XIII. Weyl's Dimension Formula

In this Section we derive the celebrated formula of Weyl for the dimensionality of an irreducible representation of a simple Lie algebra in terms of its highest weight. Our derivation is essentially that of Jacobson, which is based on the technique of Freudenthal.

We shall be considering functions defined on H_0^*. Instead of parameterizing elements of H_0^* in terms of the simple roots, it is convenient to over-parameterize by writing $\rho \in H_0^*$ as

$$\rho = \sum_{\alpha \in \Sigma} \rho_\alpha \alpha \ . \tag{XIII.1}$$

We can define the action of an element, S, of the Weyl group on a function of ρ by

$$(SF)(\rho) = F(S^{-1}\rho) \ . \tag{XIII.2}$$

As an example, consider the function known as the **character of the representation**

$$\chi(\rho) = \sum_M n_M \exp\langle M, \rho\rangle \tag{XIII.3}$$

where the sum is over all the weights of a representation and n_M is the dimensionality of the weight space for M. Now we calculate the action of $S \in W$ on χ:

$$\begin{aligned}(S\chi)(\rho) &= \sum_M n_M \exp\langle M, S^{-1}\rho\rangle \\ &= \sum_M n_M \exp\langle SM, \rho\rangle \\ &= \sum_M n_M \exp\langle M, \rho\rangle \ .\end{aligned} \tag{XIII.4}$$

Here we have used the orthogonality property of the Weyl group and the relation $n_M = n_{SM}$. Thus we see that $S\chi = \chi$, that is χ is invariant under the Weyl group.

Consider next the function

$$Q(\rho) = \prod_{\alpha>0} [\exp(\frac{1}{2}\langle\alpha, \rho\rangle) - \exp(-\frac{1}{2}\langle\alpha, \rho\rangle)] \ . \tag{XIII.5}$$

We want to determine the behavior of this function when acted upon by elements of the Weyl group. It suffices to determine the effect of the $S_i = S_{\alpha_i}$, the reflections associated with simple roots.

$$\begin{aligned}(S_iQ)(\rho) &= \prod_{\alpha>0}\left[\exp\left(\frac{1}{2}\langle\alpha, S_i^{-1}\rho\rangle\right) - \exp\left(-\frac{1}{2}\langle\alpha, S_i^{-1}\rho\rangle\right)\right] \\ &= \prod_{\alpha>0}\left[\exp\left(\frac{1}{2}\langle S_i\alpha, \rho\rangle\right) - \exp\left(-\frac{1}{2}\langle S_i\alpha, \rho\rangle\right)\right] \ .\end{aligned} \tag{XIII.6}$$

We have already seen that S_i interchanges all the positive roots except α_i whose sign it changes. Thus we see directly that

$$(S_i Q)(\rho) = -Q(\rho) \ . \tag{XIII.7}$$

Now S_i reverses the sign of α_i, but leaves every vector orthogonal to α_i unchanged. Thus $\det S_i = -1$ and we can write

$$(S_i Q) = (\det S_i) Q \ . \tag{XIII.8}$$

Indeed, every $S \in W$ is a product of S_i's, so

$$SQ = \det S Q \ . \tag{XIII.9}$$

Functions with this property are called **alternating**.

We can make alternating functions by applying the operator

$$\sigma = \sum_{S \in W} (\det S) S \tag{XIII.10}$$

for we have

$$\begin{aligned} S'\sigma &= \sum_{S \in W} S'(\det S) S \\ &= \sum_{S \in W} \det S' \det(S'S) S'S \\ &= \det S' \sigma \ . \end{aligned} \tag{XIII.11}$$

It is convenient to find a representation of the alternating function $Q(\rho)$ in the form $\sigma F(\rho)$. From the definition of $Q(\rho)$ it is clear that there must be an expansion of the form

$$Q(\rho) = \sigma \sum_{\beta} c_\beta \exp\langle \delta - \beta, \rho\rangle \tag{XIII.12}$$

where δ is half the sum of all the positive roots and where β is a sum of distinct positive roots. Now in such an expansion it is redundant to include both $\sigma\exp\langle M,\rho\rangle$ and $\sigma\exp\langle SM,\rho\rangle$ since they are equal up to a factor $\det S$. We have already seen that among all the SM, $S \in W$, the largest one has only non-negative Dynkin coefficients. Thus we need only consider terms where $\delta - \beta$ has only non-negative Dynkin coefficients. In fact, we can restrict this further because if M has a Dynkin coefficient which is zero, then $\sigma\exp\langle M,\rho\rangle = 0$. This is easy to establish since if $M(h_i) = 0$, then $S_i M = M$ and $S_i\sigma\exp\langle M,\rho\rangle = \sigma\exp\langle S_i M,\rho\rangle = \sigma\exp\langle M,\rho\rangle = \det S_i \sigma\exp\langle M,\rho\rangle = -\sigma\exp\langle M,\rho\rangle = 0$. However, we have seen that δ has Dynkin coefficients which are all unity. Now since β is a sum of positive roots it cannot have only negative Dynkin coefficients. Thus we see that the sum, Eq. (XIII.12) need include only $\beta = 0$. Comparing the coefficients of $\exp\langle\delta,\rho\rangle$ we see that $c_{\beta=0} = 1$ so

$$\begin{aligned} Q(\rho) &= \prod_{\alpha>0} [\exp\frac{1}{2}\langle\alpha,\rho\rangle - \exp -\frac{1}{2}\langle\alpha,\rho\rangle] \\ &= \sum_{S\in W} (\det S)\exp\langle S\delta,\rho\rangle \ . \end{aligned} \tag{XIII.13}$$

We shall now use these results to analyze the character. We begin with the Freudenthal recursion relation, Eq. (XI.45), together with $\sum_{k=-\infty}^{\infty} n_{M+k\alpha}\langle M + k\alpha,\alpha\rangle = 0$:

$$\begin{aligned} [\langle\Lambda+\delta,\Lambda+\delta\rangle - \langle\delta,\delta\rangle - \langle M,M\rangle] n_M \\ = \sum_{\alpha\neq 0}\sum_{k=0}^{\infty} n_{M+k\alpha}\langle M+\kappa\alpha,\alpha\rangle. \end{aligned} \tag{XIII.14}$$

Mulitplying by $\exp\langle M,\rho\rangle$ and summing over M, we have

$$[\langle\Lambda+\delta,\Lambda+\delta\rangle-\langle\delta,\delta\rangle]\chi-\sum_M n_m\langle M,M\rangle\exp\langle M,\rho\rangle$$

$$=\sum_M\sum_{\alpha\neq 0}\sum_{k=0}^{\infty} n_{M+\kappa\alpha}\langle M+\kappa\alpha,\alpha\rangle\exp\langle M,\rho\rangle\ . \quad \text{(XIII.15)}$$

Remembering that (see Eq. (IV.3))

$$\langle M,N\rangle=\sum_{\alpha\in\Sigma}\alpha(h_M)\alpha(h_N)=\sum_{\alpha\in\Sigma}\langle M,\alpha\rangle\langle\alpha,N\rangle\ , \quad \text{(XIII.16)}$$

we derive the relations

$$\frac{\partial}{\partial\rho_\alpha}\exp\langle M,\rho\rangle=\langle\alpha,M\rangle\exp\langle M,\rho\rangle\ , \quad \text{(XIII.17}a\text{)}$$

$$\sum_{\alpha\in\Sigma}\frac{\partial^2}{\partial\rho_\alpha^2}\exp\langle M,\rho\rangle=\langle M,M\rangle\exp\langle M,\rho\rangle\ . \quad \text{(XIII.17}b\text{)}$$

Inserting this into Eq. (XIII.15) gives

$$[\langle\Lambda+\delta,\Lambda+\delta\rangle-\langle\delta,\delta\rangle-\sum_{\alpha\in\Sigma}\frac{\partial^2}{\partial\rho_\alpha^2}]\chi$$

$$=\sum_M\sum_{\alpha\neq 0}\sum_{k=0}^{\infty} n_{M+\kappa\alpha}\langle M+\kappa\alpha,\alpha\rangle\exp\langle M,\rho\rangle\ . \quad \text{(XIII.18)}$$

To analyze the right hand side of Eq. (XIII.18), let us first fix $\alpha\neq 0$ and consider the SU(2) generated by $E_\alpha, E_{-\alpha}$, and H_α. The irreducible representation of the full algebra is, in general, a reducible representation of this SU(2). The dimensionality n_M is just the number of SU(2)-irreducible representations present at the weight M. Thus we can proceed by calculating the contribution of each SU(2)-irreducible representation to the sum for fixed M and α. The string of weights containing M which correspond to an SU(2) irreducible representation are distributed symmetrically about a center point, M_0 which can be expressed in terms of the highest weight in the sequence, M^*, as

$$M^0 = M^* - \alpha\frac{\langle M^*, \alpha\rangle}{\langle \alpha, \alpha\rangle}\ . \tag{XIII.19}$$

Note that $\langle M^0, \alpha\rangle = 0$.

Thus each weight in the sequence is of the form $M^0 + m\alpha$ where m is an integer or half-integer. The range of m is from $-j$ to j, where again j is an integer or half-integer. Now we can write the contribution of a single SU(2) irreducible representation to the sum as

$$\begin{aligned}
&\sum_M \sum_{k=0}^{\infty} \langle M + \kappa\alpha, \alpha\rangle \exp\langle M, \rho\rangle \\
&\qquad = \sum_m \sum_{k=0}^{j-m} \langle M^0 + m\alpha + k\alpha, \alpha\rangle \exp\langle M^0 + m\alpha, \rho\rangle \\
&\qquad = \sum_m \sum_{k=0}^{j-m} \langle \alpha, \alpha\rangle (m+k) \exp\langle M^0, \rho\rangle \exp(m\langle \alpha, \rho\rangle) \\
&\qquad = \langle \alpha, \alpha\rangle \exp\langle M^0, \rho\rangle \sum_m \sum_{k=0}^{j-m} (m+k) \exp(m\langle \alpha, \rho\rangle)\ .
\end{aligned} \tag{XIII.20}$$

The double sum is most easily evaluated by multiplying first by $\exp\langle\alpha,\rho\rangle - 1$:

$$\sum_{m=-j}^{j}\sum_{k=0}^{j-m}(k+m)\exp\langle m\alpha,\rho\rangle(\exp\langle\alpha,\rho\rangle - 1)$$

$$= \sum_{m=-j}^{j}\sum_{k=0}^{j-m}(m+k)\exp\langle(m+1)\alpha,\rho\rangle$$

$$- \sum_{m=-j-1}^{j-1}\sum_{k=0}^{j-m-1}(m+k+1)\exp\langle(m+1)\alpha,\rho\rangle$$

$$= j\exp\langle(j+1)\alpha,\rho\rangle + \sum_{m=-j}^{j-1}[j-(j-m)]\exp\langle(m+1)\alpha,\rho\rangle$$

$$+ \sum_{k=0}^{2j}(k-j)\exp\langle -j\alpha,\rho\rangle$$

$$= \sum_{m=-j}^{j} m\exp\langle(m+1)\alpha,\rho\rangle\ . \qquad \text{(XIII.21)}$$

Thus the contribution of one SU(2) irreducible representation to the original sum is

$$\langle\alpha,\alpha\rangle\exp\langle M^0,\rho\rangle\sum_{m=-j}^{j} m\exp\langle(m+1)\alpha,\rho\rangle[\exp\langle\alpha,\rho\rangle - 1]^{-1}$$

$$= \sum_{m=-j}^{j}\langle M,\alpha\rangle\exp\langle M+\alpha,\rho\rangle[\exp\langle\alpha,\rho\rangle - 1]^{-1}\ . \qquad \text{(XIII.22)}$$

Summing over all SU(2) irreducible representations, and over all the roots, we have

$$[\langle\Lambda+\delta,\Lambda+\delta\rangle - \langle\delta,\delta\rangle - \sum_{\alpha\in\Sigma}\frac{\partial^2}{\partial\rho_\alpha^2}]\chi$$

$$= \sum_{\alpha\neq 0}\sum_{M} n_M\langle\alpha,M\rangle\exp\langle M+\alpha,\rho\rangle[\exp\langle\alpha,\rho\rangle - 1]^{-1}\ . \qquad \text{(XIII.23)}$$

From the definition of Q, we see that

$$\prod_{\alpha\neq 0}[\exp\langle\alpha,\rho\rangle - 1] = \eta Q^2(\rho) \tag{XIII.24}$$

where η is $+1$ if the number of positive roots is even and -1 if it is odd. Thus

$$\frac{\partial}{\partial\rho_\beta}\log\eta Q^2(\rho) = \sum_{\alpha\neq 0}\frac{\exp\langle\alpha,\rho\rangle}{\exp\langle\alpha,\rho\rangle - 1}\langle\alpha,\beta\rangle \tag{XIII.25}$$

and

$$\frac{\partial}{\partial\rho_\beta}\exp\langle M,\rho\rangle = \langle\beta,M\rangle\exp\langle M,\rho\rangle \tag{XIII.26}$$

so

$$\begin{aligned}
&\sum_M\sum_{\alpha\neq 0} n_M\langle\alpha,M\rangle\exp\langle M+\alpha,\rho\rangle[\exp\langle\alpha,\rho\rangle - 1]^{-1}\\
&\quad= \sum_\beta\frac{\partial}{\partial\rho_\beta}\log\eta Q^2(\rho)\frac{\partial}{\partial\rho_\beta}\chi\\
&\quad= 2Q^{-1}\sum_\beta\frac{\partial}{\partial\rho_\beta}Q\frac{\partial}{\partial\rho_\beta}\chi\\
&\quad= Q^{-1}\left[\sum_\beta\frac{\partial^2}{\partial\rho_\beta^2}(Q\chi) - Q\sum_\beta\frac{\partial^2}{\partial\rho_\beta^2}\chi - \chi\sum_\beta\frac{\partial^2}{\partial\rho_\beta^2}Q\right] .
\end{aligned} \tag{XIII.27}$$

Combining these results, we have the differential equation

$$\langle\Lambda+\delta,\Lambda+\delta\rangle Q\chi = \chi\left[\langle\delta,\delta\rangle - \sum_\beta\frac{\partial^2}{\partial\rho_\beta^2}\right]Q + \sum_\beta\frac{\partial^2}{\partial\rho_\beta^2}Q\chi \ . \tag{XIII.28}$$

From the relation

$$Q(\rho) = \sum_{S\in W}(\det S)\exp\langle S\delta,\rho\rangle \ , \tag{XIII.29}$$

it follows that

$$\sum_\beta\frac{\partial^2}{\partial\rho_\beta^2}Q(\rho) = \langle\delta,\delta\rangle Q(\rho) \ , \tag{XIII.30}$$

where we have used the orthogonality of the S's. Altogether then we have

$$\sum_\beta\frac{\partial^2}{\partial\rho_\beta^2}Q\chi = \langle\Lambda+\delta,\Lambda+\delta\rangle Q\chi \ . \tag{XIII.31}$$

Now the function $Q(\rho)\chi(\rho)$ is alternating since it is the product of an alternating function and an invariant one. Since

$$\chi(\rho) = \sum_M n_M \exp\langle M, \rho\rangle \ , \tag{XIII.32}$$

and

$$Q(\rho) = \sum_{S \in W} (\det S) \exp\langle S\delta, \rho\rangle \ , \tag{XIII.33}$$

the product must be of the form

$$Q(\rho)\chi(\rho) = \sigma \sum_N c_N \exp\langle N, \rho\rangle \ , \tag{XIII.34}$$

where N is of the form $M + S\delta$ where M is a weight and where S is in the Weyl group. Substituting into the differential equation, we see that M contributes only if

$$\langle S^{-1}M + \delta, S^{-1}M + \delta\rangle = \langle \Lambda + \delta, \Lambda + \delta\rangle \ . \tag{XIII.35}$$

In fact, we can show that Eq. (XIII.35) is satisfied only when $S^{-1}M = \Lambda$. We first note that $\langle SM+\delta, SM+\delta\rangle$ is maximized for fixed M when SM has only non-negative Dynkin coefficients. Indeed if $h_i = 2h_{\alpha_i}/\langle\alpha_i,\alpha_i\rangle$ and if $M(h_i) < 0$, then $\langle S_iM+\delta, S_iM+\delta\rangle - \langle M+\delta, M+\delta\rangle = \langle -M(h_i)\alpha_i, 2M+2\delta - M(h_i)\alpha_i\rangle = -\langle\alpha_i,\alpha_i\rangle M(h_i) > 0$. Now consider $M < \Lambda$ with $M(h_i) \geq 0$. Then, by similar arguments, it is easy to show that $\langle\Lambda+\delta,\Lambda+\delta\rangle > \langle M+\delta, M+\delta\rangle$. It follows that the sum in Eq. (XIII.34) need contain only the single term for $N = \Lambda+\delta$. By comparison with the definitions of $Q(\rho)$ and $\chi(\rho)$, it is easy to see that the overall coefficient is unity, so

$$Q(\rho)\chi(\rho) = \sum_{S\in W} (\det S) \exp\langle\Lambda+\delta, S\rho\rangle \ . \tag{XIII.36}$$

This then yields Weyl's character formula

$$\chi(\rho) = \frac{\sum_{S\in W}(\det S)\exp\langle\Lambda+\delta, S\rho\rangle}{\sum_{S\in W}(\det S)\exp\langle\delta, S\rho\rangle} \ . \tag{XIII.37}$$

More useful for our purposes is the less general formula which gives the dimension of an irreducible representation. It is clear that this dimension is the value of $\chi(\rho = 0)$. This cannot be obtained simply by setting $\rho = 0$, but must be obtained as a limit. We choose $\rho = t\delta$ and let $t \to 0$. This gives

$$\begin{aligned}\chi(t\delta) &= \frac{\sum_{S\in W}(\det S)\exp\langle S(\Lambda+\delta), t\delta\rangle}{\sum_{S\in W}(\det S)\exp\langle S\delta, t\delta\rangle} \\ &= \frac{Q(t(\Lambda+\delta))}{Q(t\delta)} \\ &= \exp\langle -\delta, t\Lambda\rangle \prod_{\alpha>0} \frac{\exp\langle\alpha, t(\Lambda+\delta)\rangle - 1}{\exp\langle\alpha, t\delta\rangle - 1} \ . \end{aligned} \tag{XIII.38}$$

In this expression we can let $t \to 0$ and find the dimensionality , dim $R = \chi(0)$:

$$\dim R = \prod_{\alpha>0} \frac{\langle\alpha, \Lambda+\delta\rangle}{\langle\alpha,\delta\rangle} \ . \tag{XIII.39}$$

To evaluate this expression, we write each positive root, α, in terms of the simple roots α_i:

$$\alpha = \sum_i k^i_\alpha \alpha_i \ . \tag{XIII.40}$$

Suppose the Dynkin coefficients of Λ are $\Lambda(h_i) = \Lambda_i$, where $h_i = 2h_{\alpha_i}/\langle\alpha_i, \alpha_i\rangle$. Then we have

$$\dim R = \prod_{\alpha>0} \frac{\sum_i k^i_\alpha(\Lambda_i + 1)\langle\alpha_i, \alpha_i\rangle}{\sum_i k^i_\alpha\langle\alpha_i, \alpha_i\rangle} \ . \tag{XIII.41}$$

The algebras A_n, D_n, E_6, E_7, and E_8 have simple roots all of one size, so for them we can drop the factors of $\langle\alpha_i, \alpha_i\rangle$ in Eq. (XIII.41).

Let us illustrate this marvelous formula with a number of examples. Consider first SU(3), that is, A_2. The simple roots are all the same size so we ignore the factor $\langle\alpha_i, \alpha_i\rangle$. The positive roots are α_1, α_2, and $\alpha_1 + \alpha_2$, which we shall abbreviate here by (1), (2), and (12). Suppose the irreducible representation at hand has a highest weight with Dynkin coefficients (m_1, m_2), then we compute

$$\dim R = \left(\frac{m_1+1}{1}\right)\left(\frac{m_2+1}{1}\right)\left(\frac{m_1+m_2+2}{2}\right). \tag{XIII.42}$$

From this example and the fundamental formula, Eq. (XIII.41), we see that the rule for finding the dimensionality of an irreducible representation may be phrased as follows: *The dimension is a product of factors, one for each positive root of the algebra. Each factor has a denominator which is the number of simple roots which compose the positive root. The numerator is a sum over the simple roots in the positive root, with each simple root contributing unity plus the value of the Dynkin coefficient corresponding to the simple root. If the simple roots are not all the same size, each contribution to the numerator and to the denominator must be weighted by* $\langle\alpha_i, \alpha_i\rangle$.

Let us consider a more complicated application of Weyl's formula. The algebra G_2 has, as we have seen, fourteen roots, of which six are positive. If the simple roots are denoted α_1 and α_2 with the latter being the smaller, then the square of α_1 is three times larger than that of α_2. The positive roots are α_1, α_2, $\alpha_1+\alpha_2, \alpha_1+2\alpha_2, \alpha_1+3\alpha_2$, and $2\alpha_1+3\alpha_2$, which we denote here $(1), (2), (12), (12^2)$, (12^3), and $(1^2 2^3)$. We compute below the dimensions of the representations with the highest weights (0,1) and (1,0), where the first entry pertains to α_1 and the second to α_2.

	$(0,1)$	$(1,0)$
$(1),(2)$	$\frac{3}{3}\frac{2}{1}$	$\frac{3(2)}{3}\frac{1}{1}$
(12)	$\frac{3+2}{3+1}$	$\frac{3(2)+1}{3+1}$
(12^2)	$\frac{3+2+2}{3+1+1}$	$\frac{3(2)+1+1}{3+1+1}$
(12^3)	$\frac{3+2+2+2}{3+1+1+1}$	$\frac{3(2)+1+1+1}{3+1+1+1}$
$(1^2 2^3)$	$\frac{3+3+2+2+2}{3+3+1+1+1}$	$\frac{3(2)+3(2)+1+1+1}{3+3+1+1+1}$
	$\dim\ = 7$	$\dim\ = 14$

As yet another example, we consider SO(10), that is, D_5. The simple roots are numbered so that α_4 and α_5 are the ones which form the fork at the end of the Dynkin diagram. There are 45 roots of which 20 are positive. Below we calculate the dimensionality of the representations with highest weights (1,0,0,0,0), (0,0,0,0,1), and (0,0,0,0,2).

	$(1,0,0,0,0)$	$(0,0,0,0,1)$	$(0,0,0,0,2)$
$(1),(2),(3),(4),(5)$	2	2	3
$(12),(23),(34),(35)$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{4}{2}$
$(123),(234),(235),(345)$	$\frac{4}{3}$	$\frac{4}{3}\frac{4}{3}$	$\frac{5}{3}\frac{5}{3}$
$(1234),(2345),(1235)$	$\frac{5}{4}\frac{5}{4}$	$\frac{5}{4}\frac{5}{4}$	$\frac{6}{4}\frac{6}{4}$
$(12345),(23^2 45)$	$\frac{6}{5}$	$\frac{6}{5}\frac{6}{5}$	$\frac{7}{5}\frac{7}{5}$
$(123^2 45)$	$\frac{7}{6}$	$\frac{7}{6}$	$\frac{8}{6}$
$(12^2 3^2 45)$	$\frac{8}{7}$	$\frac{8}{7}$	$\frac{9}{7}$
	$\dim R = 10$	$\dim R = 16$	$\dim R = 126$

With little effort, we can derive a general formula for the dimensionality of an irreducible representation of SU(n), that is A_{n-1}. In the same notation as above, the roots are $(1), (2), \ldots (12), (23), \ldots (123), (234), \ldots, (123 \ldots n)$. We compute the dimensionality for the representation whose highest weight is $(m_1, m_2, \ldots m_n)$:

$(1), (2), \ldots (n)$	$\frac{m_1+1}{1} \frac{m_2+1}{1} \ldots \frac{m_n+1}{1}$
$(12), (23), \ldots (n-1\ n)$	$\frac{m_1+m_2+2}{2} \frac{m_2+m_3+2}{2} \frac{m_{n-1}+m_n+2}{2}$
$\ldots$	
$(12 \ldots n)$	$\frac{m_1+m_2+\ldots m_n+n}{n}$

It is a simple matter to multiply all these factors to find the dimension of the representation.

We can recognize the correspondence between the Dynkin notation and the more familiar Young tableaux if we start with the fundamental representation, $(1, 0, \ldots 0)$, and take the k-times anti-symmetric product of this representation with itself to obtain $(0, 0, \ldots, m_k = 1, 0 \ldots 0)$. This corresponds to the tableau with one column of k boxes. More generally, $(m_1, m_2, \ldots m_n)$ corresponds to the tableau with m_k columns of k boxes.

References

We follow closely Jacobson's version of the Freudenthal derivation of the Weyl formula, except that we have adopted a less formal language. See JACOBSON,pp. 249–259.

Exercises

1. Determine the dimensionality of the SO(10) representations $(2,0,0,0,0)$ and $(0,1,0,0,0)$.

2. Determine the dimensionality of the E_6 representations $(1,0,0,0,0,0)$ and $(0,0,0,0,0,1)$.

3. Show that the E_6 representation $(11,12,11,12,12,11)$ has a dimensionality divisible by 137.