} \tag{XIV.3}$$

Any reordering of the lowering operators in the sequence results in the quantity vanishing just as $E_{-\alpha_2} \xi_0 = 0$.

Now consider

$$E_{\alpha m}\xi_j = E_{\alpha m}E_{-\alpha_j}\xi_{j-1}. \qquad \text{(XIV.4)}$$

If $m \neq j$, $\left[E_{\alpha m}, E_{-\alpha_j}\right] = 0$ since $\alpha_m - \alpha_j$ is not a root. Thus

$$E_{\alpha m}\xi_j = E_{-\alpha_j}E_{\alpha m}\xi_{j-1} \ . \qquad \text{(XIV.5)}$$

If we continue commuting $E_{\alpha m}$ through until it hits ξ_0, we get zero since ξ_0 corresponds to the highest weight. The only alternative is that $E_{-\alpha m}$ occurs and we take the term

$$E_{-\alpha_j}\ldots[E_{\alpha m}, E_{-\alpha m}]\ldots E_{-\alpha_1}\xi_0 \ . \qquad \text{(XIV.6)}$$

But this vanishes since the commutator is just an H which we can move outside (picking up some constant terms) leaving a series of E_α's which are not in the proper order. Thus $E_{\alpha m}\xi_j = 0$ unless $m = j$.

In the event $m = j$, we compute

$$\begin{aligned} E_{\alpha_j}\xi_j &= \left[H_{\alpha_j} + E_{-\alpha_j}E_{\alpha_j}\right]\xi_{j-1} \\ &= H_{\alpha_j}\xi_{j-1} \\ &= \langle\Lambda_1 - \alpha_1 \ldots - \alpha_{j-1}, \alpha_j\rangle\xi_{j-1} \\ &= -\langle\alpha_{j-1}, \alpha_j\rangle\xi_{j-1} \qquad (j > 1) \\ E_{\alpha_1}\xi_1 &= \langle\Lambda_1, \alpha_1\rangle\xi_0 \ . \end{aligned} \qquad \text{(XIV.7)}$$

At the other end of the chain we have an analogous situation. We define

$$\begin{aligned} E_{-\alpha_k}\eta_0 &= \eta_1 \\ E_{-\alpha_{k-j}}\eta_j &= \eta_{j+1} \end{aligned} \qquad \text{(XIV.8)}$$

and find

$$
\begin{aligned}
E_{\alpha m}\eta_{k-j+1} &= 0. \qquad (m \neq j), \\
E_{\alpha_j}\eta_{k-j+1} &= -\langle\alpha_{j+1},\alpha_j\rangle\eta_{k-j} \quad (j < k) \\
E_{\alpha_k}\eta_1 &= \langle\Lambda_2,\alpha_k\rangle\eta_0 \ .
\end{aligned} \tag{XIV.9}
$$

We are now in a position to establish the existence of a vector with weight $\Lambda_1 + \Lambda_2 - \alpha_1 - \ldots - \alpha_k$ which is annihilated by every E_{α_j}, for α_j a simple root. This will then be a weight vector for the highest weight of the desired irreducible representation. The most general weight vector with this weight is, using the results of the discussion above,

$$
\zeta = \sum_{s=0}^{k} c_s \xi_s \otimes \eta_{k-s} \ . \tag{XIV.10}
$$

We simply choose the coefficients so that the vector is annihilated by every raising operator. For $j \neq 1, k$:

$$
\begin{aligned}
E_{\alpha_j}\zeta &= c_j E_{\alpha_j}\xi_j \otimes \eta_{k-j} + c_{j-1}\xi_{j-1} \otimes E_{\alpha_j}\eta_{k-j+1} \\
&= [-c_j\langle\alpha_{j-1},\alpha_j\rangle - c_{j-1}\langle\alpha_j,\alpha_{j+1}\rangle]\xi_{j-1} \otimes \eta_{k-j} \\
&= 0 \ .
\end{aligned} \tag{XIV.11}
$$

Thus for $j \neq 1, k$

$$
c_j\langle\alpha_{j-1},\alpha_j\rangle + c_{j-1}\langle\alpha_j,\alpha_{j+1}\rangle = 0 \ . \tag{XIV.12}
$$

Similarly, considering $j = 1$ and $j = k$,

$$
\begin{aligned}
c_1\langle\Lambda_1,\alpha_1\rangle - c_0\langle\alpha_1,\alpha_2\rangle &= 0 \ , \\
-c_k\langle\alpha_k,\alpha_{k-1}\rangle + c_{k-1}\langle\Lambda_2,\alpha_k\rangle &= 0 \ .
\end{aligned} \tag{XIV.13}
$$

It is clear we can solve these equations, say with $c_0 = 1$. Thus ζ exists and so does the asserted representation.

The second technique we use to reduce product representations is **Dynkin's method of parts**. This is very easy to apply. If some of the dots in a Dynkin diagram of, say, a simple Lie algebra are deleted, the result is the diagram of a semi-simple subalgebra of the original algebra. If the original diagram was marked with integers to indicate an irreducible representation, the truncated diagram will represent a particular irreducible representation of the subalgebra. Now if we consider two irreducible representations of the original algebra there are associated two irreducible representations of the subalgebra. If we compare the product representations formed by both the representations of the full algebra and those of the subalgebra, it turns out that each irreducible component of the product representation of the subalgebra has a diagram which is a "part" of a diagram of an irreducible component of the product representation of the full algebra in the sense that it is obtained again by deleting the same dots as before.

The utility of this technique lies in the possibility that the subalgebra's products may be easier to deal with than those of the full algebra. In particular, if we delete some dots so that the remaining algebra is in the series A_n, the products can be calculated using the well-known technique of Young tableaux. For example, by deleting one dot from E_6 we get A_5, D_5, $A_4 + A_1$, or $A_2 + A_2 + A_1$, each of which is somewhat easier to deal with.

Before proving the correctness of the method of parts, let us consider an example. As a preliminary, note that for D_5, i.e. SO(10), the square of the ten-dimensional representation, (1,0,0,0,0) is given by (2,0,0,0,0) + (0,1,0,0,0) + (0,0,0,0,0). This is easy to see because the first piece follows from the rule for the highest weight of a product representation. The second follows for the rule for the second highest weight, or the rule for the anti-symmetric product of an irreducible representation with itself. Use of the Weyl dimension formula reveals that the dimension of the (2,0,0,0,0) representation is 54 and that of the (0,1,0,0,0) is 45, so the remaining representation is one dimensional, i.e. it is (0,0,0,0,0). (Of course, these results can be obtained by more elementary means!) Now let us try to compute the square of the E_6 representation (1,0,0,0,0,0). This is the smallest representation of E_6, with dimension 27. Again the rules for the highest weight and second highest weight give (2,0,0,0,0,0) and (0,1,0,0,0,0) as irreducible components of the product representation. A computation reveals their dimensions to be 351 (both of them!). Thus the remaining representation is 27 dimensional. Is it (1,0,0,0,0,0) or (0,0,0,0,1,0)? Let us use the method of parts, deleting the fifth simple root to leave us with the diagram for D_5. Now we know that (1,0,0,0,0) squared in D_5 is (2,0,0,0,0) + (0,1,0,0,0) + (0,0,0,0,0). The method of parts tells us that each of these can be obtained from the irreducible representations in the E_6 product by deleting the fifth root. This clearly works for the first two. Moreover, we see that we must choose (0,0,0,0,1,0) as the final representation of E_6.

We proceed to a more formal consideration of the method of parts. Let G be a simple Lie algebra with a basis of simple roots $\{\alpha_i\}$. Select a subset, $\{\beta_j\} \subset \{\alpha_i\}$ and let G' be the semi-simple algebra which is generated by the e_{β_j}'s and $e_{-\beta_j}$'s. The Cartan subalgebra, H' of G' is contained in the Cartan subalgebra, H of G. The Dynkin diagram for G' is obtained from that of G by deleting the appropriate dots.

Suppose M is a weight of a representation of G and ϕ_M is an associated weight vector:

$$H\phi_M = M(h)\phi_M \ . \tag{XIV.14}$$

Then ϕ_M is a weight vector for the induced representation of G', since if $h' \in H'$,

$$H'\phi_M = M(h')\phi_M \ . \tag{XIV.15}$$

Now the weight in the induced representation, $\overline{M}$, has the property $\overline{M}(h') = M(h')$ but differs from M because it can be expressed entirely in terms of the β_j's. If we write

$$\begin{aligned} M = {} & [M - \sum_{i,j} \beta_i \langle M, \beta_j\rangle\langle\beta_j, \beta_i\rangle^{-1}] \\ & + \sum_{i,j} \beta_i \langle M, \beta_j\rangle\langle\beta_j, \beta_i\rangle^{-1} \ . \end{aligned} \tag{XIV.16}$$

we see that the second term is precisely $\overline{M}$. This is so because every h' is a linear combination of h_β's and because the first term vanishes on every h_β.

We see from Eq. (XIV.16) that $\overline{M}$ has exactly the same Dynkin coefficients with respect to the β's as M itself. The Dynkin coefficients with respect to G' are obtained simply by dropping the requisite coefficients from those of M. In particular, if Λ is the highest weight of a representation of G, $\overline{\Lambda}$ is a highest weight of one of the irreducible representations contained in the representation of G' arising from the representation of G. For example, the (1,1) representation of A_2 (SU(3)) contains the (1) representation of A_1 (SU(2)) when we obtain the SU(2) by deleting one dot from the Dynkin diagram of SU(3).

The representation associated with $\overline{\Lambda}$ is obtained by operating on the weight vector for Λ with all the lowering operators $E_{\alpha_i}, \alpha_i \in \Sigma'$. It is easy to see that for all vectors ϕ in this representation, if $\alpha_j \notin \Sigma'$ then $E_{\alpha_j}\phi = 0$.

Now consider two irreducible representations, R_{Λ_1} and R_{Λ_2} of G, where Λ_1 and Λ_2 are their highest weights. Associated with these are two irreducible representations of G', $R_{\Lambda'_1}$ and $R_{\Lambda'_2}$. Now consider the product representations $R_{\Lambda_1} \times R_{\Lambda_2}$ and $R_{\Lambda'_1} \times R_{\Lambda'_2}$. In general these are both reducible:

$$R_{\Lambda_1} \times R_{\Lambda_2} = R_{\Lambda_a} + R_{\Lambda_b} + \ldots \qquad \text{(XIV.15)}$$

and

$$R_{\Lambda'_1} \times R_{\Lambda'_2} = R_{\Lambda'_a} + R_{\Lambda'_b} + \ldots \ . \qquad \text{(XIV.16)}$$

We want to show that for some $\Lambda_a, \overline{\Lambda_a} = \Lambda'_a$, *etc.* Now consider the highest weight vector of $R_{\Lambda'_a}$. It is annihilated by $E_{\alpha_i}, \alpha_i \in \Sigma'$ and also by all the $E_{\alpha_j}, \alpha_j \in \Sigma$. Thus it is a highest weight also for G as well as for G', and thus defines one of the R_{Λ_a}'s. Thus every scheme of $R_{\Lambda'_1} \times R_{\Lambda'_2}$ corresponds to one of the schemes of $R_{\Lambda_1} \times R_{\Lambda_2}$ with the appropriate dots deleted.

In reducing product representations for SU(2), it is clear that the product of irreducible representations contains only either integral spin or half-integral spin representations. In Dynkin language, the Cartan matrix is the number 2. The Dynkin coefficient of the highest weight is a single integer. The lower weights are obtained by subtracting the 2 from the highest weight. Thus if the coefficient of the highest weight is odd, the coefficient of every weight is odd. The oddness or evenness of the the two irreducible representations thus determines the oddness or evenness of all the irreducible representations in their product.

The analogous concept for SU(3) is **triality**. The fundamental representation is said to have triality +1. Every weight in this representation is obtained from (1,0) by subtracting a row of the Cartan matrix. Now consider the quantity $1A_{i1} + 2A_{i2}$. For i=1, this is zero, while for i = 2, it is three. Thus if we calculate for any weight (a_1, a_2) of an irreducible representation the quantity $a_1 + 2a_2 (\text{mod } 3)$, it must be the same as it is for the highest weight. Thus for the three dimensional representation we have (1,0), (-1,1), and (0,-1), where $a_1 + 2a_2$ is 1,1, and -2. It is clear that the triality, $a_1 + 2a_2 (\text{mod } 3)$, of a representation in the product of two irreducible representations is the sum of the trialities (mod 3) of the components.

If we look at the Cartan matrix for A_n, we see that $\sum_j jA_{ij} = 0(\text{mod } n+1)$. Thus each irreducible representation of A_n can be characterized by a number $C \equiv \sum_j ja_j (\text{mod } n+1)$ where the highest weight of the representation has coefficients $(a_1, a_2, \ldots a_n)$. We see that every weight in the irreducible representation will have the same value for C. For a product of representations, R_1 and R_2,each irreducible component has the value $C \equiv C(R_1) + C(R_2)$. For example, consider $5^* \times 5^*$ in SU(5), that is (0,0,0,1) $\times$ (0,0,0,1) = (0,0,0,2) + (0,0,1,0). We have $C((0,0,0,0,1)) = 4, C((0,0,0,2)) = 8 \equiv 3(\text{mod } 5), C((0,0,1,0)) = 3$. We refer to the irreducible representations with a fixed value of C as **conjugacy classes**.

For B_n, there are two conjugacy classes which are given by $C = a_n (\text{mod } 2)$, since the last column of the Cartan matrix is given entirely by even integers. If $C = 1$, the representation is a spinor representation. This nomenclature makes sense even for B_1 which is the complex form of O(3). The spinor representations are seen to be the half-integral angular momentum representations.

For the algebras C_n, we take $C = a_1 + a_3 + a_5 + \ldots (\text{mod } 2)$ .We can see that this will work by taking the sum of the first, third, fifth, *etc.* elements in a row of the Cartan matrix and noticing that its value is always even.

The algebra D_n is slightly more complicated. First we notice that for any row of the Cartan matrix, the sum of the last two entries is an even number. Thus we can take $C_1 = a_{n-1} + a_n (\text{mod } 2)$. Next consider the case n even. If we take the sum of the first, third, fifth,... elements of a row of the Cartan matrix, it is always even. Thus for n even, we choose $C_2 = a_1 + a_3 + \ldots a_{n-1}$ (mod 2). For n odd, we take $C_2 = a_1 + a_3 + \ldots a_{n-2} + (a_{n-1} - a_n)/2(\text{mod } 2)$.

As an example, consider the SO(10) decomposition of (0,0,0,1,0) $\times$ (0,0,0,1,0), i.e. 16×16. The 16 has $C_1 = 1$ and $C_2 = \frac{1}{2}$. Thus the irreducible representations in the product will have $C_1 = 2 \equiv 0(\text{mod } 2)$ and $C_2 = 1$. The actual decomposition is (0,0,0,2,0) + (0,0,1,0,0) + (1,0,0,0,0), each piece of which has the proper values of C_1 and C_2.

An examination of the Cartan matrix for E_6 reveals that $A_{i1} + A_{i4} = A_{i2} + A_{i5}(\text{mod } 3)$, so we can take $C = a_1 - a_2 + a_4 - a_5(\text{mod } 3)$. Similarly, for E_7, we see that $C = a_4 + a_6 + a_7(\text{mod } 2)$ determines the conjugacy classes. The algebras G_2, F_4, and E_8 have only a single conjugacy class.

As an example of the techniques discussed in this chapter, let us consider the decomposition of $(2,0,0,0,0,0) \times (2,0,0,0,0,0) = 351 \times 351$ in E_6. The (2,0,0,0,0,0) is in the conjugacy class $C \equiv 2(\text{mod } 3)$, so all the components in the product will be in the conjugacy class $C \equiv 4 \equiv 1(\text{mod } 3)$. Clearly one irreducible component has highest weight (4,0,0,0,0,0). Using the rule for the second highest weight we find that $(4,0,0,0,0,0) - (2,-1,0,0,0,0) = (2,1,0,0,0,0)$ is a highest weight of an irreducible component. Next we use the method of parts, striking the sixth root to reduce E_6 to A_5. The products in A_5 may be calculated by Young tableaux, with the result $(2,0,0,0,0) \times (2,0,0,0,0) = (4,0,0,0,0) + (2,1,0,0,0) + (0,2,0,0,0)$. Thus the E_6 product contains (4,0,0,0,0,0) + (2,1,0,0,0,0) +(0,2,0,0,0,X) where X is a non-negative integer. Next, let us use the parts method, striking the fifth root to reduce E_6 to D_5. Now we must calculate $(2,0,0,0,0) \times (2,0,0,0,0) = 54 \times 54$ in D_5.

This subsidiary calculation is itself a useful example. The 54 in D_5 has $C_1 = 0$ and $C_2 = 0$, so the irreducible components of the product must have these values as well. Certainly the D_5 product contains (4,0,0,0,0) + (2,1,0,0,0) as we see using the highest weight and second highest weight procedures. Using the parts method to reduce D_5 to A_4 we see that the product must contain a term (0,2,0,0,W). Since $C_1 \equiv 0$, W is even. It is a fair bet that (0,2,0,0,2) has too high a dimension, so we guess that (0,2,0,0,0) is in the product. Using the Weyl dimension formula, we find the D_5 values, dim(4,0,0,0,0)=660, dim(2,1,0,0,0)=1386, dim(0,2,0,0,0)=770. This totals to 2816 so we need an additional 2916-2816=100. The smallest representations in the proper conjugacy class are (0,0,0,0,0), (2,0,0,0,0), (0,1,0,0,0), and (0,0,0,1,1) with dimensions 1, 54, 45, and 210 respectively. Thus we conclude that in D_5, $(2,0,0,0,0) \times (2,0,0,0,0) = (4,0,0,0,0) + (2,1,0,0,0) + (0,2,0,0,0) + (2,0,0,0,0) + (0,1,0,0,0) + (0,0,0,0,0)$.

Returning to E_6, we note that the representation identified as (0,2,0,0,0,X) must be (0,2,0,0,0,0) in order to account for the D_5 representation (0,2,0,0,0). Again, by comparison with the D_5 representations, we know that the E_6 product must contain, at a minimum, the representations (2,0,0,0,T,0), (0,1,0,0,Y,0), and (0,0,0,0,Z,0), where T,Y, and Z are non-negative integers. We can determine these integers by considering the conjugacy classes. We have $2-T \equiv 1(\mathrm{mod}\, 3), -1-Y \equiv 1(\mathrm{mod}\, 3)$, and $-Z \equiv 1(\mathrm{mod}\, 3)$. The smallest solutions are T=1, Y=1, and Z=2. Thus we guess that the E_6 decomposition is $(2,0,0,0,0,0) \times (2,0,0,0,0,0) = (4,0,0,0,0,0) + (2,1,0,0,0,0) + (0,2,0,0,0,0) + (2,0,0,0,1,0) + (0,1,0,0,1,0) + (0,0,0,0,2,0)$. Using the Weyl formula, the dimensions of these are determined to be $351 \times 351 = 19,305 + 54,054 + 34,398 + 7,722 + 7,371 + 351 = 123,201$.

This example shows that the hardest work required in reducing such products is simply the evaluation of the Weyl formula for the dimension.

One additional technique for reducing product representations is worth mentioning. We recall from Chapter XI the definition of the index of a representation, ϕ:

$$\mathrm{Tr}\ \phi(x)\phi(y) = l_\phi (x,y)_2 \qquad \text{(XIV.19)}$$

where $(\ , \ )_2$ is proportional to the Killing form but normalized so that the largest roots has a length squared of two.

Now suppose we have a representation which is the sum of two representations , ϕ_1 and ϕ_2. Then, clearly

$$l_{\phi_1+\phi_2} = l_{\phi_1} + l_{\phi_2} \ . \qquad \text{(XIV.20)}$$

On the other hand, for a product representation, we see that

$$l_{\phi_1 \otimes \phi_2} = N_{\phi_1} l_{\phi_2} + N_{\phi_2} l_{\phi_1} \qquad \text{(XIV.21)}$$

where N_{ϕ_1} and N_{ϕ_2} are the dimensions of the representations. Since we know how to compute the indices in terms of the Casimir operators, this can be used to reduce significantly the possibilities for the irreducible components of the product representation.

References

Most of this material is due to E. B. Dynkin. For the method of the second highest representation, see

DYNKIN III, especially pp. 266–268.

For the method of parts see

DYNKIN III, especially pp. 275–280.

For conjugacy classes, see

DYNKIN II, especially pp. 136–137.

Some of the same material is discussed by SLANSKY.

Exercises

1. Reduce 16×16 $[16 = (0,0,0,0,1)]$ in SO(10).

2. Reduce 14×14 $[14 = (1,0)]$ in G_2. Check using indices.

3. Reduce $27 \times 27'$ [for $27 = (1,0,0,0,0,0)$, $27' = (0,0,0,0,1,0)$] in E_6.