# I. SU(2)

A geometrical vector in three-dimensional space can be represented by a column vector whose entries are the x, y, and z components of the vector. A rotation of the vector can be represented by a three-by-three matrix. In particular, a rotation by $\phi$ about the z-axis is given by

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{I.1}$$

For small rotations,

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \approx I - i\phi T_z , \tag{I.2}$$

where $T_z$ is the matrix

$$
\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{I.3}
$$

In a similar fashion we find $T_x$ and $T_y$:

$$
T_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} , \qquad T_y = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} . \tag{I.4}
$$

By direct computation we find that the finite rotations are given as exponentials of the matrices $T_x$, $T_y$, and $T_z$. Thus we have

$$
\exp(-i\phi T_z) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{I.5}
$$

The product of two rotations like $\exp(-i\theta T_y)\exp(-i\phi T_z)$ can always be written as a single exponential, say $\exp(-i\alpha \cdot T)$ where $\alpha \cdot T = \alpha_x T_x + \alpha_y T_y + \alpha_z T_z$. Suppose we set $\exp(-i\alpha \cdot T)\exp(-i\beta \cdot T) = \exp(-i\gamma \cdot T)$ and try to calculate $\gamma$ in terms of $\alpha$ and $\beta$. If we expand the exponentials we find

$$
\begin{aligned}
[1 - i\alpha \cdot t - \tfrac{1}{2}(\alpha \cdot t)^2 + \cdots][1 - i\beta \cdot t - \tfrac{1}{2}(\beta \cdot t)^2 + \cdots] \\
= \left[1 - i(\alpha + \beta) \cdot t - \tfrac{1}{2}((\alpha + \beta) \cdot t)^2 - \tfrac{1}{2}[\alpha \cdot t, \beta \cdot t] + \cdots\right] \\
= \exp\{-i(\alpha + \beta) \cdot t - \tfrac{1}{2}[\alpha \cdot t, \beta \cdot t] + \cdots\} .
\end{aligned} \tag{I.6}
$$

To this order in the expansion, to calculate $\gamma$ we need to know the value of the commutators like $[T_x, T_y]$, but not ordinary products like $T_x T_y$. In fact, this is true to all orders (and is known as the Campbell-Baker-Hausdorff theorem[1]). It is for this reason that we can learn most of what we need to know about Lie groups by studying the commutation relations of the generators (here, the $T$'s). By direct computation we can find the commutation relations for the $T$'s:

$$[T_x, T_y] = iT_z, \quad [T_y, T_z] = iT_x, \quad [T_z, T_x] = iT_y. \tag{I.7}$$

These commutation relations which we obtained by considering geometrical rotations can now be used to form an abstract **Lie algebra**. We suppose there are three quantities $t_x$, $t_y$, and $t_z$ with a **Lie product** indicated by [ , ]

$$[t_x, t_y] = it_z, \quad [t_y, t_z] = it_x, \quad [t_z, t_x] = it_y. \tag{I.8}$$

We consider all linear combinations of the $t$'s and make the Lie product linear in each of its factors and anti-symmetric:

$$[a \cdot t + b \cdot t, c \cdot t] = [a \cdot t, c \cdot t] + [b \cdot t, c \cdot t], \tag{I.9}$$

$$[a \cdot t, b \cdot t] = -[b \cdot t, a \cdot t]. \tag{I.10}$$

It is easy to show that the **Jacobi identity** follows from Eq. (I.8):

$$[a \cdot t, [b \cdot t, c \cdot t]] + [b \cdot t, [c \cdot t, a \cdot t]] + [c \cdot t, [a \cdot t, b \cdot t]] = 0 . \tag{I.11}$$

When we speak of the abstract Lie algebra, the product $[a \cdot t, b \cdot t]$ is not to be thought of as $a \cdot t \; b \cdot t - b \cdot t \; a \cdot t$ , since the product $a \cdot t b \cdot t$ has not been defined. When we represent the algebra by matrices (as we did at the outset), then of course the ordinary product has a well-defined meaning. Nevertheless, by custom we often refer to the Lie product as a commutator.

The abstract Lie algebra derived above from the rotation group displays the features which define Lie algebras in general. A Lie algebra is a vector space, $L$, (above, the linear combinations of the $t$'s) together with a bilinear operation (from $L \times L$ into $L$ ) satisfying

$$
\begin{aligned}
[x_1+x_2,y] &= [x_1,y]+[x_2,y]\ , && x_1,x_2,y\in L\\
[ax,y] &= a\,[x,y]\ , && a\in F,\ x,y\in L\\
[x,y] &= -\,[y,x]\ , && x,y\in L\\
0 &= [x,[y,z]]+[y,[z,x]]+[z,[x,y]]\ , && x,y,z\in L\ . \qquad \text{(I.12)}
\end{aligned}
$$

Here $F$ is the field over which $L$ is a vector space. We shall always take $F$ to be the field of real numbers, $\mathcal{R}$, or the field of complex numbers, $\mathcal{C}$.

Having motivated the formal definition of a Lie algebra, let us return to the specific example provided by the rotation group. We seek the representations of the Lie algebra defined by Eq. (I.8). By a **representation** we mean a set of linear transformations (that is, matrices) $T_x$ , $T_y$ , and $T_z$ with the same commutation relations as the $t$'s. The $T$'s of Eqs. (I.3) and (I.4) are an example in which the matrices are $3\times 3$ and the representation is said to be of dimension three.

We recall here the construction which is familiar from standard quantum mechanics texts. It is convenient to define

$$t_+ = t_x + it_y\ , \qquad t_- = t_x - it_y\ , \tag{I.13}$$

so that the commutation relations become

$$[t_z,t_+] = t_+\ , \quad [t_z,t_-] = -t_-\ , \quad [t_+,t_-] = 2t_z\ . \tag{I.14}$$

We now suppose that the $t$'s are to be represented by some linear transformations: $t_x \rightarrow T_x$,$t_y \rightarrow T_y$, $t_z \rightarrow T_z$. The $T$'s act on some vector space, $V$. We shall in fact construct this space and the $T$'s directly. We start with a single vector, $v_j$ and define the actions of $T_z$ and $T_+$ on it by

$$T_z v_j = jv_j\ , \qquad T_+ v_j = 0\ . \tag{I.15}$$

Now consider the vector $T_- v_j$. This vector is an eigenvector of $T_z$ with eigenvalue $j-1$ as we see from

$$T_zT_-v_j = (T_-T_z - T_-)v_j = (j-1)T_-v_j \ . \tag{I.16}$$

Let us call this vector $v_{j-1} \equiv T_-v_j$. We proceed to define additional vectors sequentially:

$$v_{k-1} = T_-v_k \ . \tag{I.17}$$

If our space, $V$, which is to consist of all linear combinations of the $v$'s, is to be finite dimensional this procedure must terminate somewhere, say when

$$T_-v_q = 0 \ . \tag{I.18}$$

In order to determine $q$, we must consider the action of $T_+$. It is easy to see that $T_+v_k$ is an eigenvector of $T_z$ with eigenvalue $k+1$. By induction, we can show that $T_+v_k$ is indeed proportional to $v_{k+1}$. The constant of proportionality may be computed:

$$\begin{aligned} r_kv_{k+1} =& T_+v_k \\ =& T_+T_-v_{k+1} \\ =& (T_-T_+ + 2T_z)v_{k+1} \\ =& [r_{k+1} + 2(k+1)]v_{k+1} \ . \end{aligned} \tag{I.19}$$

This recursion relation for $r_k$ is easy to satisfy. Using the condition $r_j = 0$, which follows from Eq. (I.15), the solution is

$$r_k = j(j+1) - k(k+1). \tag{I.20}$$

Now we can find the value of $q$ defined by Eq. (I.18):

$$
\begin{aligned}
T_+T_-v_q =&0\\
=&(T_-T_+ + 2T_z)v_q\\
=&[j(j+1) - q(q+1) + 2q]v_q\ .
\end{aligned}
\tag{I.21}
$$

There are two roots, $q = j + 1$, and $q = -j$. The former is not sensible since we should have $q \leq j$. Thus $q = -j$, and $2j$ is integral.

In this way we have recovered the familiar representations of the rotation group, or more accurately, of its Lie algebra, Eq. (I.14). The eigenvalues of $T_z$ range from $j$ to $-j$. It is straightforward to verify that the **Casimir operator**

$$
\begin{aligned}
T^2 =&T_x^2 + T_y^2 + T_z^2\\
=&T_z^2 + \tfrac{1}{2}(T_+T_- + T_-T_+)\ ,
\end{aligned}
\tag{I.22}
$$

has the constant value $j(j+1)$ on all the vectors in $V$:

$$
\begin{aligned}
T^2 v_k =&[k^2 + \tfrac{1}{2}(r_{k-1} + r_k)]v_k\\
=&j(j+1)v_k\ .
\end{aligned}
\tag{I.23}
$$

The $2j+1$ dimensional representation constructed above is said to be **irreducible**. This means that there is no proper subspace of $V$ (that is, no subspace except $V$ itself and the space consisting only of the zero vector) which is mapped into itself by the various $T$'s. A simple example of a **reducible** representation is obtained by taking two irreducible representations on the space $V_1$ and $V_2$, say, and forming the space $V_1 \oplus V_2$. That is, the vectors, $v$, in $V$ are of the form $v = v_1 + v_2$, with $v_i \in V_i$. If $t_z$ is represented by $T_z^1$ on $V_1$ and by $T_z^2$ on $V_2$, we take the representation of $t_z$ on $V$ to be $T_z(v_1 + v_2) = T_z^1 v_1 + T_z^2 v_2$, and so on for the other components. The subspaces $V_1$ and $V_2$ are **invariant** (that is, mapped into themselves) so the representation is reducible.

A less trivial example of a reducible representation occurs in the "addition of angular momentum" in quantum mechanics. Here we combine two irreducible representations by forming the product space $V = V_1 \otimes V_2$. If the vectors $u_{1m}$ and $u_{2n}$ form bases for $V_1$ and $V_2$ respectively, a basis for $V$ is given by the quantities $u_{1m} \otimes u_{2n}$. We define the action of the $T$'s on $V$ by

$$T_z(u_{1m} \otimes u_{2n}) = (T_z^1 u_{1m}) \otimes u_{2n} + u_{1m} \otimes (T_z^2 u_{2n}) \ , \tag{I.24}$$

*etc.* If the maximum value of $T_z^1$ on $V_1$ is $j_1$ and that of $T_z^2$ on $V_2$ is $j_2$, there is an eigenvector of $T_z = T_z^1 + T_z^2$ with eigenvalue $j_1 + j_2$. By applying $T_- = T_-^1 + T_-^2$ repeatedly to this vector, we obtain an irreducible subspace, $U_{j_1+j_2}$, of $V_1 \otimes V_2$. On this space, $T^2 = (j_1+j_2)(j_1+j_2+1)$. Indeed, we can decompose $V_1 \otimes V_2$ into a series of subspaces on which $T^2$ takes the constant value $k(k+1)$ for $|j_1 - j_2| \leq k \leq j_1 + j_2$, that is $V_1 \otimes V_2 = U_{j_1+j_2} \oplus \ldots \oplus U_{|j_1-j_2|}$.

The representation of smallest dimension has $j = 1/2$. Its matrices are $2 \times 2$ and traceless. The matrices for $T_x, T_y$, and $T_z$ are **hermitian** (a hermitian matrix $M$, satisfies $M_{ji}^* = M_{ij}$ where $^*$ indicates complex conjugation). If we consider the real linear combinations of $T_x, T_y$, and $T_z$ we obtain matrices, $T$, which are traceless and hermitian. The matrices $\exp(iT)$ form a group of **unitary** matrices of determinant unity (a matrix is unitary if its adjoint - its complex conjugate transpose - is its inverse). This group is called $SU(2)$, $S$ for "special" ( determinant equal to unity), and $U$ for unitary. The rotations in three dimensions, $O(3)$, have the same Lie algebra as $SU(2)$ but are not identical as groups.

## Footnote

1. See, for example, JACOBSON, pp. 170–174.

## References

This material is familiar from the treatment of angular momentum in quantum mechanics and is presented in all the standard texts on that subject. An especially fine treatment is given in GOTTFRIED.

## Exercises

Define the standard Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad , \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad .$$

1. Prove that $t_x \to \frac{1}{2}\sigma_x, \quad t_y \to \frac{1}{2}\sigma_y$, *etc.* is a representation of SU(2).

2. Prove, if $\alpha \cdot \sigma = \alpha_x\sigma_x + \alpha_y\sigma_y + \alpha_z\sigma_z$, *etc.* then $\alpha \cdot \sigma \beta \cdot \sigma = \alpha \cdot \beta + i\alpha \times \beta \cdot \sigma$.

3. Prove that $\exp(-i\theta\sigma \cdot n/2) = \cos(\theta/2) - in \cdot \sigma \sin(\theta/2)$, where $n \cdot n = 1$.

4. Prove $\exp(-i\theta\sigma \cdot n/2)\sigma \cdot n' \exp(i\theta\sigma \cdot n/2) = \sigma \cdot n''$, where $n \cdot n = n' \cdot n' = 1$ and where $n'' = \cos\theta\ n' + n \cdot n'(1 - \cos\theta)n + \sin\theta\ n \times n'$. Interpret geometrically.

5. Prove $\exp(-i2\pi n \cdot T) = (-1)^{2j}$ where $n \cdot n = 1$ and $T^2 = j(j+1)$.