XII. The Weyl Group

The irreducible representations of SU(2) manifest a very obvious symmetry: for every state with $T_z = m$ there is a state with $T_z = -m$. This symmetry is the source of a more complex symmetry in larger algebras. The SU(2) representations are symmetric with respect to reflection about their centers. The larger algebras have reflection symmetries and the group generated by these reflections is called the **Weyl group**.

Consider an irreducible representation of a simple Lie algebra. Now if α is a root of the algebra, we can consider the SU(2) generated by $e_\alpha, e_{-\alpha}$, and h_α. The representation of the full algebra will in general be a reducible representation of this SU(2). Let M be some particular weight and consider the weights and weight spaces associated with $\ldots, M+\alpha, M, M-\alpha, \ldots$. These together form some reducible representation of the SU(2). Under the SU(2) reflection, this representation is mapped into itself. Moreover, since each weight space has a basis in which each element belongs to a distinct SU(2) representation, it is clear that the reflection will map one weight space into another of the same dimension. What is the relation between the original weight and the one into which it is mapped? This is easily inferred from geometry. The portion of M parallel to α is $\alpha\langle M,\alpha\rangle/\langle\alpha,\alpha\rangle$ and the portion perpendicular to it is then $M - \alpha\langle M,\alpha\rangle/\langle\alpha,\alpha\rangle$. The reflection changes the sign of the former and leaves unchanged the latter. Thus the reflection of the weight can be written

$$S_\alpha(M) = M - 2\frac{\langle M, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha \ , \tag{XII.1}$$

where S_α is an operator acting on the space of weights, H_0^*. It maps weights into weights whose weight spaces are of the same dimension. If we let α range over all the roots of the algebra we get a collection of reflections. By taking all combinations of these reflections applied successively, we obtain the Weyl group.

The 27-dimensional representation of SU(3) provides a good example of the symmetry at hand. The Y=0 subspace contains three SU(2) multiplets, one with T=2, one with T=1, and one with T=0. The Y=1 subspace contains two SU(2) multiplets, one with T=3/2 and one with T=1/2. The Y=2 subspace has T=1. The SU(2) reflection maps the weight diagram into itself, preserving the dimensionality of each weight space.

Rather than consider all the S_α, it turns out that it suffices to consider just those S_α where $\alpha \in \Pi$. These will also generate the full Weyl group. For SU(3) we find

$$\begin{aligned}
S_{\alpha_1}: \quad & \alpha_1 \to -\alpha_1 \\
& \alpha_2 \to \alpha_1 + \alpha_2 = \alpha_3 \\
S_{\alpha_2}: \quad & \alpha_1 \to \alpha_1 + \alpha_2 = \alpha_3 \\
& \alpha_2 \to -\alpha_2 \ .
\end{aligned} \tag{XII.2}$$

The full Weyl group for SU(3) has six elements.

We shall not need to know much about the Weyl group for specific algebras. The utility of the Weyl group is that it enables us to prove quite general propositions without actually having to consider the details of representations since it permits the exploitation of their symmetries.

Let us prove a number of useful facts about the Weyl group. First, the Weyl group is a set of orthogonal transformations on the weight space. Orthogonal transformations are those which preserve the scalar product. It is intuitively clear that reflections have this property. To prove this for the full Weyl group it suffices to prove it for the S_α which generate it. We have

$$\begin{aligned}\langle S_\alpha M, S_\alpha N\rangle &= \langle M - 2\frac{\langle M,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha, N - 2\frac{\langle N,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha\rangle \\ &= \langle M, N\rangle \ . \end{aligned} \tag{XII.3}$$

We know that the Weyl group maps weights into weights, so by taking the adjoint representation, we see that it maps roots into roots. The particular reflections S_α where α is simple have a special property. Certainly $S_\alpha(\alpha) = -\alpha$. For every other positive root, $\beta \in \Sigma^+$, $S_\alpha(\beta)$ is positive. To see this, express

$$\beta = \sum_j k_j \alpha_j \ . \tag{XII.4}$$

Now α is one of the α_j's, say $\alpha = \alpha_1$. Thus

$$\begin{aligned} S_{\alpha_1}(\beta) &= \sum_j k_j\alpha_j - 2\alpha_1 \sum_j k_j \frac{\langle\alpha_j,\alpha_1\rangle}{\langle\alpha_1,\alpha_1\rangle} \\ &= \sum_{j>1} k_j\alpha_j + \alpha_1 \times \text{something} \ . \end{aligned} \tag{XII.5}$$

Since $\beta \neq \alpha = \alpha_1$, some $k_j \neq 0$ for $j > 1$. Thus the root $S_\alpha(\beta)$ has some positive coefficient in its expansion in terms of simple roots. But this is enough to establish that all the coefficients are positive and hence so is the root.

The Weyl group provides the means to prove the relation used in the preceding section, that the Dynkin coefficients of $\delta = \frac{1}{2}\sum_{\alpha>0}\alpha$ are all unity. Let α_i be one of the simple roots. By the orthogonality of the Weyl reflections, $\langle S_{\alpha_i}\delta, \alpha_i\rangle = \langle\delta, -\alpha_i\rangle$. On the other hand, S_{α_i} interchanges all the positive roots except α_i itself, so $S_{\alpha_i}\delta = \delta - \alpha_i$. Thus

$$
\begin{aligned}
\langle \delta - \alpha_i, \alpha_i \rangle &= \langle \delta, -\alpha_i \rangle \\
2\langle \delta, \alpha_i \rangle &= \langle \alpha_i, \alpha_i \rangle
\end{aligned}
\tag{XII.6}
$$

as we wished to show.

Finally, consider all the weights M' which can be obtained by acting on the weight M with an element $S \in W$, the Weyl group. We claim that the M' which is the highest has Dynkin coefficients which are all non-negative. Suppose M^* is the highest of these weights SM, and further suppose that the Dynkin coefficient $2\langle M^*, \alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle < 0$. Then $S_{\alpha_i} M^* = M^* - 2\alpha_i \langle M^*, \alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle$ is an even higher weight, providing a contradiction.

References

The Weyl group is covered by JACOBSON, pp. 240–243.

Exercise

1. Find the elements of the Weyl group for G_2 and their multiplication table.