XI. Casimir Operators and Freudenthal's Formula

One of the most familiar features of the analysis of SU(2) is the existence of an operator $T^2 = T_x^2 + T_y^2 + T_z^2$ which commutes with all the generators, T_x, T_y, and T_z. It is important to note that T^2 really has meaning only for representations, and not as an element of the Lie algebra since products like $t_x t_x$ are not defined for the algebra itself. Products like $T_x T_x$ are defined for representations since then T_x is a linear transformation of a vector space into itself, and can be applied twice. We seek here the generalization of T^2 for an arbitrary simple Lie algebra.

It is well-known that $T^2 = \frac{1}{2}(T_+T_- + T_-T_+) + T_zT_z$. This is the form which is easiest to relate to the forms we have used to describe Lie algebras in general. We might guess that the generalization will be roughly of the form

$$\mathcal{C} = \sum_{j,k} H_j M_{jk} H_k + \sum_{\alpha \neq 0} E_\alpha E_{-\alpha} \tag{XI.1}$$

where $H_j = H_{\alpha_j}$ and the α_j are a basis of simple roots. The matrix M is to be determined by requiring that $\mathcal{C}$ commute with all the generators of the algebra. The normalizations of the e_α are chosen so $(e_\alpha, e_\beta) = \delta_{\alpha,-\beta}$, and thus

$$[e_\alpha, e_{-\alpha}] = h_\alpha \ , \tag{XI.2a}$$

$$[E_\alpha, E_{-\alpha}] = H_\alpha \ . \tag{XI.2b}$$

Let us define $N_{\alpha\beta}$ by

$$[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta} = -N_{\beta\ \alpha} e_{\alpha+\beta} . \tag{XI.3}$$

It is clear that $\mathcal{C}$ in Eq. (XI.1) already commutes with all the generators of the Cartan subalgebra since $[H_i, H_j] = 0$ and $[E_\alpha E_{-\alpha}, H_i] = 0$. It remains to calculate the commutator of $\mathcal{C}$ with E_β. We begin with the second term in $\mathcal{C}$:

$$\begin{aligned}\left[\sum_{\alpha\neq 0} E_\alpha E_{-\alpha}, E_\beta\right] &= \sum_{\substack{\alpha\neq 0\\ \alpha\neq\beta}} E_\alpha N_{-\alpha,\beta} E_{\beta-\alpha}\\ &\quad + \sum_{\substack{\alpha\neq 0\\ \alpha\neq -\beta}} N_{\alpha\beta} E_{\alpha+\beta} E_{-\alpha} + E_\beta H_{-\beta} + H_{-\beta} E_\beta \ .\end{aligned} \tag{XI.4}$$

We can obtain the necessary relation between the coefficients $N_{\alpha\beta}$ using the invariance of the Killing form:

$$\begin{aligned}(e_\alpha, [e_\beta, e_\gamma]) &= N_{\beta,\gamma}\, \delta_{-\alpha,\beta+\gamma} = -N_{-\alpha-\beta,\beta}\ \delta_{\alpha+\beta,-\gamma}\\ &= ([e_\alpha, e_\beta], e_\gamma)\\ &= N_{\alpha,\beta} \delta_{\alpha+\beta,-\gamma}\\ N_{\alpha,\beta} &= -N_{-\alpha-\beta,\beta} \ .\end{aligned} \tag{XI.5}$$

Thus we have

$$\begin{aligned}\sum_{\substack{\alpha\neq 0\\ \alpha\neq -\beta}} N_{\alpha\beta} E_{\alpha+\beta} E_{-\alpha} &= \sum_{\substack{\alpha'\neq 0\\ \alpha'\neq\beta}} N_{\alpha'-\beta,\beta} E_{\alpha'} E_{\beta-\alpha'}\\ &= \sum_{\substack{\alpha'\neq 0\\ \alpha'\neq\beta}} -N_{-\alpha',\beta} E_{\alpha'} E_{\beta-\alpha'} \ ,\end{aligned} \tag{XI.6}$$

so the piece of the commutator of E_β with $\mathcal{C}$ we have calculated is

$$\left[\sum_{\alpha\neq 0} E_\alpha E_{-\alpha}, E_\beta\right] = E_\beta H_{-\beta} + H_{-\beta} E_\beta \ . \tag{XI.7}$$

We now arrange the matrix M so that the remainder of the commutator of $\mathcal{C}$ with E_β just cancels this.

$$\left[\sum_{j,k} H_j M_{jk} H_k, E_\beta\right] = \sum_{j,k} \left[\langle \alpha_k, \beta\rangle H_j M_{jk} E_\beta + \langle \beta, \alpha_j\rangle M_{jk} E_\beta H_k\right] \ . \tag{XI.8}$$

Now suppose that β has the expansion in terms of simple roots

$$\beta = \sum_l k_l \alpha_l \ . \tag{XI.9}$$

Then the coefficients are given by

$$k_l = \sum_j \langle \beta, \alpha_j\rangle \mathcal{A}^{-1}_{jl} \tag{XI.10}$$

where the matrix $\mathcal{A}$ is

$$\mathcal{A}_{pq} = \langle \alpha_p, \alpha_q\rangle \tag{XI.11}$$

and $\mathcal{A}^{-1}$ is its inverse. Now

$$
\begin{aligned}
H_\beta &= \sum_l k_l H_l \\
&= \sum_{j,k} \langle \beta, \alpha_j \rangle \mathcal{A}^{-1}_{jk} H_k \ .
\end{aligned} \tag{XI.12}
$$

Thus if we take the matrix M to be $\mathcal{A}^{-1}$, then the second portion of the commutator is

$$
\left[\sum_{j,k} H_j \mathcal{A}^{-1}_{jk} H_k, E_\beta \right] = H_\beta E_\beta + E_\beta H_\beta \ , \tag{XI.13}
$$

just cancelling the first part. Altogether then

$$
\mathcal{C} = \sum_{j,k} H_j \mathcal{A}^{-1}_{jk} H_k + \sum_{\alpha \neq 0} E_\alpha E_{-\alpha} \ . \tag{XI.14}
$$

Consider SU(2) as an example. Our standard commutation relations are

$$
[t_z, t_+] = t_+; \quad [t_z, t_-] = -t_-; \quad [t_+, t_-] = 2t_z \ , \tag{XI.15}
$$

from which we find

$$
(t_+, t_-) = \mathrm{Tr}\,\mathrm{ad}\,t_+ \mathrm{ad}\,t_- = 4 \ . \tag{XI.16}
$$

Thus to obtain the normalization we have used in deriving the Casimir operator, we must set

$$
t'_+ = \frac{1}{2} t_+ \ ; \quad t'_- = \frac{1}{2} t_- \ . \tag{XI.17}
$$

so that

$$
(t'_+, t'_-) = 1 \ . \tag{XI.18}
$$

Now we regard t'_+ as the e_α. The corresponding h_α is accordingly

$$h_\alpha = [t'_+, t'_-] = \frac{1}{2} t_z. \tag{XI.19}$$

It is straightforward to compute

$$\langle \alpha, \alpha \rangle = (h_\alpha, h_\alpha) = \frac{1}{4}(t_z, t_z) = \frac{1}{2} \; . \tag{XI.20}$$

It follows that the 1×1 matrix $M = \mathcal{A}^{-1}$ is simply 2. Altogether then, we find

$$\begin{aligned} \mathcal{C} &= 2H_\alpha H_\alpha + E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha \\ &= \frac{1}{2} T_z T_z + \frac{1}{4}(T_+ T_- + T_- T_+) \; . \end{aligned} \tag{XI.21}$$

This differs from the conventional SU(2) Casimir operator by an overall factor of $\frac{1}{2}$, a result simply of our need to establish some initial normalization in Eq. (XI.1). The importance of the Casimir operator is that since it commutes with all the generators, including the raising and lowering operators, it has the same value on every vector of an irreducible representation, since every such vector can be obtained by applying lowering operators to the highest weight vector. In fact, we can find the value of the Casimir operator on an irreducible representation by considering its action on the highest weight vector. Suppose the highest weight is Λ and that ϕ_Λ is a vector with this weight. Then for every positive root α we know that $E_\alpha \phi_\Lambda = 0$ since otherwise it would have weight $\Lambda + \alpha$. On the other hand, we can compute $E_\alpha E_{-\alpha} \phi_\Lambda = (E_{-\alpha} E_\alpha + H_\alpha) \phi_\Lambda = \Lambda(h_\alpha) \phi_\Lambda = \langle \Lambda, \alpha \rangle \phi_\Lambda$, if α is positive. Thus we have

$$\begin{aligned} \mathcal{C} \phi_\Lambda &= \sum_{j,k} H_j \mathcal{A}^{-1}_{jk} H_k \phi_\Lambda + \sum_{\alpha > 0} \langle \Lambda, \alpha \rangle \phi_\Lambda \\ &= \sum_{j,k} \langle \Lambda \alpha_j \rangle \mathcal{A}^{-1}_{jk} \langle \Lambda, \alpha_k \rangle \phi_\Lambda + \sum_{\alpha > 0} \langle \Lambda, \alpha \rangle \phi_\Lambda \; . \end{aligned} \tag{XI.22}$$

Thus on this irreducible representation

$$\begin{aligned}\mathcal{C} &= \sum_{j,k}\langle\Lambda\alpha_j\rangle\mathcal{A}^{-1}_{jk}\langle\Lambda,\alpha_k\rangle + \sum_{\alpha>0}\langle\Lambda,\alpha\rangle \\ &= \langle\Lambda,\Lambda\rangle + \langle\Lambda,2\delta\rangle\end{aligned} \tag{XI.23}$$

where we have introduced the element of H_0^*

$$\delta = \frac{1}{2}\sum_{\alpha>0}\alpha \ . \tag{XI.24}$$

A few comments are in order concerning normalizations. We have derived our scalar product from the Killing form. As we saw in Chapter IV, all invariant bilinear forms are proportional to the Killing form if the algebra is simple. Suppose we define a second form by

$$(x,y)' = c(x,y) = c\mathrm{Tr}\,\mathrm{ad}\,x\mathrm{ad}\,y \ . \tag{XI.22}$$

Now since h_ρ is defined by

$$(h_\rho,k) = \rho(k) \tag{XI.23}$$

we define h'_ρ by

$$(h'_\rho,k)' = \rho(k) \tag{XI.24}$$

so that

$$h'_\rho = \frac{1}{c}h_\rho \tag{XI.25}$$

and

$$\langle \rho, \tau \rangle' \equiv (h'_\rho, h'_\tau)' = \frac{1}{c}\langle \rho, \tau \rangle \ . \tag{XI.26}$$

We now compare the commutation relations as expressed using the two different scalar products. We have

$$[h_\alpha, e_\beta] = \beta(h_\alpha)e_\beta = (h_\beta, h_\alpha)e_\beta = \langle \alpha, \beta \rangle e_\beta \tag{XI.27}$$

which becomes

$$[h'_\alpha, e_\beta] = \beta(h'_\alpha)e_\beta = (h'_\alpha, h'_\beta)' e_\beta = \langle \alpha, \beta \rangle' e_\beta \ . \tag{XI.28}$$

Thus the commutation relations look the same for this new scalar product. A new Casimir operator (which is just a multiple of the old one) can be chosen so that its value is just $\langle \Lambda, \Lambda + 2\delta \rangle'$. In this way, we can choose a scalar product with any desired normalization and have the computations go through just as before. For some applications, it is traditional to use a scalar product which gives the largest root a length squared equal to 2. We indicate this scalar product by $\langle\ ,\ \rangle_2$.

One particular way to choose an invariant bilinear form is to take the trace of two representation matrices. That is, if ϕ is a representation $\phi(e_\alpha) = E_\alpha$, *etc.*, then we define

$$((x, y)) = \mathrm{Tr}\, \phi(x)\phi(y) \ . \tag{XI.29}$$

The invariance of this form follows from the invariance of traces under cyclic permutation. We know then that $((\ ,\ ))$ is proportional to the Killing form and to the form $(\ ,\ )_2$ which yields $\langle\ ,\ \rangle_2$. The constant of proportionality to the latter is called the **index of the representation**, l_ϕ:

$$((x, y)) = l_\phi (x, y)_2 \ . \tag{XI.30}$$

Now we can evaluate l_ϕ by considering $\mathcal{C}$ with the normalization appropriate to $(\ ,\ )_2$. For a representation with highest weight Λ, $\mathcal{C} = \langle\Lambda, \Lambda + 2\delta\rangle_2$. If we take $\mathrm{Tr}\,\mathcal{C}$, we get $N_\Lambda\langle\Lambda, \Lambda+2\delta\rangle_2$, where N_Λ is the dimension of the representation. On the other hand, replacing $((\ ,\ ))$ by $l_\phi(\ ,\ )_2$ yields $l_\phi N_{adj}$ where N_{adj} is the dimension of the algebra, that is, the dimension of the adjoint representation. Thus

$$l_\phi = \frac{N_\Lambda\langle\Lambda, \Lambda+2\delta\rangle_2}{N_{adj}} \ . \tag{XI.31}$$

We shall see some applications of the index in later chapters.

One particular application of the Casimir operator is in deriving **Freudenthal's Recursion Formula** for the dimensionality of a weight space. Previously, we developed an algorithm for determining all the weights of an irreducible representation, but without ascertaining the dimensionality of each weight space, an omission which we now rectify. Subsequently, this result will be used to derive Weyl's formula for the dimension of an irreducible representation.

We consider an irreducible representation whose highest weight is Λ and seek the dimensionality of the space with weight M. Now we know the constant value of $\mathcal{C}$ on the whole carrier space of the representation, so we can calculate the trace of $\mathcal{C}$ restricted to the space with weight M:

$$\mathrm{Tr}_M\mathcal{C} = n_M\langle\Lambda, \Lambda+2\delta\rangle \ . \tag{XI.32}$$

Here n_M is the dimensionality of the space with weight M, that is, the quantity we wish to compute. Now we calculate the same quantity another way. The first part of $\mathcal{C}$ gives us

$$\begin{aligned}\mathrm{Tr}_M\sum_{j,k} H_j\mathcal{A}^{-1}_{jk}H_k &= \sum_{j,k}\langle\alpha_j, M\rangle\mathcal{A}^{-1}_{jk}\langle\alpha_k, M\rangle n_M \\ &= n_M\langle M, M\rangle \ . \end{aligned}\tag{XI.33}$$

What remains is

$$\mathrm{Tr}_M \sum_{\alpha>0} (E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha) \ , \tag{XI.34}$$

where our normalization is $(e_\alpha, e_{-\alpha}) = 1$ so

$$[E_\alpha, E_{-\alpha}] = H_\alpha, \qquad [H_\alpha, E_\alpha] = \langle \alpha, \alpha \rangle E_\alpha \ . \tag{XI.35}$$

Now usually for SU(2) we have

$$[T_+, T_-] = 2T_z, \qquad [T_z, T_+] = T_+ \tag{XI.36}$$

and

$$T^2 = T_z^2 + \frac{1}{2}[T_+ T_- + T_- T_+] \ . \tag{XI.37}$$

We want to exploit our understanding of SU(2) so we consider the SU(2) generated by $E_\alpha, E_{-\alpha}$, and H_α. The correspondence which gives the right normalization is

$$T_z = \frac{H_\alpha}{\langle \alpha, \alpha \rangle}, \qquad T_+ = \sqrt{\frac{2}{\langle \alpha, \alpha \rangle}} E_\alpha \ , \qquad T_- = \sqrt{\frac{2}{\langle \alpha, \alpha \rangle}} E_{-\alpha} \ . \tag{XI.38}$$

Now consider the weight space associated with the weight M. The full irreducible representation contains, in general, many irreducible representations of the SU(2) associated with the root α. We can pick a basis for the weight space for weight M so that each basis vector belongs to a distinct irreducible representation of the SU(2). Each such irreducible representation is characterized by an integer or half-integer, t which is the maximal eigenvalue of T_z. Moreover, the usual Casimir operator, Eq. (XI.37), then has the value $t(t+1)$. If ϕ_t is an appropriate weight vector then we can write

$$\left[\frac{H_\alpha H_\alpha}{\langle\alpha,\alpha\rangle^2}+\frac{1}{\langle\alpha,\alpha\rangle}E_\alpha E_{-\alpha}+\frac{1}{\langle\alpha,\alpha\rangle}E_{-\alpha}E_\alpha\right]\phi_t=t(t+1)\phi_t \tag{XI.39}$$

so that

$$\left[E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha\right]\phi_t=\langle\alpha,\alpha\rangle t(t+1)\phi_t-\frac{\langle M,\alpha\rangle^2}{\langle\alpha,\alpha\rangle}\phi_t \tag{XI.40}$$

where we have used the fact that ϕ_t has weight M. The particular weight vector ϕ_t belongs to a series of weight vectors which form a basis for an irreducible representation of the SU(2) described above. Suppose the highest weight in this series is $M+k\alpha$, and the associated weight vector is $\phi_{M+k\alpha}$. Then

$$T_z\phi_{M+k\alpha}=t\phi_{M+k\alpha}=\frac{H_\alpha}{\langle\alpha,\alpha\rangle}\phi_{M+k\alpha}=\frac{\langle\alpha,M+k\alpha\rangle}{\langle\alpha,\alpha\rangle}\phi_{M+k\alpha}\ . \tag{XI.41}$$

Thus we can indentify

$$t=\frac{\langle\alpha,M+k\alpha\rangle}{\langle\alpha,\alpha\rangle}\ . \tag{XI.42}$$

This result can now be inserted in Eq. (XI.39) to find

$$\left[E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha\right]\phi_t=[k(k+1)\langle\alpha,\alpha\rangle+(2k+1)\langle M,\alpha\rangle]\phi_t\ . \tag{XI.43}$$

Each of our basis vectors for the space with weight M has associated with it a value of k. In fact, more than one basis vector may have the same value of k. A moment's reflection reveals that the number of such basis vectors is precisely the difference between the dimension of the space with weight $M+(k+1)\alpha$ and that with weight $M+k\alpha$. Accordingly, we write

$$\begin{aligned}&\mathrm{Tr}_M\sum_{\alpha>0}\left[E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha\right]\\&=\sum_{k\geq 0}\left(n_{M+k\alpha}-n_{M+(k+1)\alpha}\right)[k(k+1)\langle\alpha,\alpha\rangle+(2k+1)\langle M,\alpha\rangle]\\&=n_M\langle M,\alpha\rangle+\sum_{k=1}^{\infty}n_{M+k\alpha}[2k\langle\alpha,\alpha\rangle+2\langle M,\alpha\rangle]\ .\end{aligned} \tag{XI.44}$$

Combining this result with Eqs. (XI.32) and (XI.33), we find

$$\begin{aligned}\mathrm{Tr}_M\mathcal{C} &= n_M\langle\Lambda,\Lambda+2\delta\rangle\\ &= n_M\langle M,M\rangle\\ &\quad+\sum_{\alpha>0}\left[n_M\langle M,\alpha\rangle+\sum_{k=1}^{\infty}2n_{M+k\alpha}\langle M+k\alpha,\alpha\rangle\right] .\end{aligned} \tag{XI.45}$$

This relation may be solved for n_M in terms of the higher weights:

$$n_M=\frac{\sum_{\alpha>0}\sum_{k=1}^{\infty}2n_{M+k\alpha}\langle M+k\alpha,\alpha\rangle}{\langle\Lambda+M+2\delta,\Lambda-M\rangle} . \tag{XI.46}$$

The highest weight always has a space of dimension one. Using Freudenthal's formula, Eq. (XI.45), we can determine the dimensionality of the spaces of the lower weights successively. The denominator is most easily evaluated by expressing the first factor by its Dynkin coefficients and the second factor in terms of simple roots. As we shall demonstrate later, the Dynkin coefficients of δ are $(1,1,\ldots)$. Since Λ and M appear in the table of weights expressed in Dynkin coefficients, it is easy then to find $\Lambda+M+2\delta$. Similarly, $\Lambda-M$ is easily determined from the table of weights. If the Dynkin coefficients of $\Lambda+M+2\delta$ are

$$(\Lambda+M+2\delta)=(a_1,a_2,\ldots) \tag{XI.47}$$

and

$$\Lambda-M=\sum_i k_i\alpha_i \tag{XI.48}$$

then

$$\langle\Lambda+M+2\delta,\Lambda-M\rangle=\sum_i a_ik_i\frac{1}{2}\langle\alpha_i,\alpha_i\rangle . \tag{XI.49}$$

The quantity in the numerator can be evaluated fairly easily as well. For a given positive root, α, we check to see whether $M' = M + k\alpha$ is also a weight. If it is, then M' and M lie in a string of weights separated by α's. Let the highest and lowest weights in the string be $M' + p\alpha$ and $M' - m\alpha$. Then by Eq. (V.22),

$$2\langle M', \alpha\rangle = (m - p)\langle \alpha, \alpha\rangle \ . \tag{XI.50}$$

Let us consider an application of the Freudenthal formula to SU(3). The 27-dimensional representation has the Dynkin representation (2,2). It is the representation with highest weight in the product of two adjoint representations.

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\boxed{2\ \ 2}$$

$$\boxed{0\ \ 3} \qquad \boxed{3\ \ 0}$$

$$\boxed{-2\ \ 4} \qquad \boxed{1\ \ 1} \qquad \boxed{4\ \ -2}$$

$$\boxed{-1\ \ 2} \qquad \boxed{2\ \ -1}$$

$$\boxed{-3\ \ 3} \qquad \boxed{0\ \ 0} \qquad \boxed{3\ \ -3}$$

$$\boxed{-2\ \ 1} \qquad \boxed{1\ \ -2}$$

$$\boxed{-4\ \ 2} \qquad \boxed{-1\ \ -1} \qquad \boxed{2\ \ -4}$$

$$\boxed{-3\ \ 0} \qquad \boxed{0\ \ -3}$$

$$\boxed{-2\ \ -2}$$

First note that the weights (0,3) and (3,0) clearly are one dimensional since there is only one way to reach them by lowering operators from the highest weight. Thus the first ambiguity is for the weight (1,1). We compute with M=(1,1)

$$\begin{aligned} \Lambda + M + 2\delta &= (5,5) \\ \Lambda - M &= \alpha_1 + \alpha_2 \\ \langle \Lambda + M + 2\delta, \Lambda - M \rangle &= 10 \cdot \frac{1}{2} \langle \alpha_1, \alpha_1 \rangle \end{aligned} \tag{XI.51}$$

where we have used the relation $\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle$.

To compute the numerator, we remember that there are three positive roots α_1, α_2, and $\alpha_1 + \alpha_2$. The weight preceding (1,1) by α_1 is (3,0). For this weight, m=3 and p=0. Similarly for (0,3) which precedes (1,1) by α_2. The weight preceding (1,1) by $\alpha_1 + \alpha_2$ is (2,2). For this weight, m=4 and p =0. Remembering that all the roots of SU(3) have the same size, we have for the numerator $(3+3+4)\langle \alpha_1, \alpha_1 \rangle$ and thus

$$N_{(1,1)} = 2 \ . \tag{XI.52}$$

For the weight (2,-1) we have

$$\begin{aligned} \Lambda + M + 2\delta &= (6,3) \\ \Lambda - M &= \alpha_1 + 2\alpha_2 \\ \langle \Lambda + M + 2\delta, \Lambda - M \rangle &= (6+6) \cdot \frac{1}{2} \cdot \langle \alpha_1, \alpha_1 \rangle \ . \end{aligned} \tag{XI.53}$$

The numerator receives a contribution of $4\langle \alpha_1, \alpha_1 \rangle$ from the root α_1. For the root α_2 there are two preceding weights to consider. The weight (0,3) contributes $3\langle \alpha_1, \alpha_1 \rangle$. The weight (1,1) contributes $2(2-1)\langle \alpha_1, \alpha_1 \rangle$, where the factor of two comes from the dimensionality of the weight space for (1,1). For the root $\alpha_1 + \alpha_2$, there is a contribution $3\langle \alpha_1, \alpha_1 \rangle$. Altogether, then,

$$N_{(2,-1)} = \frac{4+3+2+3}{12 \cdot \frac{1}{2}} = 2 \ . \tag{XI.54}$$

References

The derivation of Freudenthal's formula follows that given by JACOBSON, pp. 243–249. The index of a representation is developed by DYNKIN II, see in particular pp. 130–134.

Exercises

1. Find the index of the seven dimensional representation of G_2.

2. Find the dimensionalities of all the weight spaces of the 27-dimensional representation of SU(3).

3. Show that the index of the k dimensional representation of $SU(2)$ is $(k-1)k(k+1)/6$.