X. More on Representations

We continue our discussion of Section IV. As before, a representation is a mapping of the elements of the Lie algebra into linear operators,

$$\begin{aligned} e_\alpha &\rightarrow E_\alpha \\ h_i &\rightarrow H_i \end{aligned} \tag{X.1}$$

which preserves the commutation relations of the Lie algebra. The operators E and H act on a vector space with elements generically denoted ϕ. We can select a basis in which the H's are diagonal.[1] Thus we can write

$$H\phi^M = M(h)\phi^M \tag{X.2}$$

where $M \in H_0^*$ is called a **weight** and ϕ^M is a **weight vector**.

The weights come in sequences with successive weights differing by roots of the Lie algebra. We have seen that if a complete string of roots is $M + p\alpha$, $\ldots M$, $\ldots M - m\alpha$, then (see Eq. (V.22))

$$m - p = 2\frac{\langle M, \alpha \rangle}{\langle \alpha, \alpha \rangle} \; . \tag{X.3}$$

A finite dimensional irreducible representation must have a highest weight, that is, a weight Λ such that every other weight is less than Λ, where the ordering is determined in the usual fashion (That is, we pick a basis of roots and order it. A weight is positive if, when expanded in this ordered basis, the first non-zero coefficient is positive, and we say $M_1 > M_2$ if $M_1 - M_2 > 0$.)

Let $\{\alpha_i\}$ be a basis of simple roots and let Λ be the highest weight of an irreducible representation. Then $\Lambda + \alpha_i$ is not a weight. Thus by Eq. (X.3),

$$\Lambda_i = 2\frac{\langle \Lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \geq 0 \; . \tag{X.4}$$

Each greatest weight, Λ, is thus specified by a set of non-negative integers called **Dynkin coefficients**:

$$\Lambda_i = 2\frac{\langle \Lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \; . \tag{X.5}$$

We could use the inverse of the Cartan matrix to determine the precise expansion of Λ in terms of the simple roots, but this is rarely worthwhile.

Given the Dynkin coefficients of the highest weight, it is easy to determine the full set of weights in the irreducible representation, expressed again in terms of their Dynkin coefficients. The algorithm is similar to that we used to find all the roots of a Lie algebra from its Cartan matrix. Given a weight, M, we need to determine whether $M - \alpha_j$ is also a weight. Since we begin with the highest weight and work down, we know the value of p in Eq. (X.3). We shall keep track of the integers

$$M_i = 2\frac{\langle M, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \; . \tag{X.6}$$

If

$$m_j = p_j + M_j > 0 \ , \tag{X.7}$$

then we know we can subtract the root α_j from the root M to obtain another root. We record the Dynkin coefficients of $M - \alpha_j$ by subtracting from M_i the quantities A_{ji}. This is most easily carried out by writing the Cartan matrix at the top of the computation.

Consider an example for SU(3) (or A_2 in the other notation). Let us determine the weights corresponding to the irreducible representation whose highest weight has Dynkin coefficients (1,0).

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\boxed{1 \ \ 0}$$

$$\boxed{-1 \ \ 1}$$

$$\boxed{0 \ \ -1}$$

The Dynkin coefficients are entered in the boxes and successive rows are obtained by subtracting the appropriate row of the Cartan matrix. It is easy to see that the highest weight here can be expanded in simple roots as

$$\Lambda = \tfrac{2}{3}\alpha_1 + \tfrac{1}{3}\alpha_2 \ . \tag{X.8}$$

Thus the weights of this three dimensional representation are

$$\tfrac{2}{3}\alpha_1 + \tfrac{1}{3}\alpha_2 \ ; \quad -\tfrac{1}{3}\alpha_1 + \tfrac{1}{3}\alpha_2 \ ; \quad -\tfrac{1}{3}\alpha_1 - \tfrac{2}{3}\alpha_2 \ .$$

Of course, if we had started with Dynkin coefficients (0,1), we would have obtained a three dimensional representation with weights

$$\tfrac{1}{3}\alpha_1 + \tfrac{2}{3}\alpha_2\ ; \quad \tfrac{1}{3}\alpha_1 - \tfrac{1}{3}\alpha_2\ ; \quad -\tfrac{2}{3}\alpha_1 - \tfrac{1}{3}\alpha_2\ .$$

Actually, we have relied on our previous knowledge of SU(3) to assert that these representations are three dimensional. All we have seen is that there are three different weights. It is often the case that a weight may correspond to more than one (linearly independent) weight vector, so that the **weight space** may be more than one dimensional. Consider for example the SU(3) representation with Dynkin coefficients (1,1), the familiar adjoint representation:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\boxed{1\ \ 1}$$

$$\boxed{-1\ \ 2} \qquad \boxed{2\ \ -1}$$

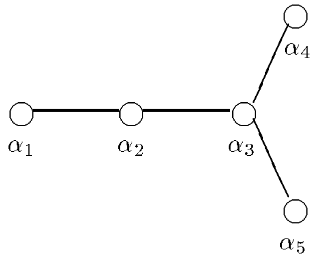
$$\boxed{0\ \ 0}$$

$$\boxed{1\ \ -2} \qquad \boxed{-2\ \ 1}$$

$$\boxed{-1\ \ -1}$$

This representation is eight dimensional. The weight with Dynkin coefficients (0,0) corresponds to a two dimensional space. Indeed, since this is the adjoint representation, we recognize that this space coincides with the Cartan subalgebra. The procedure for determining the dimensionality of a weight space will be discussed later.

As two additional examples, consider the representations of SO(10) (D_5) specified by the Dynkin coefficients (1,0,0,0,0) and (0,0,0,0,1) where the simple roots are numbered:



We have then the schemes:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

$$\boxed{1\ \ 0\ \ 0\ \ 0\ \ 0}$$

$$\boxed{-1\ \ 1\ \ 0\ \ 0\ \ 0}$$

$$\boxed{0\ \ -1\ \ 1\ \ 0\ \ 0}$$

$$\boxed{0\ \ 0\ \ -1\ \ 1\ \ 1}$$

$$\boxed{0\ \ 0\ \ 0\ \ -1\ \ 1} \qquad \boxed{0\ \ 0\ \ 0\ \ 1\ \ -1}$$

$$\boxed{0\ \ 0\ \ 1\ \ -1\ \ -1}$$

$$\boxed{0\ \ 1\ \ -1\ \ 0\ \ 0}$$

$$\boxed{1\ \ -1\ \ 0\ \ 0\ \ 0}$$

$$\boxed{-1\ \ 0\ \ 0\ \ 0\ \ 0}$$

and

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

$$\boxed{0\ \ 0\ \ 0\ \ 0\ \ 1}$$

$$\boxed{0\ \ 0\ \ 1\ \ 0\ \ -1}$$

$$\boxed{0\ \ 1\ \ -1\ \ 1\ \ 0}$$

$$\boxed{1\ \ -1\ \ 0\ \ 1\ \ 0} \qquad \boxed{0\ \ 1\ \ 0\ \ -1\ \ 0}$$

$$\boxed{-1\ \ 0\ \ 0\ \ 1\ \ 0} \qquad \boxed{1\ \ -1\ \ 1\ \ -1\ \ 0}$$

$$\boxed{-1\ \ 0\ \ 1\ \ -1\ \ 0} \qquad \boxed{1\ \ 0\ \ -1\ \ 0\ \ 1}$$

$$\boxed{-1\ \ 1\ \ -1\ \ 0\ \ 1} \qquad \boxed{1\ \ 0\ \ 0\ \ 0\ \ -1}$$

$$\boxed{0\ \ -1\ \ 0\ \ 0\ \ 1} \qquad \boxed{-1\ \ 1\ \ 0\ \ 0\ \ -1}$$

$$\boxed{0\ \ -1\ \ 1\ \ 0\ \ -1}$$

$$\boxed{0\ \ 0\ \ -1\ \ 1\ \ 0}$$

$$\boxed{0\ \ 0\ \ 0\ \ -1\ \ 0}$$

New representations may be obtained by taking products of representations. This procedure when applied to SU(2) is the familiar addition of angular momentum in quantum mechanics. Suppose we have two representations, $x \to X^{(1)}$ and $x \to X^{(2)}$ where $x \in L$ and $X^{(1)}$ and $X^{(2)}$ are linear operators on vector spaces whose basis elements will be denoted by ϕ and η respectively:

$$X^{(1)}\phi_i = \sum_j X^{(1)}_{ij}\phi_j \tag{X.9a}$$

$$X^{(2)}\eta_i = \sum_j X^{(2)}_{ij}\eta_j \ . \tag{X.9b}$$

Here $X^{(1)}_{ij}$ and $X^{(2)}_{ij}$ are coefficients, not operators. We can define a product representation on the product space whose basis elements are of the form $\phi_i \otimes \eta_j$ as follows:

$$X\phi_i \otimes \eta_j = \sum_k X^{(1)}_{ik}\phi_k \otimes \eta_j + \sum_l \phi_i \otimes X^{(2)}_{jl}\eta_l \ . \tag{X.10}$$

For the rotation group, we might write $J = L + S$ and ϕ and η might represent the spatial and spin parts of the wave function.

If x is an element of the Cartan subalgebra we indicate it by h and its representation by H. If ϕ and η are weight vectors with weights $M^{(1)}$ and $M^{(2)}$, then $\phi\otimes\eta$ is a weight vector of the product representation with weight $M^{(1)}+M^{(2)}$, as we see from Eq. (X.10). If the highest weights of the two representations are $\Lambda^{(1)}$ and $\Lambda^{(2)}$, then the highest weight of the product representation is $\Lambda^{(1)} + \Lambda^{(2)}$.

Our construction of the weights of an irreducible representation from the Dynkin coefficients of its highest weight shows that all the weights are determined by the highest weight. It is also possible to show that the weight space of the highest weight is always one-dimensional for an irreducible representation. Thus each product of irreducible representations contains one irreducible representation whose highest weight is the sum of the highest weights of the two representations forming it.

As an example, consider again SU(3). The three dimensional representation may be represented by (1,0) and the other three dimensional representation by (0,1). Their product must contain the representation (1,1), which is in fact the eight dimensional representation.

Consider the special case in which the representations being multiplied are identical. The product space can be broken into two parts, a symmetric part with basis $\phi_i \otimes \phi_j + \phi_j \otimes \phi_i$, and an anti-symmetric part, with basis $\phi_i \otimes \phi_j - \phi_j \otimes \phi_i$. If the highest weight of the representation carried by ϕ is Λ, then the highest weight carried by the symmetric space is 2Λ. The anti-symmetric space does not contain the vector with this weight since it is symmetric . The highest weight in the anti-symmetric space is found by taking the sum of the highest and the next-to-highest weights.

Again, a simple example may be taken from SU(3). Consider 3×3 (i.e. $(1,0)\times(1,0)$). The second highest weight in (1,0) is (-1,1). Thus the anti-symmetric space carries the representation whose highest weight is (1,0)+(-1,1)=(0,1). This is the 3^*. The symmetric space carries the (2,0), the 6 of SU(3). In general, however, the product contains more than two irreducible components.

It is possible to extend the anti-symmetrization procedure by taking the n-fold anti-symmetric product of a given representation. It is clear that the three fold anti-symmetric product will contain a representation whose highest weight is the sum of the three highest weights of the irreducible representation from which it is made, and so on. Similarly, the n-fold symmetric product will contain an irreducible representation whose highest weight is n-times the highest weight of the initial irreducible representation.

These procedures are especially easy to apply to A_n, beginning with the **fundamental representation**,$(1,0,\ldots)$. Calculating the weights of this representation, we quickly see that the two-fold anti-symmetrization yields a highest weight $(0,1,0,\ldots)$, the three-fold anti-symmetrization $(0,0,1,0,\ldots)$, and so on.

In fact, combining these operations we can produce any of the irreducible representations of A_n. To produce the representation with highest weight (m_1, m_2, m_3, $\ldots$), we take the m_1-fold symmetric product of $(1,0,0,\ldots)$ and the m_2-fold symmetric product of $(0,1,0,\ldots)$ and form their product. The irreducible representation with highest weight in the product is $(m_1, m_2, 0, \ldots)$. We continue in this fashion to build (m_1, m_2, m_3, $\ldots$).

The representations with Dynkin coefficients all equal to zero except for one entry of unity are called **basic representations**. It is clear that every representation can be formed from basic representations simply using the highest weights of product representations. Moreover, for A_n, we have seen that every basic representation can be obtained from a single fundamental representation.

Consider, on the other hand, B_3, (O(7)). We display the weights of the representations (1,0,0) and (0,0,1).

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\boxed{1 \quad 0 \quad 0}$$

$$\boxed{-1 \quad 1 \quad 0}$$

$$\boxed{0 \quad -1 \quad 2}$$

$$\boxed{0 \quad 0 \quad 0}$$

$$\boxed{0 \quad 1 \quad -2}$$

$$\boxed{1 \quad -1 \quad 0}$$

$$\boxed{-1 \quad 0 \quad 0}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\boxed{0\ \ 0\ \ 1}$$

$$\boxed{0\ \ 1\ \ -1}$$

$$\boxed{1\ \ -1\ \ 1}$$

$$\boxed{-1\ \ 0\ \ 1} \qquad \boxed{1\ \ 0\ \ -1}$$

$$\boxed{-1\ \ 1\ \ -1}$$

$$\boxed{0\ \ -1\ \ 1}$$

$$\boxed{0\ \ 0\ \ -1}$$

We see that the twice anti-symmetric product of (1,0,0) contains (0,1,0), but the three times anti-symmetric product is (0,0,2). Thus we cannot build all the basic representations from (1,0,0). Nor can they all be built from (0,0,1). We must begin with both the (0,0,1) and (1,0,0) to generate all the basic representations.

Analogous considerations establish that a single representation will generate all representations for the C_n algebras, but three initial representations are necessary for the D_n algebras.

Footnote

1. See JACOBSON, p 113.

References

Representations are discussed at length in the appendix to DYNKIN III. For $SU(n)$, Young tableaux are the most effective procedure. They are explained in GEORGI. For a mathematical exposition, see BOERNER. For a very practical exposition, see SCHENSTED.

Exercises

1. Find the weight scheme for the representations $(1,0$ and $(0,1)$ of B_2.

2. Find the weight scheme for $(1,0)$ of G_2.

2. Find the weight scheme for $(1,0)$ and $(0,1)$ of F_4.