8. Diophantine approximations, substitutions, and fractals

The aim of this chapter is to associate in a constructive way with a (primitive, unimodular, Pisot type) substitution σ a domain X_σ with fractal boundary, generalizing Rauzy's famous construction of the Rauzy fractal [292]. We thus obtain a geometric representation of such a substitution as a domain exchange and even under some extra hypotheses, as a rotation on a torus (also called a quasi-periodic motion). Two dynamical systems act on the set X_σ: the dynamics (of the shift) corresponds to the exchange of domains, whereas the dynamics of the substitution is given by a Markov endomorphism with structure matrix equal to the incidence matrix of the substitution.

The domain X_σ is not only interesting from the viewpoint of fractal geometry but also in the sense of ergodic theory and number theory (see for instance [292, 186, 188] and Chap. 7). We illustrate this study with some substitutions such as the "modified Jacobi-Perron substitutions". We also end each section by evoking the results in the one-dimensional case corresponding to those obtained in the two-dimensional case. We end this chapter by surveying some applications of these constructions in Diophantine approximation. Most of the results of this chapter are issued from [37, 38], where the interested reader can find the corresponding proofs.

8.1 Substitutions and domains with fractal boundary

The aim of this section is to fix the framework in which we will be working in this chapter. We introduce in particular the domain of $\mathbb{R}^2$ with fractal boundary associated with a substitution over a three-letter alphabet.

8.1.1 First definitions and some examples

Let $\mathcal{A}^\star = \bigcup_{n=0}^{\infty} \{1,2,3\}^n$ be the set of finite words over the alphabet $\mathcal{A} = \{1,2,3\}$. Let us recall that a *substitution* in an endomorphism σ from $\mathcal{A}^*$ to $\mathcal{A}^*$, that is:

$$\forall\, W = w_1 \cdots w_k \in \mathcal{A}^*, \quad \sigma(W) = \sigma(w_1)\sigma(w_2)\cdots\sigma(w_k).$$

Therefore, the substitution σ is determined by $\sigma(i)$, for $i = 1, 2, 3$.

Matrix of substitution. There is a convenient way of associating with a substitution a map in $\mathbb{Z}^3$, by abelianization of the free monoid. Let $f : \mathcal{A}^* \to \mathbb{Z}^3$ denote the *canonical homomorphism* (also called the *abelianization map*) associated with σ. If $\{\boldsymbol{e}_i | i = 1, 2, 3\}$ is the canonical basis of $\mathbb{Z}^3 \subset \mathbb{R}^3$, the map f is defined as follows:

$$\begin{aligned} f(i) &= \boldsymbol{e}_i, \quad i = 1, 2, 3, \\ f(\varepsilon) &= \mathbf{0}, \\ f(W) &= \textstyle\sum_{i=1}^{k} f(w_i), \text{ for every } W = w_1 \cdots w_k \in \mathcal{A}^*. \end{aligned}$$

For any substitution $\sigma : \mathcal{A}^* \to \mathcal{A}^*$, let $\mathbf{M}_\sigma$ be the 3×3 matrix:

$$\mathbf{M}_\sigma = [f(\sigma(1)), f(\sigma(2)), f(\sigma(3))] .$$

Lemma 8.1.1. *The commutative relation holds:*

$$\begin{array}{ccc} \mathcal{A}^* & \xrightarrow{\sigma} & \mathcal{A}^* \\ f \downarrow & & \downarrow f \\ \mathbb{Z}^3 & \xrightarrow{L_\sigma} & \mathbb{Z}^3 \end{array} .$$

The matrix L_σ is called the *matrix of* σ or *abelianization of* σ.

Let us introduce the families of substitutions we will consider in this chapter.

Example 8.1.1 (Rauzy substitution [292]). The *Rauzy substitution* is defined by:

$$\sigma(1) = 12, \ \sigma(2) = 13, \ \sigma(3) = 1.$$

The matrix of this substitution is

$$\mathbf{M}_\sigma = \begin{bmatrix} 1\,1\,1 \\ 1\,0\,0 \\ 0\,1\,0 \end{bmatrix} .$$

Example 8.1.2 (Modified Jacobi-Perron substitution [188]). For every $a \in \mathbb{N}^+$ and $\delta \in \{0, 1\}$, let us define the substitution $\sigma_{(a,\delta)}$ as follows:

$$\sigma_{(a,0)} : \begin{array}{l} 1 \longrightarrow \overbrace{11 \cdots 1}^{a \ times} 2 \\ 2 \longrightarrow 3 \\ 3 \longrightarrow 1 \end{array} , \quad \sigma_{(a,1)} : \begin{array}{l} 1 \longrightarrow \overbrace{11 \cdots 1}^{a \ times} 3 \\ 2 \longrightarrow 1 \\ 3 \longrightarrow 2 \end{array} .$$

The substitutions $\sigma_{(a,\delta)}$, for $a \in \mathbb{N}^+$ and $\delta \in \{0, 1\}$, are called *modified Jacobi-Perron substitutions* and are connected in a natural way with the modified Jacobi-Perron algorithm: for more details concerning this algorithm, see Sec. 8.6.

For every $a \in \mathbb{N}^+$, the matrices of the modified Jacobi-Perron substitutions $\sigma_{(a,0)}$ and $\sigma_{(a,1)}$ are the following:

$$\mathbf{M}_{\sigma_{(a,0)}} = \begin{bmatrix} a & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{\sigma_{(a,1)}} = \begin{bmatrix} a & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Definition 8.1.1. *A substitution $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ is said to be* unimodular and of Pisot type *if*

1. $\det \mathbf{M}_\sigma = \pm 1$ *(*unimodular condition*),*
*2. the eigenvalues λ,λ' and λ'' of $\mathbf{M}_\sigma$ satisfy $\lambda > 1 > \mid \lambda' \mid, \mid \lambda'' \mid$ (*Pisot condition*).*

Throughout this chapter, substitutions will always be unimodular, and of Pisot type. Let us recall that a substitution of Pisot type is primitive: there exists N such that $\mathbf{M}_\sigma^N > O$.

Contracting plane. Let ${}^t(1, \alpha, \beta)$ and ${}^t(1, \gamma, \delta)$ be the right and left eigenvectors of $\mathbf{M}_\sigma$ for the dominant eigenvalue λ:

$$\mathbf{M}_\sigma \ {}^t(1, \alpha, \beta) = \lambda \ {}^t(1, \alpha, \beta) \quad \text{and} \quad {}^t\mathbf{M}_\sigma \ {}^t(1, \gamma, \delta) = \lambda \ {}^t(1, \gamma, \delta).$$

Let $\mathcal{P}_{\gamma,\delta}$ be the plane defined by

$$\mathcal{P}_{\gamma,\delta} := \left\{ \boldsymbol{x} \in \mathbb{R}^3 \mid \left\langle \boldsymbol{x}, \ {}^t(1, \gamma, \delta) \right\rangle = 0 \right\}. \tag{8.1}$$

Lemma 8.1.2. *The plane $\mathcal{P}$ is the* contracting invariant plane *with respect to $\mathbf{M}_\sigma$. More precisely, we first have $\mathbf{M}_\sigma \mathcal{P}_{\gamma,\delta} = \mathcal{P}_{\gamma,\delta}$. Let $\boldsymbol{v}' = {}^t(1, \alpha', \beta')$ and $\boldsymbol{v}'' = {}^t(1, \alpha'', \beta'')$ be the right eigenvectors of $\mathbf{M}_\sigma$ associated respectively with eigenvalues λ' and λ''. Secondly, we have:*

- *If $\lambda', \lambda'' \in \mathbb{R}$, then*

$$\boldsymbol{v}', \boldsymbol{v}'' \in \mathcal{P}_{\gamma,\delta} \ \textit{and} \ [\mathbf{M}_\sigma \boldsymbol{v}', \mathbf{M}_\sigma \boldsymbol{v}''] = [\boldsymbol{v}', \boldsymbol{v}''] \begin{bmatrix} \lambda' & 0 \\ 0 & \lambda'' \end{bmatrix}.$$

- *If $\overline{\lambda'} = \lambda'' \in \boldsymbol{C}$, let $\boldsymbol{u}' := \frac{1}{2}(\boldsymbol{v}' + \boldsymbol{v}'')$ and $\boldsymbol{u}'' := \frac{1}{2i}(\boldsymbol{v}' - \boldsymbol{v}'')$. Then*

$$\boldsymbol{u}', \boldsymbol{u}'' \in \mathcal{P}_{\gamma,\delta}, \qquad \exists\, \theta \ \textit{with} \ [\mathbf{M}_\sigma \boldsymbol{u}', \mathbf{M}_\sigma \boldsymbol{u}''] = \frac{1}{\sqrt{\lambda}} [\boldsymbol{u}', \boldsymbol{u}''] \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The proof is left as an exercise.

Fixed point. Let σ be a unimodular substitution of Pisot type, such that $\sigma(1) = 1 \cdot W_1$, that is 1 is the first letter of $\sigma(1)$. Then the sequence $u \in \{1, 2, 3\}^{\mathbb{N}}$ defined by $u = \lim_{n \to \infty} \sigma^n(1)$ exists and satisfies the relation $\sigma(u) = u$. We call it a *fixed point* of σ. Let u_j, $j \in \mathbb{N}$, be the letters of u :

$$u = \lim_{n \to \infty} \sigma^n(1) = u_0 u_1 \cdots u_k \cdots.$$

8.1.2 Domain with fractal boundary associated with a substitution

Let us associate with the fixed point u (and to the dynamical system $(\overline{\mathcal{O}}, T)$ generated by u) a domain included in the contracting plane $\mathcal{P}_{\gamma,\delta}$ by projecting the broken lines in $\mathbb{R}^3$ corresponding to the images under the canonical homomorphism of the prefixes of the sequence u.

Let σ be a unimodular substitution of Pisot type such that $\sigma(1)$ begins with 1. The map $\pi : \mathbb{R}^3 \to \mathcal{P}_{\gamma,\delta}$ is defined to be the *projection* along the vector $(1, \alpha, \beta)$.

We associate with the substitution σ a domain X_σ with a fractal boundary, which admits a natural decomposition into three sets X_i, $i = 1, 2, 3$ as follows:

$$\text{the set } X_\sigma \text{ is the closure of } \bigcup_{N=1}^{\infty} Y_N, \quad \text{with } Y_N = \left\{ \pi(\textstyle\sum_{j=1}^{k} \boldsymbol{e}_{u_j}) \;\middle|\; k = 1, \cdots, N \right\}. \tag{8.2}$$

$$\text{For } i = 1, 2, 3,\ X_i \text{ is the closure of } \bigcup_{N=1}^{\infty} Y_{N,i} \quad \text{with } Y_{N,i} = \left\{ \pi \textstyle\sum_{j=1}^{k} \boldsymbol{e}_{u_j} \;\middle|\; s_k = i,\, k = 1, \cdots, N \right\}.$$

For the Rauzy substitution (Example 8.1.1), and the modified Jacobi-Perron substitution (Example 8.1.2) with $a = 1$ and $\delta = 0$, the domains X_σ are shown on Fig. 8.1.

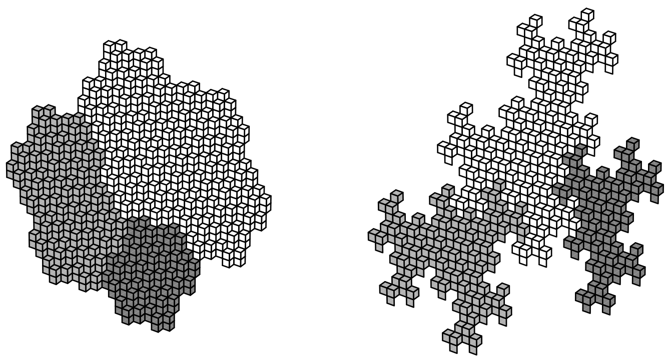


Fig. 8.1. The domain X_σ with fractal boundary for the Rauzy substitution and the modified Jacobi-Perron substitution $\sigma_{(1,0)}$.

Our aim now is to give an explicit device of construction for the set X_σ by introducing in the next section a notion of generalized substitution.

8.1.3 The two-dimensional case

While in this chapter we focus on substitutions over a three-alphabet alphabet, the results are satisfied by substitutions over a general alphabet $\{1, 2 \cdots, d\}$. In particular, the simplest case of a two-letter alphabet is fundamental. At the end of each section of this chapter, we will survey the corresponding results in the two-dimensional case.

The definition of unimodular substitutions of Pisot type over the alphabet $\{1, 2\}$ is the same as in the three-dimensional case, except that there is just one eigenvalue of modulus strictly less than 1.

Let ${}^t(1, \alpha)$ and ${}^t(1, \gamma)$ be respectively the right and left eigenvectors of the matrix $\mathbf{M}_\sigma$ for the dominant eigenvalue λ:

$$\mathbf{M}_\sigma \ {}^t(1, \alpha) = \lambda \ {}^t(1, \alpha) \qquad {}^t\mathbf{M}_\sigma \ {}^t(1, \gamma) = \lambda \ {}^t(1, \gamma).$$

The *contracting invariant line* l_σ with respect to $\mathbf{M}_\sigma$ is defined as

$$l_\sigma = \{ \boldsymbol{x} \mid \langle \boldsymbol{x} \in \mathbb{R}^2, \ {}^t(1, \gamma) \rangle = 0 \}.$$

The *projection* along ${}^t(1, \alpha)$ is denoted $\pi : \mathbb{R}^2 \to l_\sigma$.

For any unimodular substitution σ of Pisot type such that $\sigma(1)$ begins with 1, one can obtain here also some projective sets Y_i and $Y_{N,i}$ for $i = 1, 2$, and their limit X_σ and X_i for $i = 1, 2$, defined respectively as $X_i = \bigcup_{N=1}^{\infty} Y_{N,i}$, for $i = 1, 2$, and $X_\sigma = \bigcup_{N=1}^{\infty} Y_N$.

Example 8.1.3 (Continued fraction substitutions). For every $a \in \mathbb{N}^+$, let us define the substitutions σ_a over a two-letter alphabet by

$$\sigma_a(1) = \overbrace{11 \cdots 1}^{a \ times} 2, \qquad \sigma_a(2) = 1.$$

The substitutions σ_a, with $a \in \mathbb{N}^+$, are called *continued fraction substitutions* (the definition of the continued fraction algorithm will be given in Sec. 8.6).

The matrix of σ_a is

$$\mathbf{M}_{\sigma_a} = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}.$$

Usually, the sets X_σ and X_i, $i = 1, 2$ associated with the continued fraction substitutions are intervals.

8.2 Generalized substitutions

8.2.1 Definitions

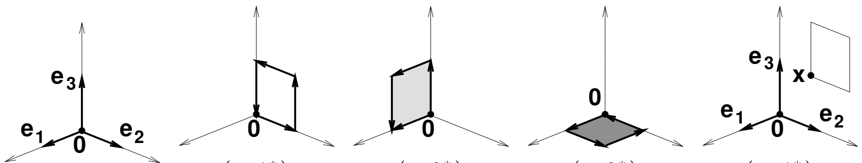
In this section, we build a linear map associated with the substitution σ: this map, which can be considered as a generalized substitution, acts over translated faces of the unit cube. It can be considered as a dual map of the map obtained by acting on the graphic representations of the words through the abelianization map f, replacing each segment parallel to $\boldsymbol{e}_i$ by a broken line corresponding to the word $\sigma(i)$. For more details, see [37, 38]. The purpose of the introduction of such a device is first to get an explicit way of approximating by iteration the contracting plane, and second, to recover in an explicit way after a suitable renormalization the domain X_σ associated with the substitution σ.

We illustrate this notion through the examples of the Rauzy substitution and the modified Jacobi-Perron substitutions. We end this section by considering the two-dimensional case.

Basic squares. Let $\boldsymbol{x} \in \mathbb{Z}^3$ and $i \in \{1,2,3\}$. Translate the unit cube of $\mathbb{R}^3$ by the vector $\boldsymbol{x}$. Then the face of the translated cube which contains $\boldsymbol{x}$ and is orthogonal to the vector $\boldsymbol{e}_i$ is denoted $(\boldsymbol{x}, i^*)$. More precisely, let (j,k) be such $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$, then:

$$(\boldsymbol{x}, i^*) := \{\boldsymbol{x} + \lambda \boldsymbol{e}_j + \mu \boldsymbol{e}_k \mid 0 \le \lambda \le 1, \quad 0 \le \mu \le 1\}.$$

Such a set is called a *basic square*. Illustrations of some basic squares are given on Fig. 8.2.



$(\mathbf{0}, 1^*)$ $(\mathbf{0}, 2^*)$ $(\mathbf{0}, 3^*)$ $(\boldsymbol{x}, 1^*)$

Fig. 8.2. The basic squares $(0, i^*)$, $i = 1,2,3$ and $(x, 1^*)$.

The $\mathbb{Z}$-module generated by the set of basic squares. Our next step will be now to define a linear map which acts on the set of these basic squares. The idea is to substitute (according to a scheme given by the substitution σ) to a basic square a union of basic squares. When iterating such a device, overlap problems of basic squares may occur. We thus need to introduce a suitable formalism. Let us equip the set of basic squares of a structure of $\mathbb{Z}$-module.

Let Λ denote the set of all the basic squares $(\boldsymbol{x}, i^*)$:

$$\Lambda = \left\{ (\boldsymbol{x}, i^*) \;\middle|\; \boldsymbol{x} \in \mathbb{Z}^3, i \in \{1,2,3\} \right\}.$$

Let $\mathcal{G}$ be the $\mathbb{Z}$-module generated by Λ:

$$\mathcal{G} = \left\{ \textstyle\sum_{\lambda \in \Lambda} m_\lambda \lambda \;\middle|\; m_\lambda \in \mathbb{Z},\ \# \{\lambda \mid m_\lambda \neq 0\} < +\infty \right\}.$$

The set $\mathcal{G}$ is the set of formal sums of weighted faces.

The lattice $\mathbb{Z}^3$ acts on $\mathcal{G}$ by translation:

$$\forall\, \boldsymbol{y}, \boldsymbol{x} \in \mathbb{Z}^3,\ \forall i \in \{1,2,3\}, \qquad \boldsymbol{y} + (\boldsymbol{x}, i^*) := (\boldsymbol{y} + \boldsymbol{x}, i^*).$$

Let us define now a map Θ acting on the set $\mathcal{G}$.

Definition 8.2.1. *The* endomorphism Θ *of* $\mathcal{G}$ *associated with* σ *is defined as follows:*

$$\begin{cases} \forall i \in \mathcal{A}, & \Theta(\mathbf{0}, i^*) := \sum_{j=1}^{3} \sum_{W : \sigma(j) = Y \cdot i \cdot W} \left(\mathbf{M}_\sigma^{-1} \left(f(W) \right), j^* \right), \\ \forall \boldsymbol{x} \in \mathbb{Z}^3,\ \forall i \in \mathcal{A}, & \Theta(\boldsymbol{x}, i^*) := \mathbf{M}_\sigma^{-1} \boldsymbol{x} + \Theta(\mathbf{0}, i^*), \\ \forall \sum_{\lambda \in \Lambda} m_\lambda \lambda \in \mathcal{G}, & \Theta \sum_{\lambda \in \Lambda} m_\lambda \lambda := \sum_{\lambda \in \Lambda} m_\lambda \Theta(\lambda). \end{cases}$$

The formalism in this definition is quite cumbersome. For a better understanding of the map Θ, see Fig. 8.6 and see also the next sections, where we apply the above formulas in some basic cases.

Note that since the substitution is unimodular, $\mathbf{M}_\sigma^{-1}$ maps $\mathbb{Z}^3$ to $\mathbb{Z}^3$.

Let us introduce the stepped surface associated with a plane on which the endomorphism Θ acts, in order to discuss the geometrical properties of Θ.

Stepped surface of a plane. Let $0 < \gamma, \delta < 1$. Let $\mathcal{P}_{\gamma,\delta}$ be the plane orthogonal to the vector ${}^t(1, \gamma, \delta)$:

$$\mathcal{P}_{\gamma,\delta} := \left\{ \boldsymbol{x} \;\middle|\; \left\langle \boldsymbol{x},\ {}^t(1,\gamma,\delta) \right\rangle = 0 \right\}.$$

For each γ, δ, the *stepped surfaces* $\mathcal{S}^+_{\gamma,\delta}$ and $\mathcal{S}^-_{\gamma,\delta}$ associated with $\mathcal{P}_{\gamma,\delta}$ are defined as the surfaces consisting of basic squares which are the best ahead and above approximations of the plane $\mathcal{P}_{\gamma,\delta}$. More precisely, let $\mathbf{S}^+_{\gamma,\delta}$ and $\mathbf{S}^-_{\gamma,\delta}$ gather the faces of the translates with integer vertices of the unit cube which intersect $\mathcal{P}_{\gamma,\delta}$:

$$\mathbf{S}^+_{\gamma,\delta} := \left\{ (\boldsymbol{x}, i^*) \;\middle|\; \left\langle \boldsymbol{x},\ {}^t(1,\gamma,\delta) \right\rangle > 0,\ \left\langle (\boldsymbol{x} - \boldsymbol{e}_i),\ {}^t(1,\gamma,\delta) \right\rangle \leq 0 \right\},$$

$$\mathbf{S}^-_{\gamma,\delta} := \left\{ (\boldsymbol{x}, i^*) \;\middle|\; \left\langle \boldsymbol{x},\ {}^t(1,\gamma,\delta) \right\rangle \geq 0, \left\langle (\boldsymbol{x} - \boldsymbol{e}_i),\ {}^t(1,\gamma,\delta) \right\rangle < 0 \right\}.$$

Then the stepped surfaces are unions of such basic squares:

$$\mathcal{S}^+_{\gamma,\delta} := \bigcup_{(\boldsymbol{x},i^*)\in \mathbf{S}^+_{\gamma,\delta}} (\boldsymbol{x}, i^*), \qquad \mathcal{S}^-_{\gamma,\delta} := \bigcup_{(\boldsymbol{x},i^*)\in \mathbf{S}^-_{\gamma,\delta}} (\boldsymbol{x}, i^*).$$

In other words, a basic square $(\boldsymbol{x}, i^*)$ belongs for instance to $\mathcal{S}^+_{\gamma,\delta}$ if the interior of the unit cube with upper corner at x intersects the plane $\mathcal{P}_{\gamma,\delta}$, and its lower face orthogonal to $\boldsymbol{e}_i$ does not intersect the plane, except maybe on the boundary. The stepped surface is thus the best approximation of the plane by faces contained in the closed upper half-space $\{\boldsymbol{x} | \langle \boldsymbol{x},\ {}^t(1,\gamma,\delta) \rangle \geq 0\}$.

Let $\Gamma^+_{\gamma,\delta}$ and $\Gamma^-_{\gamma,\delta}$ be the subset of $\mathcal{G}$ consisting in all finite unions of elements of $\mathbf{S}^+_{\gamma,\delta}$ and $\mathbf{S}^-_{\gamma,\delta}$:

$$\Gamma^+_{\gamma,\delta} := \left\{ \textstyle\sum_{\lambda\in\mathbf{S}^+_{\gamma,\delta}} n_\lambda \lambda \;\middle|\; n_\lambda \in \{0,1\},\ \#\{\lambda \mid n_\lambda = 1\} < +\infty \right\}$$

$$\Gamma^-_{\gamma,\delta} := \left\{ \textstyle\sum_{\lambda\in\mathbf{S}^-_{\gamma,\delta}} n_\lambda \lambda \;\middle|\; n_\lambda \in \{0,1\},\ \#\{\lambda \mid n_\lambda = 1\} < +\infty \right\}.$$

Thus elements of $\Gamma^+_{\gamma,\delta}$ are finite parts of the stepped surface $\mathcal{S}^+_{\gamma,\delta}$. We call them *patches* of the stepped surface. The stepped surface has the following property of invariance with respect to Θ:

Theorem 8.2.1 ([38]). *For each unimodular substitution σ of Pisot type, let Θ be the endomorphism of $\mathcal{G}$ associated with σ and $\Gamma^+_{\gamma,\delta}$ (respectively $\Gamma^-_{\gamma,\delta}$) be the patch of the stepped surface associated with the contracting invariant plane $\mathcal{P}_{\gamma,\delta}$. Then $\Gamma^+_{\gamma,\delta}$ and $\Gamma^-_{\gamma,\delta}$ are invariant with respect to the endomorphism Θ, that is*

$$\textstyle\sum_{\lambda\in\mathbf{S}^+_{\gamma,\delta}} n_\lambda\lambda \in \Gamma^+_{\gamma,\delta} \quad \textit{implies} \quad \Theta\left(\sum_{\lambda\in\mathbf{S}^+_{\gamma,\delta}} n_\lambda\lambda\right) \in \Gamma^+_{\gamma,\delta},$$

$$\textstyle\sum_{\lambda\in\mathbf{S}^-_{\gamma,\delta}} n_\lambda\lambda \in \Gamma^-_{\gamma,\delta} \quad \textit{implies} \quad \Theta\left(\sum_{\lambda\in\mathbf{S}^-_{\gamma,\delta}} n_\lambda\lambda\right) \in \Gamma^-_{\gamma,\delta}.$$

This theorem means that one can apply Θ to any finite part of the stepped surface and obtain a new finite part of the stepped surface. Let us define a partial relationship of inclusion between parts of the stepped surface.

Notation. For $\gamma = \sum_{\lambda\in\mathbf{S}^+_{\gamma,\delta}} n_\lambda\lambda$, $\delta = \sum_{\lambda\in\mathbf{S}^+_{\gamma,\delta}} m_\lambda\lambda \in \Gamma^+_{\gamma,\delta}$, $\gamma \succ \delta$ means that $n_\lambda \neq 0$ if $m_\lambda \neq 0$, i.e., the patch δ is a subpatch of γ (See Figure 8.3).

Let $\mathcal{U}'$ denote the above faces of the unit cube, that is the set of faces which are in contact with the origin $(0,0,0)$, and $\mathcal{U}$ the ahead faces, that is the set of faces which are in contact with the point $(1,1,1)$. Next lemma means that applying Θ to these patches generates new patches which contain the initial half-unit cube.

Lemma 8.2.1. *Let*

$$\mathcal{U} := \sum_{i=1,2,3} (\boldsymbol{e}_i, i^*) \qquad \textit{and} \qquad \mathcal{U}' := \sum_{i=1,2,3} (\mathbf{0}, i^*).$$

Then we have

$$\Theta(\mathcal{U}) \succ \mathcal{U} \quad \textit{and} \quad \Theta(\mathcal{U}') \succ \mathcal{U}'.$$

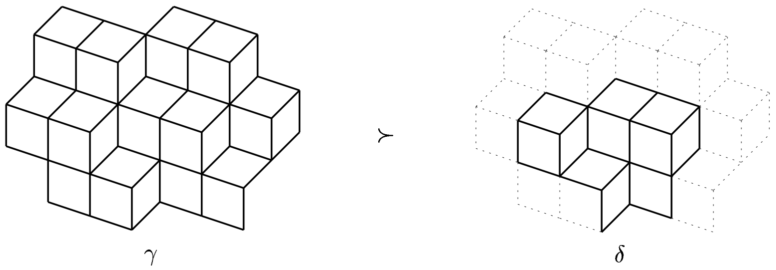


Fig. 8.3. Two patches γ and δ such that $\gamma \succ \delta$.

8.2.2 First example: the Rauzy substitution

Let us recall that the Rauzy substitution is defined as $\sigma(1) = 12$, $\sigma(2) = 13$ and $\sigma(3) = 1$.

Let $\boldsymbol{f}_i$ be the column vectors of the matrix $\mathbf{M}_\sigma^{-1}$:

$$\mathbf{M}_\sigma^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} = [\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3] .$$

Then the endomorphism Θ is defined by the following formulas:

$$\Theta : \begin{array}{l} (\mathbf{0}, 1^*) \longrightarrow (\mathbf{0}, 3^*) + (\boldsymbol{f}_2, 1^*) + (\boldsymbol{f}_3, 2^*) \\ (\mathbf{0}, 2^*) \longrightarrow (\mathbf{0}, 1^*) \\ (\mathbf{0}, 3^*) \longrightarrow (\mathbf{0}, 2^*) \end{array}$$

The way Θ acts on the faces of the unit cube is shown in Fig. 8.4. In Fig. 8.5, one sees the successive images of the half unit cube $\mathcal{U}' = (\mathbf{0}, 1^*) \cup (\mathbf{0}, 2^*) \cup (\mathbf{0}, 3^*)$ through Θ. Since $\Theta(\mathcal{U}')$ contains $\mathcal{U}'$, the sequence of patches $\Theta^n(\mathcal{U}')$ is strictly increasing. An idea of the limit of this sequence is given in Fig. 8.6.

8.2.3 Second example: the modified Jacobi-Perron substitutions

These substitutions $\sigma_{(a,0)}$ and $\sigma_{(a,1)}$ (see Example 8.1.2) are defined by:

$$\sigma_{(a,0)} : \begin{array}{l} 1 \longrightarrow \overbrace{11 \cdots 1}^{a \ times} 2 \\ 2 \longrightarrow 3 \\ 3 \longrightarrow 1 \end{array} ,$$

$$\sigma_{(a,1)} : \begin{array}{l} 1 \longrightarrow \overbrace{11 \cdots 1}^{a \ times} 3 \\ 2 \longrightarrow 1 \\ 3 \longrightarrow 2 \end{array} .$$

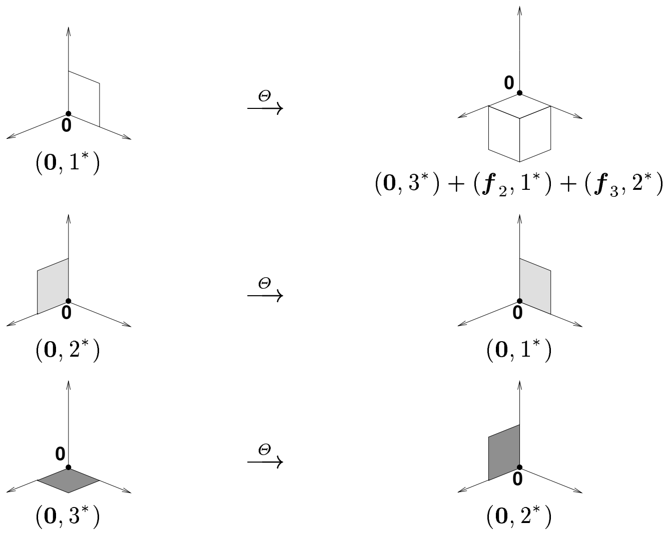


Fig. 8.4. Endomorphism Θ for the Rauzy substitution.

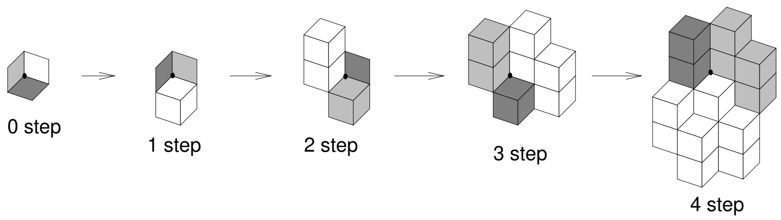


Fig. 8.5. Illustration of $\bigcup_{i=1,2,3} \Theta^n(0, i^*)$ for Rauzy substitution, $n = 0, 1, 2, 3, 4$.

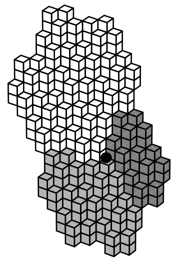


Fig. 8.6. Illustration of $\bigcup_{i=1,2,3} \Theta^8(0, i^*)$ for Rauzy substitution.

The associated endomorphisms $\Theta_{(a,0)}$ and $\Theta_{(a,1)}$ are the following:

$$\Theta_{(a,0)} : \begin{array}{l} (\mathbf{0},1^*) \longrightarrow (\mathbf{0},3^*) + \sum_{1\leq k\leq a}((\boldsymbol{e}_1 - k\boldsymbol{e}_3),1^*) \\ (\mathbf{0},2^*) \longrightarrow (\mathbf{0},1^*) \\ (\mathbf{0},3^*) \longrightarrow (\mathbf{0},2^*) \end{array},$$

$$\Theta_{(a,1)} : \begin{array}{l} (\mathbf{0},1^*) \longrightarrow (\mathbf{0},2^*) + \sum_{1\leq k\leq a}((\boldsymbol{e}_1 - k\boldsymbol{e}_2),1^*) \\ (\mathbf{0},2^*) \longrightarrow (\mathbf{0},3^*) \\ (\mathbf{0},3^*) \longrightarrow (\mathbf{0},1^*) \end{array}.$$

The way $\Theta_{(a,\varepsilon)}$ (for $\varepsilon = 0,1$) acts on the faces of the unit cube is shown on Fig. 8.7 and 8.8. In Fig. 8.9, one sees the successive images of the half cube $\mathcal{U}' = (\mathbf{0},1^*) \cup (\mathbf{0},2^*) \cup (\mathbf{0},3^*)$ through Θ. Similarly as in the case of the Rauzy substitution, since $\Theta(\mathcal{U}')$ contains $\mathcal{U}'$, then the sequence of patches $\Theta^n(\mathcal{U}')$ is strictly increasing. An idea of the limit of this sequence is given in Fig. 8.10.

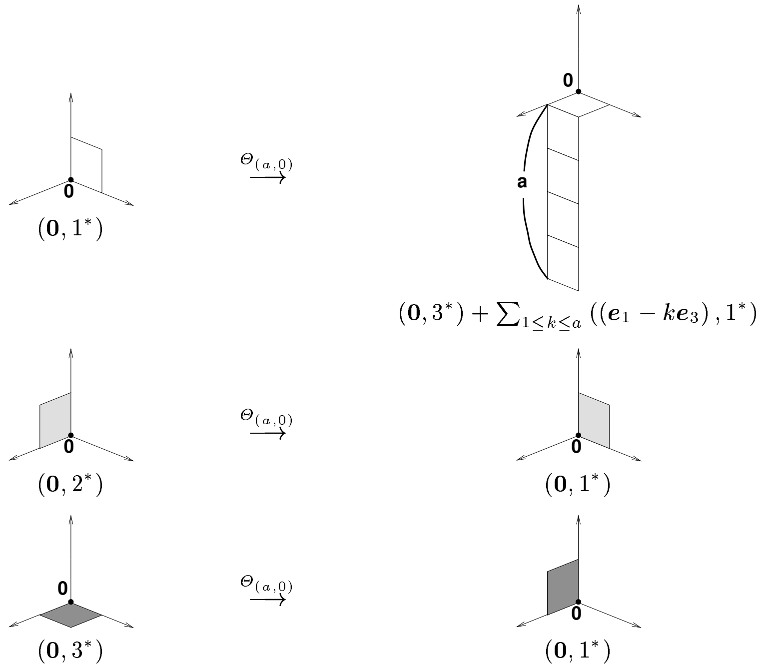


Fig. 8.7. The map Θ for the modified Jacobi-Perron substitutions $\sigma_{(a,0)}$.

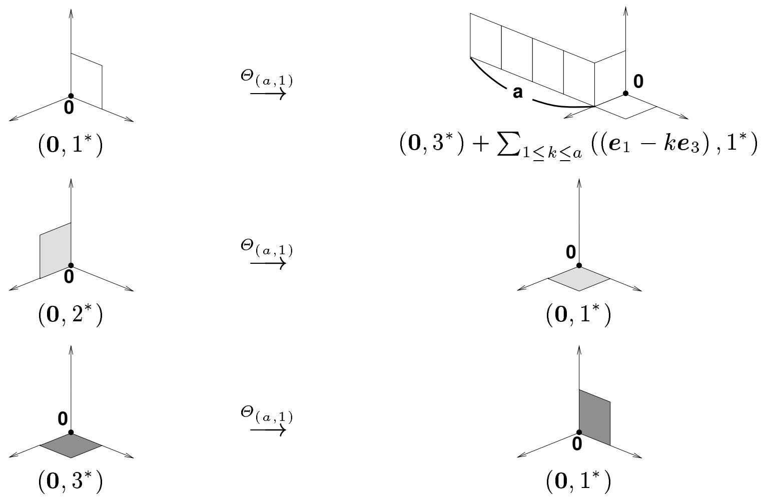


Fig. 8.8. The map Θ for the modified Jacobi-Perron substitutions $\sigma_{(a,1)}$.

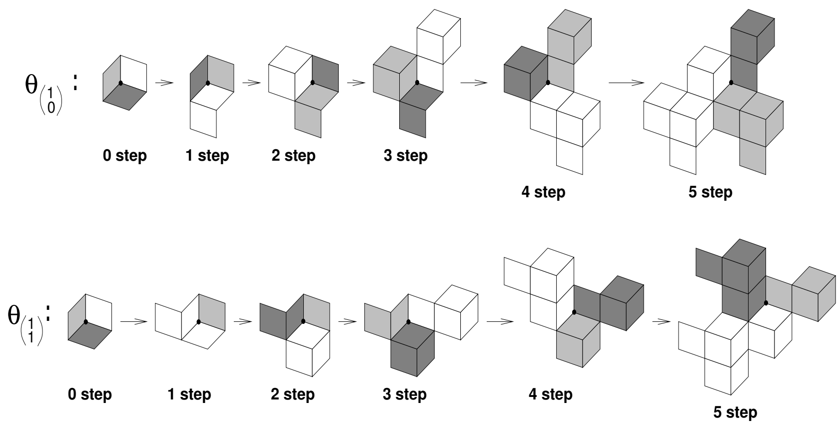
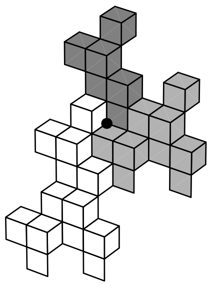


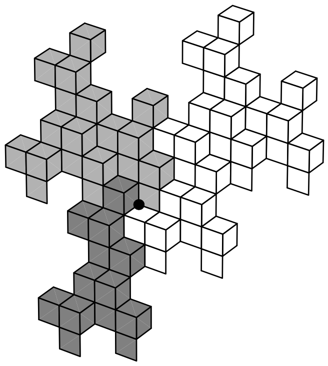
Fig. 8.9. Illustration of $\bigcup_{i=1,2,3} \Theta^n(o, i^*), n = 0, 1, \cdots, 5$ for the modified Jacobi-Perron substitutions

8.2.4 Third example: a non-simply connected example

Simple connectedness. In the case of the Rauzy and the modified Jacobi-Perron substitutions, the sets $\Theta^n(\mathbf{0}, i^*)$, $i = 1, 2, 3$ are simply connected. Geometrical properties of the sets $\Theta^n(\mathbf{0}, i^*)$, $i = 1, 2, 3$ are studied in [187, 188] and [157]. It can be remarked that, in general, the sets $\Theta^n(\mathbf{0}, i^*)$, for $i = 1, 2, 3$ are not simply connected.



$\bigcup_{i=1,2,3} \Theta^{8}_{(1,0)}\,(\mathbf{0}, i^*)$



$\bigcup_{i=1,2,3} \Theta^{10}_{(1,0)}\,(\mathbf{0}, i^*)$

Fig. 8.10. Modified Jacobi-Perron substitutions

Example 8.2.1 (A non-simply connected example). Let σ be the substitution defined by:

$$\sigma(1) = 21,\ \sigma(2) = 13,\ \sigma(3) = 1.$$

If $\mathbf{M}_\sigma$ denotes the matrix of this substitution, let $\boldsymbol{f}_i$ be the columns of its inverse:

$$\mathbf{M}_\sigma^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} = [\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3].$$

The associated endomorphism Θ is the following:

$$\Theta : \begin{array}{l} (\mathbf{0}, 1^*) \longrightarrow (\mathbf{0}, 1^*) + (\mathbf{0}, 3^*) + (\boldsymbol{f}_3, 2^*) \\ (\mathbf{0}, 2^*) \longrightarrow (\boldsymbol{f}_1, 1^*) \\ (\mathbf{0}, 3^*) \longrightarrow (\mathbf{0}, 2^*) \end{array}$$

The way Θ acts on the faces of the unit cube is shown on Fig. 8.11. The successive images of the half unit cube $\mathcal{U}' = (\mathbf{0}, 1^*) \cup (\mathbf{0}, 2^*) \cup (\mathbf{0}, 3^*)$ under the action of Θ are given on Fig. 8.12. Here again, $\Theta(\mathcal{U}')$ contains $\mathcal{U}'$, hence the sequence of patches $\Theta^n(\mathcal{U}')$ converges, see Fig. 8.13. These figures show that for this substitution, the sets $\Theta^n(\mathbf{0}, i^*)$, for $i = 1, 2, 3$, are not simply connected.

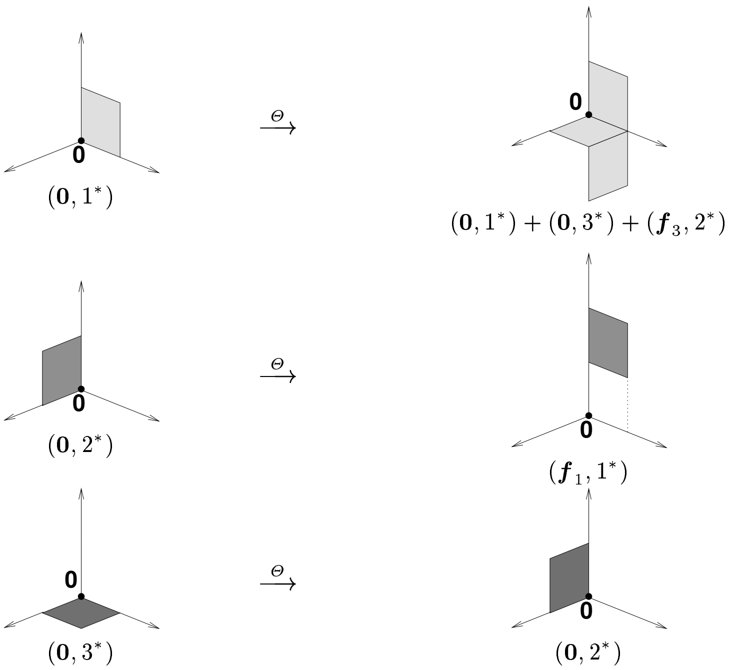


Fig. 8.11. The endomorphism Θ for a substitution which generates a non-simply connected stepped surface.

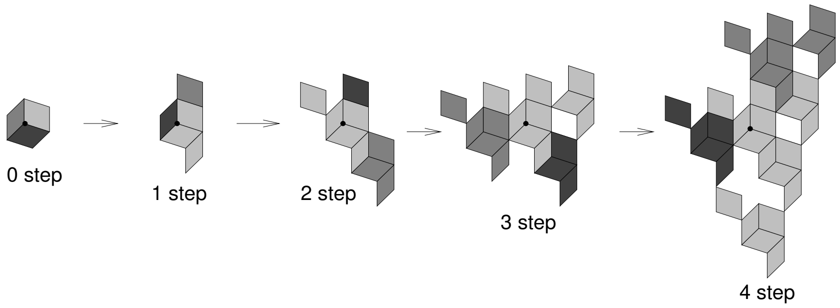


Fig. 8.12. Convergence of $\bigcup_{i=1,\dots,n} \Theta^n(0, i^*)$ towards a non-simply connected stepped surface.

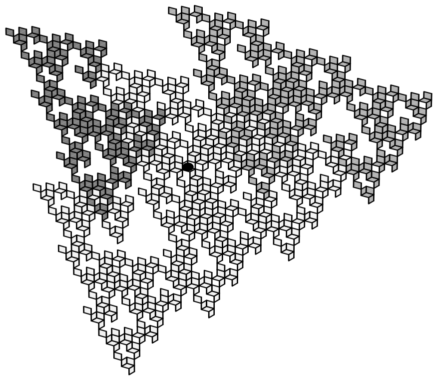


Fig. 8.13. The non-simply connected set $\bigcup_{i=1,2,3} \Theta^{10}(0, i^*)$.

8.2.5 The two-dimensional case

Endomorphism associated with a two-letter substitution. We will consider now a two-letter alphabet $\{1,2\}$. The cube faces are to be replaced by translates of the unit segment. More precisely, for any $\boldsymbol{x} \in \mathbb{Z}^2$ and $i \in \{1,2\}$, let $(\boldsymbol{x}, i^*)$ be

$$(\boldsymbol{x}, i^*) = \{\boldsymbol{x} + \lambda \boldsymbol{e}_j \mid 0 \leq \lambda \leq 1\}, \text{ with } (i,j) \in \{(1,2),(2,1)\}.$$

By analogy with the three-dimensional case, Λ denotes the set which contains these segments:

$$\Lambda = \{(\boldsymbol{x}, i^*) | \ \boldsymbol{x} \in \mathbb{Z}^2, \ i \in \{1,2\}\}.$$

The $\mathbb{Z}$-module generated by the translated segments is defined by:

$$\mathcal{G} = \{\textstyle\sum_{\lambda \in \Lambda} m_\lambda \lambda \mid m_\lambda \in \boldsymbol{Z}, \ \#\{\lambda \mid m_\lambda \neq 0\} < +\infty\}.$$

Then the *endomorphism of $\mathcal{G}$ associated with σ*, denoted by Θ, is defined as follows:

$$\begin{cases} \forall i \in \mathcal{A}, & \Theta(\mathbf{0}, i^*) := \sum_{j=1}^{2} \sum_{W:,\sigma(j)=Y \cdot i \cdot W} \left(\mathbf{M}_\sigma^{-1}\left(f(W)\right), j^*\right), \\ \forall \boldsymbol{x} \in \mathbb{Z}^2, \ \forall i \in \mathcal{A}, & \Theta(\boldsymbol{x}, i^*) := \mathbf{M}_\sigma^{-1}\boldsymbol{x} + \Theta(\mathbf{0}, i^*), \\ \forall \sum_{\lambda \in \Lambda} m_\lambda \lambda \in \mathcal{G}, & \Theta \sum_{\lambda \in \Lambda} m_\lambda \lambda := \sum_{\lambda \in \Lambda} m_\lambda \Theta(\lambda). \end{cases}$$

Stepped curve of a line. For any $0 < \gamma < 1$, let l_γ be the line which admits ${}^t(1,\gamma)$ as a normal vector:

$$l_\gamma = \{\boldsymbol{x} \mid \langle \boldsymbol{x},\ {}^t(1,\gamma)\rangle = 0\}.$$

The *stepped curves* $\mathcal{S}^+$ and $\mathcal{S}^-$ associated with the line l_γ are the unions of segments which approximate the best the line l_γ. More precisely, let $\mathbf{S}^+$ and $\mathbf{S}^-$ be the set of these segments:

$$\mathbf{S}^+ := \{(\boldsymbol{x}, i^*) \mid \langle \boldsymbol{x},\ {}^t(1,\gamma)\rangle > 0,\ \langle (\boldsymbol{x} - \boldsymbol{e}_i),\ {}^t(1,\gamma)\rangle \le 0\}$$

$$\mathbf{S}^- := \{(\boldsymbol{x}, i^*) \mid \langle \boldsymbol{x},\ {}^t(1,\gamma)\rangle \ge 0,\ \langle (\boldsymbol{x} - \boldsymbol{e}_i),\ {}^t(1,\gamma)\rangle < 0\}$$

Then one obtains two stepped curves when considering the union of the supports of the elements of $\mathbf{S}^+$ and $\mathbf{S}^-$:

$$\mathcal{S}^+ := \bigcup_{(\boldsymbol{x},i^*)\in\mathbf{S}^+} (\boldsymbol{x}, i^*), \quad \text{and} \quad \mathcal{S}^- := \bigcup_{(\boldsymbol{x},i^*)\in\mathbf{S}^-} (\boldsymbol{x}, i^*).$$

Let Γ^+ and Γ^- be the sets of the finite unions of elements of $\mathbf{S}^+$ and $\mathbf{S}^-$, i.e.:

$$\Gamma^+ := \{\textstyle\sum_{\lambda\in\mathbf{S}^+} n_\lambda \lambda \mid n_\lambda \in \{0,1\},\ \#\{\lambda \mid n_\lambda = 1\} < +\infty\}$$

$$\Gamma^- := \{\textstyle\sum_{\lambda\in\mathbf{S}^-} n_\lambda \lambda \mid n_\lambda \in \{0,1\},\ \#\{\lambda \mid n_\lambda = 1\} < +\infty\}.$$

Then Theorem 8.2.1 holds according to this framework.

Example 8.2.2 (Continued fraction substitutions). Let us recall that the continued fraction substitutions are defined as (see Example 8.1.3):

$$\sigma_a(1) = \overbrace{11\cdots1}^{a\ times}2 \qquad \sigma_a(2) = 1.$$

The endomorphism Θ_a associated with σ_a is the following:

$$\Theta_a : \begin{array}{l} (\mathbf{0}, 1^*) \longrightarrow (\mathbf{0}, 2^*) + \sum_{k=1}^{a}((\boldsymbol{e}_1 - k\boldsymbol{e}_2), 1^*) \\ (\mathbf{0}, 2^*) \longrightarrow (\mathbf{0}, 1^*) \end{array}$$

Figure 8.14 illustrates how Θ_a acts on the unit segments $(\mathbf{0}, 1^*)$ and $(\mathbf{0}, 2^*)$. In the special case $a = 1$, Fig. 8.15 illustrates the iteration of Θ_1 on the unit half square $(\mathbf{0}, 1^*) \cup (\mathbf{0}, 2^*)$.

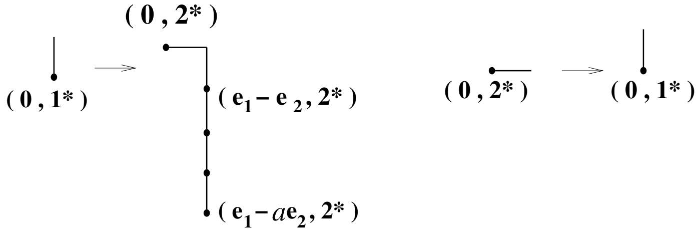


Fig. 8.14. Action of Θ on unit segments for the continued fraction substitutions.

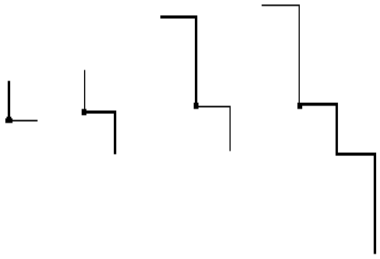


Fig. 8.15. $\bigcup_{i=1,2} {\Theta_1}^n(0, i^*)$, $n = 0, 1, 2, 3$.

Example 8.2.3. Let $\sigma = {\sigma_1}^2$ be the substitution defined by:

$$\sigma(1) = 121 \quad \sigma(2) = 12.$$

Let $\mathbf{M}_\sigma$ denote the incidence matrix of this substitution, and let $\boldsymbol{f}_1, \boldsymbol{f}_2$ be the column vectors of its inverse:

$$\mathbf{M}_\sigma^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = [\boldsymbol{f}_1, \boldsymbol{f}_2] \,.$$

Then the endomorphism Θ associated with this substitution is the following:

$$\Theta : \begin{array}{l} (\mathbf{0}, 1^*) \longrightarrow (\mathbf{0}, 1^*) + (\boldsymbol{f}_1 + \boldsymbol{f}_2, 1^*) + (\boldsymbol{f}_2, 2^*) \\ (\mathbf{0}, 2^*) \longrightarrow (\boldsymbol{f}_1, 1^*) + (\mathbf{0}, 2^*) \end{array}$$

Figure 8.16 illustrates how Θ acts on the unit segments $(\mathbf{0}, 1^*)$ and $(\mathbf{0}, 2^*)$ and Fig. 8.17 illustrates the iteration of Θ on the unit half square $(\mathbf{0}, 1^*) \cup (\mathbf{0}, 2^*)$.

Simple connectedness. On Example 8.1.3 and Example 8.2.3, one can remark that the sets $\Theta^n(\mathbf{0}, i^*)$, $i = 1, 2$ are simply connected. This property does not hold any more in the general case as illustrated by the following example.

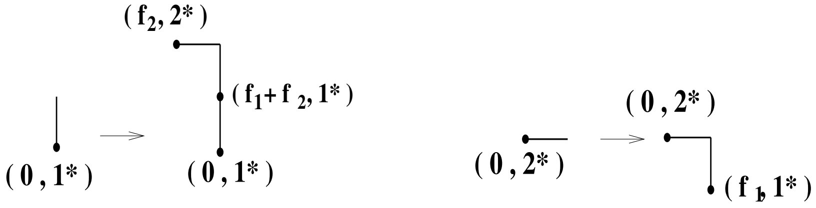


Fig. 8.16. Action of Θ on unit segments for the substitution of Example 8.2.3.

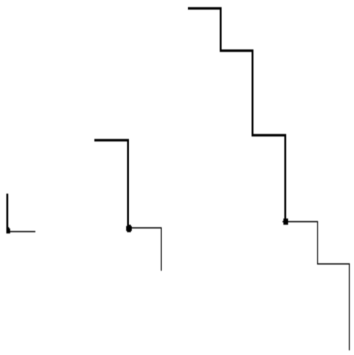


Fig. 8.17. The set $\bigcup_{i=1,2} \Theta^n(0, i^*), n = 0, 1, 2$.

Example 8.2.4. [A non-simply connected example] Let σ be the substitution defined by

$$\sigma(1) = 112 \quad \sigma(2) = 21.$$

Let $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ be the column vectors of the inverse of the incidence matrix of σ:

$$\mathbf{M}_\sigma^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = [\boldsymbol{f}_1, \boldsymbol{f}_2] .$$

Then the endomorphism Θ associated with σ is given by

$$\Theta : \begin{array}{l} (\mathbf{0}, 1^*) \longrightarrow (\mathbf{0}, 2^*) + (\boldsymbol{f}_2, 1^*) + (\boldsymbol{f}_1 + \boldsymbol{f}_2, 1^*) \\ (\mathbf{0}, 2^*) \longrightarrow (\mathbf{0}, 1^*) + (\boldsymbol{f}_1, 2^*). \end{array}$$

The action Θ on unit segments is described in Fig. 8.18. The iteration of Θ on the half unit square is illustrated in Fig. 8.19. It will be noted that the sets $\Theta^n(\mathbf{0}, i^*)$, $i = 1, 2$ are not simply connected.

The simple connectedness is characterized by the following property in the case of two-letter substitutions (see also Chap. 9):

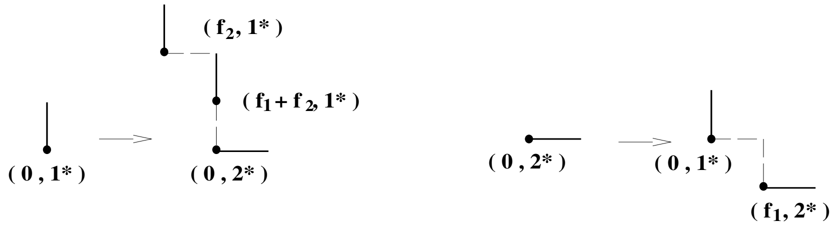


Fig. 8.18. The map Θ.

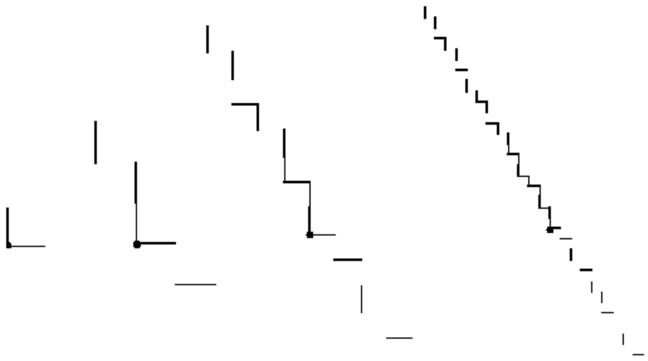


Fig. 8.19. The set $\bigcup_{i=1,2} \Theta^n(0, i^*), n = 0, 1, 2, 3$.

Theorem 8.2.2 ([371, 134]). *The sets $\Theta^n(\mathbf{0}, i^*)$, $i = 1, 2$ are simply connected for every integer n if and only if the substitution σ is an invertible substitution, that is, if there exists an automorphism $\theta : F_2 \to F_2$ of the free group F_2 of rank 2 generated by $\{1, 2\}$, such that*

$$\sigma \circ \theta = \theta \circ \sigma = Id.$$

8.3 Dynamical systems associated with the stepped surface

There are two different methods of constructing the Rauzy fractal: one can first use convergent series associated with the substitution, as is done in the seminal paper of Rauzy (this is also the approach developed in [86, 87]). One can also approximate it by exduction/induction. The aim of this section is to try to formalize this approach.

8.3.1 Exchanges of domains associated with a substitution

Let σ be a Pisot substitution over a three-letter alphabet and Θ be the endomorphism associated with σ.

The aim of this section is to build a sequence of domains D_n which tile the stepped surface associated with the contracting plane $\mathcal{P}_{\gamma,\delta}$ and a sequence of dynamical systems W_n on D_n. The domains D_n are obtained by iterating Θ; we thus obtain a sequence of larger and larger pieces of the stepped surface.

Some special domains in the contracting plane. For any letter $i = 1, 2, 3$, we define the sets $D_n^{(i)}$ and $D_n^{(i)'}$ as the projections on the contracting plane $\mathcal{P}_{\gamma,\delta}$, of the n-th iterate of Θ of the unit basic squares orthogonal to the vector $\boldsymbol{e}_i$, i.e., $(\boldsymbol{e}_i, i^*)$ and $(\boldsymbol{0}, i^*)$:

Definition 8.3.1. *Let*

$$D_n^{(i)} := \pi\left(\Theta^n(\boldsymbol{e}_i, i^*)\right), \text{ and } D_n^{(i)'} := \pi\left(\Theta^n(0, i^*)\right), \text{ for } i = 1, 2, 3.$$

We denote by D_n and D'_n the union of these domains. Thus, if $\mathcal{U}$ and $\mathcal{U}'$ are the above and ahead half unit cubes, that is

$$\mathcal{U} = \textstyle\sum_{i=1,2,3} (\boldsymbol{e}_i, i^*), \qquad \mathcal{U}' = \textstyle\sum_{i=1,2,3} (\boldsymbol{0}, i^*),$$

then we have

$$D_n := \pi\left(\Theta^n(\mathcal{U})\right), \quad D'_n := \pi\left(\Theta^n(\mathcal{U}')\right).$$

Note that these unions are disjoint up to a set of zero measure.

Inclusion relations between these domains. By definition, we have $\mathcal{U} \in \Gamma^+_{\alpha,\beta}$ and $\mathcal{U}' \in \Gamma^-_{\alpha,\beta}$. Thus, Theorem 8.2.1 and Lemma 8.2.1 imply that

$$\Theta^n(\mathcal{U}) \succ \Theta^{n-1}(\mathcal{U}) \quad \text{and} \quad \Theta^n(\mathcal{U}') \succ \Theta^{n-1}(\mathcal{U}').$$

Therefore, for every n, the domain D_n contains the domain D_{n-1}, and D'_n contains D'_{n-1}.

One can remark that $\pi(\mathcal{U})$ and $\pi(\mathcal{U}')$ are just the above and ahead projection of the unit cube on a plane which crosses this cube, which implies that they are equal. We deduce from this (see [37, 38] for more details), that we have more generally:

$$D_n = D'_n.$$

The following theorem means that it is possible to translate each part $D_n^{(i)}$ of D_n by a specified vector without changing the general aspect of D_n. In other words one performs on D_n an exchange of domains (the domains being the sets $D_n^{(i)}$, for $i = 1, 2, 3$).

Theorem 8.3.1. *Let* $\boldsymbol{f}_1^{(n)}$, $\boldsymbol{f}_2^{(n)}$ *and* $\boldsymbol{f}_3^{(n)}$ *be the column vectors of the matrix* $\mathbf{M}_\sigma^{-n}$, *that is* $\mathbf{M}_\sigma^{-n} = \left[\boldsymbol{f}_1^{(n)}, \boldsymbol{f}_2^{(n)}, \boldsymbol{f}_3^{(n)}\right]$.

Then the following dynamical system W_n *on* D_n *is well-defined (up to a set of measure* 0*):*

$$\begin{array}{rcl} D_n & \xrightarrow{W_n} & D_n \\ \boldsymbol{x} & \mapsto & \boldsymbol{x} - \pi(\boldsymbol{f}_i^{(n)}) \quad \text{if} \quad \boldsymbol{x} \in D_n^{(i)}, \ i = 1, 2, 3. \end{array}$$

All the dynamical systems (D_n, W_n), *for* $n \geq 0$, *are isomorphic.*

For a proof of this theorem, see [37].

For the **Rauzy substitution**, defined in Example 8.1.1 by $\sigma(1) = 12$, $\sigma(2) = 13$ and $\sigma(3) = 1$, the successive sets D_n, for $n = 0, 1, 2, 3$, appear twice on each horizontal floor of Fig. 8.20.

Each vertical arrow shows the action of Θ: the set D_0 goes onto D_1, then onto D_2 and so on. The result of the action of the domain exchange W_n on the right domain D_n divided into its three parts $D_n^{(i)}$, is shown on the domain D_n which is on the left handside of the arrow. Note that for $i = 1, 2, 3$, $D_n^{(i)}$ is mapped onto $D_n^{(i)'}$.

Figure 8.21 gives the equivalent illustrations for the **modified Jacobi-Perron substitution** $\sigma_{(1,0)}$, defined in Example 8.1.2 by $\sigma_{(1,0)}(1) = 12$, $\sigma_{(1,0)}(2) = 3$ and $\sigma_{(1,0)}(3) = 1$.

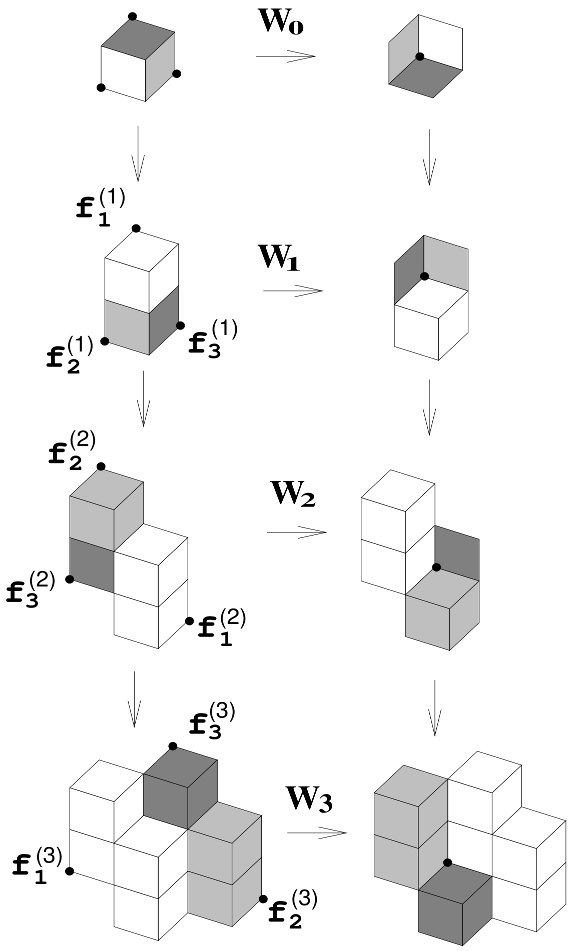


Fig. 8.20. Successives exchanges of domains for the Rauzy substitution.

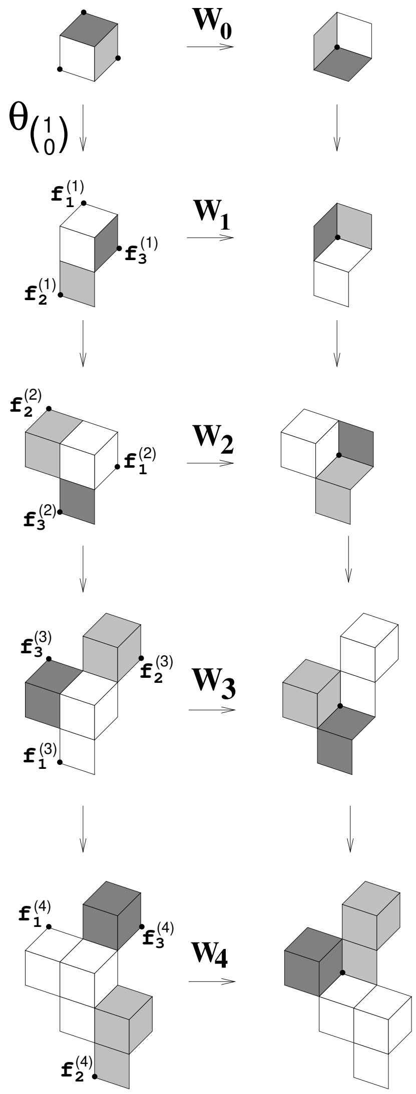


Fig. 8.21. Successive exchanges of domains for a modified Jacobi-Perron substitution.

8.3.2 Induced transformations

Our motivation here is to give a geometric representation by means of the sets D_n of the action of the shift on the substitutive dynamical system. Since each domain D_n contains the preceding one D_{n-1}, the map W_n is defined on D_{n-1}. The next theorem means that the first return map of W_n on D_{n-1} is nothing other than the domain exchange W_{n-1}. For a definition of the first return map, see Definition 3.3.2.

Theorem 8.3.2. *The induced transformation $W_n|_{D_{n-1}}$ of W_n into D_{n-1} coincides with W_{n-1}. More precisely, let $s_k^{(n,i)}$ denote the letters of $\sigma^n(i)$:*

$$\sigma^n(i) = s_1^{(n,i)} s_2^{(n,i)} \cdots s_{l(n,i)}^{(n,i)}, \quad i = 1, 2, 3.$$

Then the following relation holds:

$$W_n^{k-1}\left(D_{n-1}^{(i)}\right) \subset D_n^{(s_k^{(n,i)})}, \quad k = 1, 2, \cdots, l(n,i) \tag{8.3}$$

$$W_n^{l(n,i)}\left(D_{n-1}^{(i)}\right) = D_{n-1}^{(i)'}. \tag{8.4}$$

Indeed, according to (8.3), the successive iterates of the domain exchange W_n map the sub-domain $D_{n-1}^{(i)}$ of D_{n-1} onto the sub-domains $D_n^{(j)}$ of D_n, the index j being a letter of $\sigma^n(i)$ depending on the index of iteration. All the sub-domains $D_n^{(j)}$ are disjoint from D_{n-1} before the $l(n,i)$-th iteration, where $l(n,i)$ denote the length of $\sigma^n(i)$. For this iteration, (8.4) means that $D_{n-1}^{(i)}$ is mapped on D_{n-1}, and that its image is precisely the set $D_{n-1}^{(i)'}$, that is $W_{n-1}(D_{n-1}^{(i)})$. Thus, the first return map of W_n on D_{n-1} (i.e., its induced transformation) is exactly the domain exchange W_{n-1}.

An illustration of this theorem for $n = 3$ is given in Fig. 8.22, in the case of the **Rauzy substitution** (see Example 8.1.1).

The commutative diagram represents the action of the domain exchanges W_2 and W_3, respectively on the domains D_2 and D_3, the set D_3 containing the set D_2.

The pictures on the right hand side of the figure represent the successive actions of W_3 on $D_2^{(1)}$. Initially, $D_2^{(1)}$ is included in $D_3^{(1)}$ (because $\sigma(1)$ starts with 1). Then W_3 maps this set inside $D_3^{(2)}$ (because $\sigma(1)$ continues with 2). And since $\sigma(1)$ is of length 2, W_3 maps $W_2(D_2^{(1)})$ inside D_2, more precisely onto $D_2^{(1)'}$.

Since every domain D_n contains the first domain D_0, we deduce from Theorem 8.3.2 the following result.

Corollary 8.3.1. *Let us denote the letters occurring in σ^n by:*

$$\sigma^n(i) = s_1^{(n,i)} s_2^{(n,i)} \cdots s_{l(n,i)}^{(n,i)}, \quad i = 1, 2, 3.$$

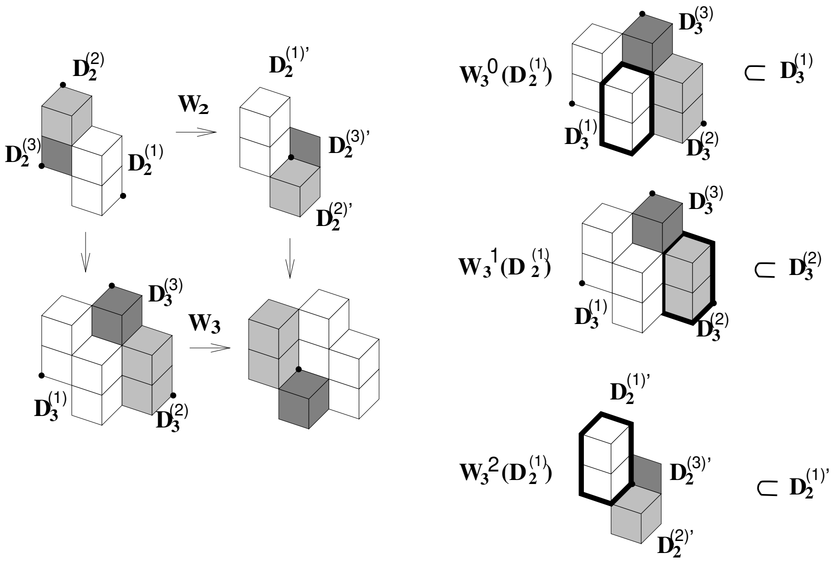


Fig. 8.22. Illustration of Theorem 8.3.2 for the Rauzy substitution

Then we have

$$W_n^{k-1}\left(D_0^{(i)}\right) \subset D_n^{\left(s_k^{(n,i)}\right)},\ k=1,2,\cdots,l(n,i) \tag{8.5}$$

$$W_n^{l(n,i)}\left(D_0^{(i)}\right) = D_0^{(i)'}. \tag{8.6}$$

In particular, we have

$$\left\{W_n^k(\mathbf{0}) \mid k=1,2,\cdots,l(n,1)\right\} = \left\{ -\pi\sum_{j=1}^{k} \boldsymbol{f}^{(n)}_{s_j^{(n,1)}} \;\middle|\; k=1,2,\cdots,l(n,1)\right\}, \tag{8.7}$$

where $\mathbf{M}_\sigma^{-n} = \left[\boldsymbol{f}_1^{(n)}, \boldsymbol{f}_2^{(n)}, \boldsymbol{f}_3^{(n)}\right]$.

Formulas (8.5) and (8.6) are illustrated in Fig. 8.23 for the **Rauzy substitution** in the case $n=3$. Indeed, on the top of this figure one finds the action of the domain exchange W_3 on D_3. Then are drawn the successive images of $D_0^{(1)}$ through the iterations of W_3. These images are successively included into the sets $D_3^{(j)}$, where j follows the sequence 1213121, that is $\sigma^3(1)$. In this way, these iterations of W_3 on $D_0^{(1)}$ go successively onto all the projections of the unit-cube that D_3 contains. On the 7-th iteration (because $\sigma^3(1)$ is of length 3), $D_0^{(1)}$ comes back into D_0, and more precisely onto $D_0^{(1)'}$.

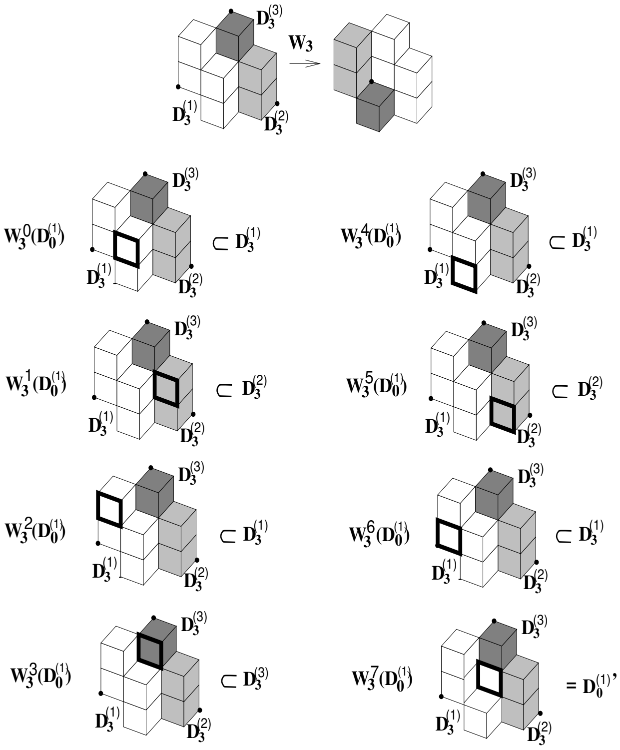


Fig. 8.23. Action of the iterates of W_3 on $D_0^{(1)}$ for the Rauzy substitution

Formula (8.7) is illustrated in Fig. 8.24 for the Rauzy substitution where $\sigma^3(1) = 1213121$ and $\left[\boldsymbol{f}_1^{(3)}, \boldsymbol{f}_2^{(3)}, \boldsymbol{f}_3^{(3)}\right] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$.

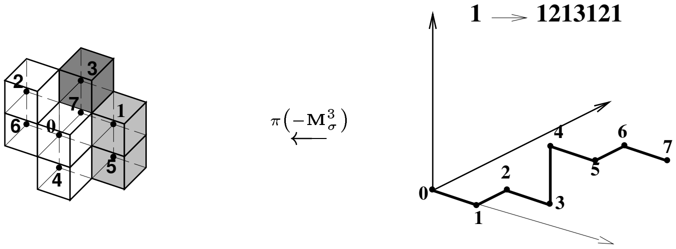


Fig. 8.24. Illustration of (8.7) for the Rauzy substitution

Quasi periodic motion. The domain exchange transformation $W_0 : D_0 \to D_0$ (Fig. 8.25) coincides with the following quasi-periodic motion, that is,

$$W_0 : \begin{array}{rcl} D_0 & \longrightarrow & D_0 \\ \boldsymbol{x} & \mapsto & \boldsymbol{x} - \pi(\boldsymbol{e}_1) \quad (\mathrm{mod}\ \boldsymbol{L}_0) \end{array}$$

where $\boldsymbol{L}_0 = \{n\pi(\boldsymbol{e}_2 - \boldsymbol{e}_1) + m\pi(\boldsymbol{e}_3 - \boldsymbol{e}_1) \mid m, n \in \mathbb{Z}\}$ (see Fig. 8.26).

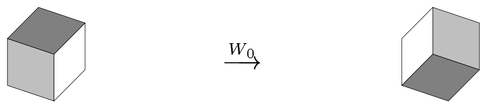


Fig. 8.25. Action of W_0.

Moreover, it can be shown that W_0 is isomorphic to the quasi-periodic motion given by the rotation $T_{\alpha,\beta}$ on $\mathbb{T}^2$ (see [156]):

$$T_{\alpha,\beta}(x, y) = \left(x + \frac{\alpha}{1+\alpha+\beta},\ x + \frac{\beta}{1+\alpha+\beta}\right) \quad (mod \quad 1)$$

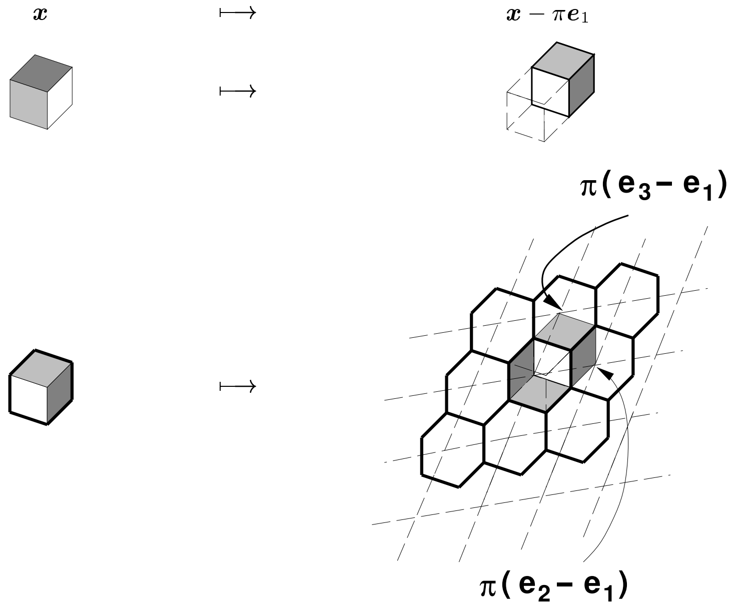


Fig. 8.26. Illustration of the quasi-periodic motion.

8.3.3 The two-dimensional case

For two-letter substitutions, let us introduce analogously the unions of intervals on the line l_γ defined by:

$$D_n^{(i)} := \pi\left(\Theta^n(\boldsymbol{e}_i, i^*)\right), \qquad D_n^{(i)'} := \pi\left(\Theta^n(\boldsymbol{0}, i^*)\right), \quad i = 1, 2.$$

$$D_n := \pi\left(\Theta^n\left(\textstyle\sum_{i=1,2}(\boldsymbol{e}_i, i^*)\right)\right), \qquad D'_n := \pi\left(\Theta^n\left(\textstyle\sum_{i=1,2}(\boldsymbol{0}, i^*)\right)\right).$$

Then the following "union of intervals" exchange on D_n is well defined:

$$\begin{array}{ccl} D_n & \xrightarrow{W_n} & D_n \\ \boldsymbol{x} & \mapsto & \boldsymbol{x} - \pi \boldsymbol{f}_i^{(n)} \quad \text{, if } \boldsymbol{x} \in D_n^{(i)}, \end{array}$$

and this map is isomorphic to the interval exchange

$$\begin{array}{ccl} D_0 & \xrightarrow{W_0} & D_0 \\ \boldsymbol{x} & \mapsto & \boldsymbol{x} - \pi \boldsymbol{e}_i \quad \text{if } \boldsymbol{x} \in D_0^{(i)} \end{array}$$

where $\mathbf{M}_\sigma^{-n} = \left[\boldsymbol{f}_1^{(n)}, \boldsymbol{f}_2^{(n)}\right]$.

The map W_n is illustrated on Fig. 8.27, for the substitution of Example 8.2.3 defined by $\sigma(1) = 121$ and $\sigma(2) = 12$.

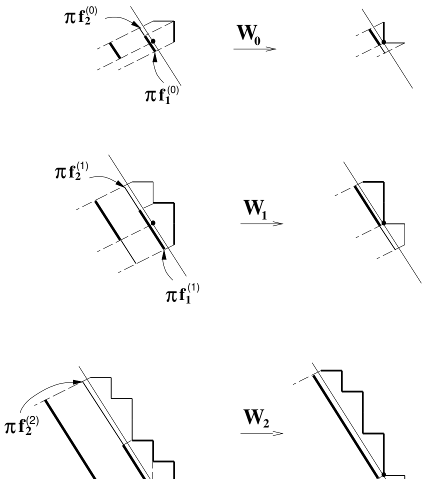


Fig. 8.27. The maps W_0, W_1 and W_2 for the substitution $\sigma(1) = 121$, $\sigma(2) = 12$.

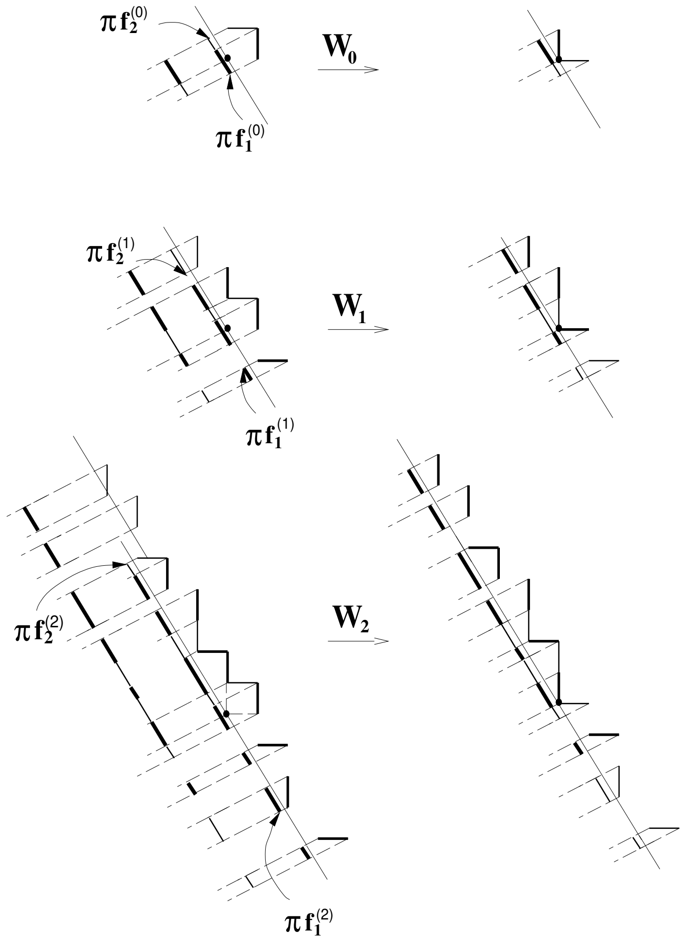


Fig. 8.28. The maps W_0, W_1 and W_2 for the substitution $\sigma(1) = 121$, $\sigma(2) = 21$.

The map W_n is illustrated on Fig. 8.28 for the substitution of Example 8.2.4 defined by $\sigma(1) = 121$ and $\sigma(2) = 21$.

The statements of Theorem 8.3.2 and Corollary 8.3.1 also hold in the case of a two-letter alphabet.

8.4 Renormalization and realization of substitutive dynamical systems

Our aim is to find a domain exchange (and even a translation on the torus $\mathbb{T}^2$) which describes the dynamics of the substitution. We first prove that for (unimodular, Pisot) substitutions the sequence of sets D_n, when properly renormalized, converges to a compact space X_σ in the Hausdorff topology. We then equip this limit set with two transformations producing a realization of the substitutive dynamical system.

Renormalized limit sets. The sequence of sets D_n is increasing. But all these sets are included in the contracting hyperplane $\mathcal{P}_{\delta,\gamma}$ of the matrix $\mathbf{M}_\sigma$. Therefore, the action of $\mathbf{M}_\sigma$ on D_n is a contraction.

A simple computation (see [37, 38]) proves that $\mathbf{M}_\sigma^n$ contracts the sets D_n in such a way that the limit set exists in the sense of the Hausdorff metric on the family of compact subsets of $\mathcal{P}_{\gamma,\delta}$:

$$\begin{aligned} X_\sigma &:= \lim_{n\to\infty} \mathbf{M}_\sigma^n \left(\pi\left(\Theta^n(\mathcal{U})\right)\right) \left(= \lim_{n\to\infty} \mathbf{M}_\sigma^n \left(\pi\left(\Theta^n(\mathcal{U}')\right)\right)\right), \\ X_i &:= \lim_{n\to\infty} \mathbf{M}_\sigma^n \left(\pi\left(\Theta^n(\boldsymbol{e}_i, i^*)\right)\right), \\ X_i' &:= \lim_{n\to\infty} \mathbf{M}_\sigma^n \left(\pi\left(\Theta^n(\mathbf{0}, i^*)\right)\right). \end{aligned}$$

We can now equip X_σ with two dynamical systems: the first one is a quasi-periodic motion (isomorphic with a rotation in $\mathbb{T}^2$) which describes the shift on the dynamical system generated by the fixed point u of σ; the second one is a Markov endomorphism with structure matrix $\mathbf{M}_\sigma$ and describes the action of σ on this dynamical system.

Then the following theorem holds.

Theorem 8.4.1. *Let σ be a Pisot substitution and let us assume that there exist $N > 0$, $N \in \mathbb{N}$ and $i \in \mathcal{A}$ such that $\Theta^N(\boldsymbol{e}_i, i^*) \succ \mathcal{U}$.*

Then the limit sets X_σ and X_i satisfy the following properties:

(1) *The set X_σ is the union disjoint in measure (i.e., up to a set of zero measure) of the sets X_i, that is $\mu(\ X_i \bigcap X_j\ ) = 0$, $(i \neq j\ )$ where μ denotes the Lebesgue measure:*

$$X_\sigma = \bigcup_{i=1,2,3} X_i \quad \textit{(disjoint in measure)}\ ,$$

The set X_σ generates a periodic tiling of the contracting plane $\mathcal{P}_{\gamma,\delta}$, for the lattice $\boldsymbol{L}_0 = \{n\pi(\boldsymbol{e}_2 - \boldsymbol{e}_1) + m\pi(\boldsymbol{e}_3 - \boldsymbol{e}_1) \mid m, n \in \boldsymbol{Z}\}$. More precisely, $\mu(\ (X + \boldsymbol{z}) \bigcap (X + \boldsymbol{z'})\ ) = 0$ for every $\boldsymbol{z} \neq \boldsymbol{z}' \in \boldsymbol{L}_0$.

$$\mathcal{P}_{\gamma,\delta} = \bigcup_{\boldsymbol{z}\in\boldsymbol{L}_0} (X + \boldsymbol{z}) \quad \textit{(disjoint in measure)}\ .$$

(2) *The following transformation $W : X_\sigma \to X_\sigma$ is well-defined and is isomorphic to*

$$W_0 : D_0 \to D_0,$$

$$\boldsymbol{x} \mapsto \boldsymbol{x} - \pi(\boldsymbol{e}_i),\ \textit{if} \quad \boldsymbol{x} \in X_i.$$

(3) *The induced transformation* $W|_{\mathbf{M}_\sigma X_\sigma}$ *is isomorphic to* W *and satisfies*

$$W^{k-1}(X_i^{(1)}) \subset X_{s_k^{(i)}}, \quad k = 1, 2, \cdots, l(i),$$
$$W^{l(i)}(X_i^{(1)}) = X_i^{(1)'},$$

where $\sigma(i) = s_1^{(i)} \cdots s_{l(i)}^{(i)}$ *and* $X_i^{(1)} := \mathbf{M}_\sigma X_i \quad (X_i^{(1)'} := \mathbf{M}_\sigma X_i')$.

(4) *The following transformation* $T : X_\sigma \longrightarrow X_\sigma$ *is well defined:*

$$T\boldsymbol{x} = \mathbf{M}_\sigma^{-1}\boldsymbol{x} - \mathbf{M}_\sigma^{-1}\left(f(W)\right) + \boldsymbol{e}_j \quad , \textit{ if} \quad \mathbf{M}_\sigma^{-1}\boldsymbol{x} \in \mathbf{M}_\sigma^{-1}\left(f(W)\right) + X_j,$$

where the word W *is given by the following formula:*

$$\Theta(\mathbf{0}, i^*) := \sum_{j=1,2,3} \sum_{W:\sigma(j)=Y\cdot i\cdot W} \left(\mathbf{M}_\sigma^{-1}\left(f(W)\right), j^*\right).$$

The transformation T *is a Markov endomorphism, the structure matrix of which is* $\mathbf{M}_\sigma$*.*

Remark. Theorem 8.4.1 requires the assumption that the sequences of approximations we build for the domain exchange contains a fixed open set, and more precisely, that there exists N and i such that

$$\Theta^N(\boldsymbol{e}_i, i^*) \succ \mathcal{U}.$$

This sufficient condition guarantees that the substitution system $(\overline{\mathcal{O}}(u), T)$ is isomorphic to a toral translation. It will be noted that this condition can be explicitly checked.

In all known examples, the substitution system is in fact isomorphic to a toral translation. We do not know of any example which does not verify the previous assumption. We believe that this assumption holds for any Pisot substitution. However, it is still an open problem, even in the case of a two-letter alphabet.

Interpretation of the theorem. If a substitution satisfies this assumption, Theorem 8.4.1 means that the dynamical system associated with the substitution has discrete spectrum and is moreover isomorphic to a quasi-periodic motion, i.e., to a rotation on the torus $\mathbb{T}^2$.

The transformation W on the set X_σ is illustrated in Fig. 8.29, for the **Rauzy substitution** of Example 8.1.1.

The transformation W on the set X_σ is illustrated in Fig. 8.30, for the **modified Perron-Frobenius substitution** $\sigma_{(0,1)}$ of Example 8.1.2.

For the substitution of Example 8.2.1, the transformation W on the non-simply connected set X_σ is illustrated in Fig. 8.31.

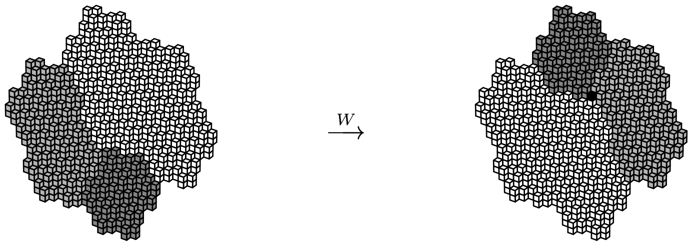


Fig. 8.29. The domain exchange W isomorphic to the dynamical system associated with the Rauzy substitution.

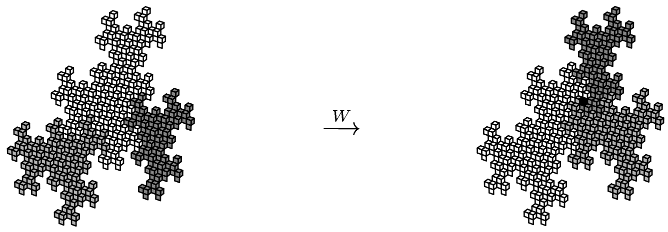


Fig. 8.30. The domain exchange W isomorphic to the dynamical system associated with a modified Jacobi-Perron substitution.

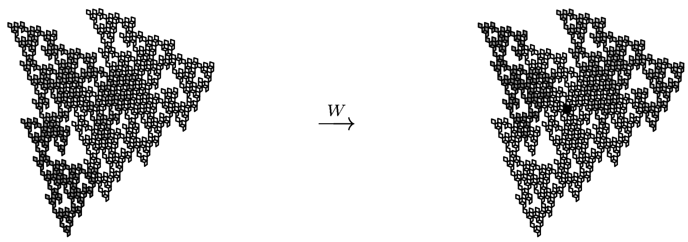


Fig. 8.31. The map W.

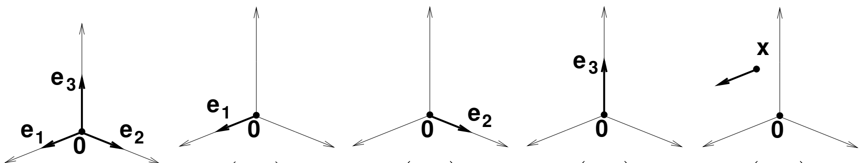
8.5 Fractal boundary

In each example, the boundary of the domain X_σ seems to be fractal.

To observe the properties of these boundaries, let us introduce some notation.

Unit intervals. Let $(\boldsymbol{x}, i) \in \mathbb{Z}^3 \times \{1,2,3\}$ be the following intervals, illustrated on Fig. 8.32:

$$(\boldsymbol{x}, i) = \{\boldsymbol{x} + \lambda \boldsymbol{e}_i \mid 0 \le \lambda < 1\}, \quad i = 1,2,3.$$



$(\boldsymbol{0},1)$ $(\boldsymbol{0},2)$ $(\boldsymbol{0},3)$ $(\boldsymbol{x},1)$

Fig. 8.32. Illustration of $(0, i)$, $i = 1,2,3$ and $(x, 1)$.

Boundary map. Let us define the *boundary map* ∂ on $\mathbb{Z}^3 \times \{1,2,3\}$ by

$$\partial(\boldsymbol{x}, i^*) := (\boldsymbol{x}, j) + (\boldsymbol{x} + \boldsymbol{e}_j, k) - (\boldsymbol{x}, k) - (\boldsymbol{x} + \boldsymbol{e}_k, j),$$

where $\{i,j,k\} \in \{\{1,2,3\},\{2,3,1\},\{3,1,2\}\}$.

This map extends in a natural way to $\mathcal{G}$ as follows: given $\gamma = \sum_{\lambda \in \Lambda} m_\lambda \lambda \in \mathcal{G}$, we set

$$\partial\gamma = \sum_{\lambda \in \Lambda} m_\lambda \partial\lambda.$$

The set of points $\partial\gamma$, for $\gamma \in \mathcal{G}$, is denoted by $\mathcal{G}_1$. Then $\mathcal{G}_1$ is a $\mathbb{Z}$-module.

Boundary endomorphism of $\mathcal{G}_1$.

Definition 8.5.1. *An endomorphism θ of $\mathcal{G}_1$ is called a* boundary endomorphism *associated with Θ if the following commutative relationship is satisfied:*

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\Theta} & \mathcal{G} \\ \partial \downarrow & & \downarrow \partial \\ \mathcal{G}_1 & \xrightarrow{\theta} & \mathcal{G}_1 \end{array}.$$

Roughly speaking, the boundary of the image under Θ of a weighted sum of faces is the image of the boundary of this sum under the map θ, which acts on weighted sums of unit intervals.

The following theorem proves the existence of such a boundary endomorphism. Let us first fix some notation:

$$\boldsymbol{y} + (\boldsymbol{x}, i) := (\boldsymbol{y} + \boldsymbol{x}, i);$$

$$\forall\, t, u \in \{1,2,3\}, \quad (\boldsymbol{x}, t \wedge u) = \begin{cases} (\boldsymbol{x}, s) & \text{if } (t,u) \in \{(2,3),(3,1),(1,2)\} \\ -(\boldsymbol{x}, s) & \text{if } (t,u) \in \{(3,2),(1,3),(2,1)\} \end{cases}$$

where $\{s,t,u\} = \{1,2,3\}$.

Theorem 8.5.1 ([38]). *Let σ be a Pisot substitution, and write $\sigma(i) = s_1^{(i)} \cdots s_{l(i)}^{(i)}, \quad i = 1,2,3.$*

Let θ be defined on the unit intervals $(\mathbf{0}, i)$ by

$$\theta(\mathbf{0}, i) := \sum_{\substack{1 \le t \le 3 \\ 1 \le u \le 3}} \sum_{\substack{s_l{}^{(t)} = j \\ s_m^{(u)} = k}} \left(\mathbf{M}_\sigma^{-1} \left(f\left(S_l^{(t)} \right) \right) + \mathbf{M}_\sigma^{-1} \left(f\left(S_m^{(u)} \right) \right), t \wedge u \right)$$

where • $\{i,j,k\} \in \{\{1,2,3\},\{2,3,1\},\{3,1,2\}\}$,
• *$S_l^{(t)}$ in the suffix of $s_l^{(t)}$ in $\sigma(t)$, i.e., $\sigma(t) = P_l{}^{(t)} s_l{}^{(t)} S_l{}^{(t)}$.*

We extend the definition of the map θ to $\mathcal{G}_1$ as follows:

$$\theta(\boldsymbol{x}, i) := \mathbf{M}_\sigma^{-1}\boldsymbol{x} + \theta(\mathbf{0}, i),$$
$$\theta\Big(\sum_{\lambda \in \Lambda} n_\lambda \lambda\Big) := \sum_{\lambda \in \Lambda} n_\lambda \theta(\lambda).$$

Then the map θ is a boundary endomorphism of Θ.

For the **Rauzy substitution** of Example 8.1.1, the map θ is defined as follows:

$$\begin{aligned} \theta(\mathbf{0}, 1) &= (\mathbf{0}, 1 \wedge 2) = (\mathbf{0}, 3), \\ \theta(\mathbf{0}, 2) &= \big(\mathbf{M}_\sigma{}^{-1}(\boldsymbol{e}_2), 2 \wedge 1\big) + (\mathbf{0}, 2 \wedge 3) \\ &= -\,([1,0,-1],3) + (\mathbf{0}, 1), \\ \theta(\mathbf{0}, 3) &= \big(\mathbf{M}_\sigma{}^{-1}(\boldsymbol{e}_3), 2 \wedge 1\big) + (\mathbf{0}, 3 \wedge 1) \\ &= -\,([0,1,-1],3) + (\mathbf{0}, 2). \end{aligned}$$

This map θ is represented in Fig. 8.33.

Computation of Haussdorf dimension. Using the boundary endomorphism θ, if θ satisfies some conditions with respect to the cancellation, then we can calculate explicitly the dimension of X_σ (see [186], [188] and [187]).

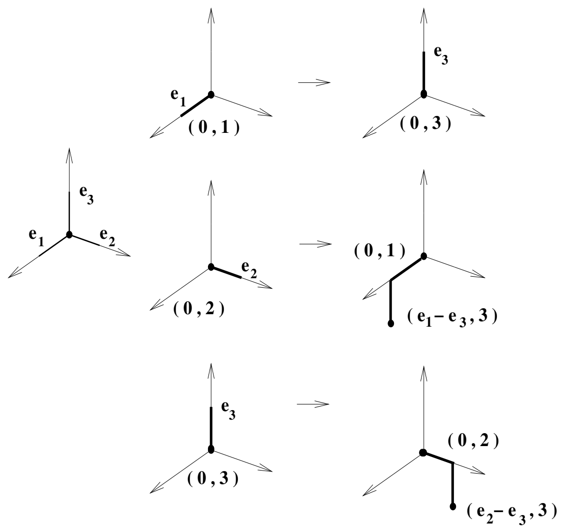


Fig. 8.33. Boundary endomorphism for the Rauzy substitution.

8.6 Diophantine algorithm and substitution

The aim of this section is to apply the previous constructions to the modified Jacobi-Perron substitutions and to make explicit the connection with the corresponding simultaneous continued fraction algorithm. We first introduce this algorithm, and then we discuss its natural extension. We conclude by stating the previous results in the one-dimensional case.

The classical Jacobi-Perron algorithm is an example of multi-dimensional generalization of the regular continued fraction expansion. This algorithm generates a sequence of simultaneous rational approximations of a pair of points with the same denominator (see for instance [52, 77, 315]). The modified Jacobi-Perron algorithm introduced by E. V. Podsypanin in [278] shares this property. Moreover, both algorithms are shown to be strongly convergent (in the sense of Brentjes [77]) almost everywhere with exponential rate (see also [214]); the Jacobi-Perron case is dealt with in [319] and the modified Jacobi-Perron in [154] (see also for a simpler proof [240]). Both Jacobi-Perron algorithms (the classical one and the modified one) are known to have an invariant ergodic probability measure equivalent to the Lebesgue measure (see [315, 317]). However, this measure is not known explicitly in the classical case (the density of the measure is shown to be a piecewise analytical function in [79]), whereas it is known explicitly for the modified case [316, 318, 154].

8.6.1 The modified Jacobi-Perron algorithm

Let X be equal to $X = [0,1) \times [0,1)$ and

$$X_0 = \{(\alpha,\beta) \mid \alpha \geq \beta\},$$
$$X_1 = \{(\alpha,\beta) \mid \alpha < \beta\}.$$

Let us define the transformation T on X by

$$T(\alpha,\beta) := \begin{cases} \left(\dfrac{\beta}{\alpha}, \dfrac{1}{\alpha} - \left[\dfrac{1}{\alpha}\right]\right) & \text{if } (\alpha,\beta) \in X_0 - \{(0,0)\}, \\ \left(\dfrac{1}{\beta} - \left[\dfrac{1}{\beta}\right], \dfrac{\alpha}{\beta}\right) & \text{if } (\alpha,\beta) \in X_1, \\ (0,0) & \text{if } (\alpha,\beta) = (0,0). \end{cases}$$

By using the integer value functions

$$a(\alpha,\beta) := \begin{cases} \left[\dfrac{1}{\alpha}\right] & \text{if } (\alpha,\beta) \in X_0, \\ \left[\dfrac{1}{\beta}\right] & \text{if } (\alpha,\beta) \in X_1, \end{cases}$$

$$\varepsilon(\alpha,\beta) := \begin{cases} 0 & \text{if } (\alpha,\beta) \in X_0, \\ 1 & \text{if } (\alpha,\beta) \in X_1, \end{cases}$$

on $X_\sigma - \{(0,0)\}$, we define for each $(\alpha,\beta) \in X - \{(0,0)\}$ a sequence of digits ${}^t(a_n, \varepsilon_n)$ by

$${}^t(a_n, \varepsilon_n) := {}^t\left(a\left(T^{n-1}(\alpha,\beta)\right), \varepsilon\left(T^{n-1}(\alpha,\beta)\right)\right), \text{ if } T^{n-1}(\alpha,\beta) \neq (0,0).$$

We denote

$$(\alpha_n, \beta_n) := T^n(\alpha,\beta).$$

Definition 8.6.1. *The triple* $(X, T, (a(\alpha,\beta), \varepsilon(\alpha,\beta)))$ *is called the* modified Jacobi-Perron algorithm.

8.6.2 The natural extension

Let us introduce a transformation $(\overline{X}, \overline{T})$ called a *natural extension* of the modified Jacobi-Perron algorithm. Roughly speaking, we try to make the map T one-to-one by considering a larger space.

Let $\overline{X} = X \times X$ and $\overline{T}$ be the transformation defined on $\overline{X}$ by

$$\overline{T}(\alpha,\beta,\gamma,\delta) = \begin{cases} \left(\dfrac{\beta}{\alpha}, \dfrac{1}{\alpha} - a_1, \dfrac{\delta}{a_1+\gamma}, \dfrac{1}{a_1+\gamma}\right) & \text{if } (\alpha,\beta) \in X_0 - \{(0,0)\}, \\ \left(\dfrac{1}{\beta} - a_1, \dfrac{\alpha}{\beta}, \dfrac{1}{a_1+\delta}, \dfrac{\gamma}{a_1+\delta}\right) & \text{if } (\alpha,\beta) \in X_1, \\ (0,0,\gamma,\delta) & \text{if } (\alpha,\beta) = (0,0). \end{cases}$$

Then we know that the transformation $\overline{T}$ is bijective from $(X - \{(0,0)\}) \times X$ to $X \times (X - \{(0,0)\})$.

We denote

$$(\alpha_n, \beta_n, \gamma_n, \delta_n) := \overline{T}^n(\alpha,\beta,\gamma,\delta).$$

8.6.3 A matricial point of view

For every $a \in \mathbb{N}^+$, let us introduce the following family of matrices:

$$A_{(a,0)} = \begin{bmatrix} a & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_{(a,1)} = \begin{bmatrix} a & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then we have the following formulas:

$$\begin{pmatrix} 1 \\ \alpha_n \\ \beta_n \end{pmatrix} = \frac{1}{\theta\theta_1 \cdots \theta_{n-1}} A^{-1}_{(a_n,\varepsilon_n)} A^{-1}_{(a_{n-1},\varepsilon_{n-1})} \cdots A^{-1}_{(a_1,\varepsilon_1)} \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ \gamma_n \\ \delta_n \end{pmatrix} = \frac{1}{\eta\eta_1 \cdots \eta_{n-1}} \, {}^tA_{(a_n,\varepsilon_n)} \, {}^tA_{(a_{n-1},\varepsilon_{n-1})} \cdots {}^tA_{(a_1,\varepsilon_1)} \begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix}$$

$$\text{where} \quad \theta_k = \max(\alpha_k, \beta_k), \qquad \eta_k = \begin{cases} a_k + \gamma_{k-1} & \text{if } (\alpha_{k-1}, \beta_{k-1}) \in X_0 \\ a_k + \delta_{k-1} & \text{if } (\alpha_{k-1}, \beta_{k-1}) \in X_1 \end{cases}.$$

8.6.4 A geometrical point of view

Let $\mathcal{P}_{\gamma_n,\delta_n}$ be the orthogonal plane with respect to ${}^t(1, \gamma_n, \delta_n)$:

$$\mathcal{P}_{\gamma_n,\delta_n} = \{ \boldsymbol{x} \mid \langle \boldsymbol{x}, {}^t(1, \gamma_n, \delta_n) \rangle = 0 \}$$

and let us introduce the linear map $\varphi_{(a_n,\varepsilon_n)} : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix $A_{(a_n,\varepsilon_n)}$ in the canonical basis of $\mathbb{R}^3$. Then we see that the following relationship is true:

$$\varphi^{-1}_{(a_n,\varepsilon_n)} \mathcal{P}_{\gamma_{n-1},\delta_{n-1}} = \mathcal{P}_{\gamma_n,\delta_n}.$$

Link with the modified Jacobi-Perron substitutions. The endomorphisms $\Theta_{(a,\varepsilon)}$ associated with the modified Jacobi-Perron substitutions $\sigma_{(a,\varepsilon)}$ satisfy the following property:

$$\Theta_{(a_n,\varepsilon_n)} \cdots \Theta_{(a_2,\varepsilon_2)} \Theta_{(a_1,\varepsilon_1)}(\mathcal{U}) \in \Gamma^+_{\gamma_n,\delta_n}.$$

Let us consider the renormalization of the set

$$\pi_n \Theta_{(a_n,\varepsilon_n)} \cdots \Theta_{(a_2,\varepsilon_2)} \Theta_{(a_1,\varepsilon_1)}(\mathcal{U}),$$

that is,

$$\lim_{n\to\infty} \mathbf{M}_{(a_1,\varepsilon_1)} \cdots \mathbf{M}_{(a_n,\varepsilon_n)} \pi_n \Theta_{(a_n,\varepsilon_n)} \cdots \Theta_{(a_1,\varepsilon_1)}(\mathcal{U})$$

where $\pi_n : \mathbb{R}^3 \to \mathcal{P}_{\gamma_n,\delta_n}$ in the projection along ${}^t(1, \alpha_n, \beta_n)$. The properties of this set are stated in the following theorem.

Theorem 8.6.1 ([185]). *For almost every $\gamma, \delta \in [0,1) \times [0,1)$ the following limit sets exist*

$$X^{(i)}_{\alpha,\beta,\gamma,\delta} := \lim_{n\to\infty} \mathbf{M}^{-1}_{(a_1,\varepsilon_1)} \cdots \mathbf{M}^{-1}_{(a_n,\varepsilon_n)} \pi_n \Theta_{(a_n,\varepsilon_n)} \cdots \Theta_{(a_1,\varepsilon_1)}(\boldsymbol{e}_i, i^*),$$
$$X^{(i)'}_{\alpha,\beta,\gamma,\delta} := \lim_{n\to\infty} \mathbf{M}^{-1}_{(a_1,\varepsilon_1)} \cdots \mathbf{M}^{-1}_{(a_n,\varepsilon_n)} \pi_n \Theta_{(a_n,\varepsilon_n)} \cdots \Theta_{(a_1,\varepsilon_1)}(\mathbf{0}, i^*).$$

They satisfy the following properties:

(1) $X_{\alpha,\beta,\gamma,\delta} = \bigcup_{i=1,2,3} X^{(i)}_{\alpha,\beta,\gamma,\delta}$ *is a periodic tiling on* $\mathcal{P}_{\gamma,\delta}$, *that is,*

$$\bigcup_{\boldsymbol{z}\in \boldsymbol{L}_0} (X_{\alpha,\beta,\gamma,\delta} + \boldsymbol{z}) = \mathcal{P}_{\gamma,\delta};$$
$$\mathit{int}(X_{\alpha,\beta,\gamma,\delta} + \boldsymbol{z}) \bigcap \mathit{int}(X_{\alpha,\beta,\gamma,\delta} + \boldsymbol{z}') = \phi \quad (\boldsymbol{z} \neq \boldsymbol{z}').$$

(2) the following domain exchange transformation $W_{\alpha,\beta,\gamma,\delta}$ *is well-defined:*

$$W_{\alpha,\beta,\gamma,\delta} : X_{\alpha,\beta,\gamma,\delta} \to X_{\alpha,\beta,\gamma,\delta}$$

$$\boldsymbol{x} \mapsto \boldsymbol{x} - \pi_0(\boldsymbol{e}_i) \ , \ \mathit{if} \ \ \boldsymbol{x} \in X^{(i)}_{\alpha,\beta,\gamma,\delta}.$$

Moreover, consider the following sets:

$$X_n := \bigcup_{i=1,2,3} X^{(i)}_{\alpha_n,\beta_n,\gamma_n,\delta_n} \subset \mathcal{P}_{\gamma_n,\delta_n},$$
$$\widehat{X}_n := \mathbf{M}^{-1}_{(a_1,\varepsilon_1)} \cdots \mathbf{M}_{(a_n,\varepsilon_n)}(X_n) \subset X_{\alpha,\beta,\gamma,\delta},$$
$$\widehat{X}_{n,i} := \mathbf{M}_{(a_1,\varepsilon_1)} \cdots \mathbf{M}^{-1}_{(a_n,\varepsilon_n)}(X^{(i)}_{\alpha_n,\beta_n,\gamma_n,\delta_n}) \subset X_{\alpha,\beta,\gamma,\delta}.$$

Then

$$W^{k-1}_{\alpha,\beta,\gamma,\delta}\left(X^{(i)}_{\alpha_n,\beta_n,\gamma_n,\delta_n}\right) \subset X^{s_k^{(n,i)}}_{\alpha,\beta,\gamma,\delta}, \quad 1 \le k \le q_n + p_n + r_n,$$
$$W^{q_n+p_n+r_n}_{\alpha,\beta,\gamma,\delta}\left(X^{(i)}_{\alpha_n,\beta_n,\gamma_n,\delta_n}\right) = X^{(i)'}_{\alpha,\beta,\gamma,\delta},$$

where $\sigma_{(a_1,\varepsilon_1)} \cdots \sigma_{(a_n,\varepsilon_n)}(i) = s_1^{(n,i)} \cdots s^{(n,i)}_{l(n,i)}$,

$$\begin{bmatrix} q_n & * & * \\ p_n & * & * \\ r_n & * & * \end{bmatrix} := A_{(a_1,\varepsilon_1)} A_{(a_2,\varepsilon_2)} \cdots A_{(a_n,\varepsilon)}.$$

8.6.5 The one-dimensional case and usual continued fractions

In the one-dimensional case, we consider the usual continued fraction algorithm

$$T\alpha = \frac{1}{\alpha} - \left[\frac{1}{\alpha}\right] \quad \text{on } [0,1] \text{ if } \alpha \neq 0, \text{ otherwise } T(0) = 0,$$

and the continued fraction expansion:

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{1}{\ddots + \cfrac{1}{a_n + T^n\alpha}}}}}$$

where $a_k := \left[\frac{1}{T^{k-1}\alpha}\right]$.

Instead of the plane $\mathcal{P}_{\gamma,\delta}$, we introduce the line l_γ,

$$l_\gamma := \left\{ {}^t(x,y) \;\middle|\; \left\langle {}^t(x,y),\ {}^t(1,\gamma) \right\rangle = 0 \right\}$$

and the stepped curves $\mathcal{S}_\gamma$. We also introduce the natural extension $\overline{T}$ of T, defined by

$$\overline{T}(\alpha,\gamma) := \left(\frac{1}{\alpha} - a_1, \frac{1}{a_1+\gamma} \right).$$

Let us introduce the linear map $\varphi_a : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix in the canonical basis of $\mathbb{R}^3$:

$$\mathbf{A}_a = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then the map φ_a satisfies

$$\varphi_{a_n}^{-1} l_{\gamma_{n-1}} = l_{\gamma_n}$$

where

$$(\alpha_n, \gamma_n) := T^n(\alpha,\gamma).$$

Moreover, we already know that

$$\Theta_{a_n}\left(\mathcal{S}_{\gamma_{n-1}}\right) = \mathcal{S}_{\gamma_n}$$

where Θ_a was defined in Sec. 8.2.5.

8.7 Applications

8.7.1 Quasi-periodic tiling related to the stepped surface

Let $\mathcal{P}_{\gamma,\delta}$, $0 < \gamma, \delta < 1$ be the plane given by

$$\mathcal{P}_{\gamma,\delta} = \left\{ \boldsymbol{x} \;\middle|\; \left\langle \boldsymbol{x},\ {}^t(1,\gamma,\delta) \right\rangle = 0 \right\},$$

and let $\mathcal{S}_{\gamma,\delta}$ be the stepped surface associated with $\mathcal{P}_{\gamma,\delta}$.

Let $\pi : \mathbb{R}^3 \longrightarrow \mathcal{P}_{\gamma,\delta}$ be the projection along ${}^t(1,\alpha,\beta)$. Then we have a tiling of $\pi(\mathcal{S}_{\alpha,\beta})$ generated by three parallelograms $\pi(\mathbf{0}, i^*)$, $(i = 1, 2, 3)$ and their translates. Let us denote the above tiling by $\mathcal{T}_{\alpha,\beta}(= \pi\mathcal{S}_{\alpha,\beta})$.

Let Γ_n be the family of patches which are generated by n parallelograms and which are simply connected, that is,

$$\Gamma_n = \{\gamma \mid \gamma \prec \mathcal{T}_{\alpha,\beta},\ \#\gamma = n,\ \gamma \text{ is simply connected } \}.$$

Definition 8.7.1. *A tiling $\mathcal{T}$ of a plane is said to be* quasi-periodic *if for any $n > 0$, there exists $R > 0$ such that any configuration $\gamma \in \Gamma_n$ occurs somewhere in any neighborhood of radius R of any point.*

Theorem 8.7.1. *Let $(1,\gamma,\delta)$ be linearly independent with respect to $\mathbb{Q}$, then the tiling $\mathcal{T}_{\gamma,\delta}$ is a quasi-periodic tiling.*

The essential idea for the proof of this result comes from the following fact: for each $(1,\gamma,\delta)$ there exists a sequence

$$\begin{pmatrix} a_1 & a_2 & \cdots & \cdots \\ \varepsilon_1 & \varepsilon_2 & \cdots & \cdots \end{pmatrix}$$

such that the stepped surface of $\mathcal{P}_{\gamma,\delta}$ is given by

$$\lim_{n\to\infty} \Theta_{(a_1,\varepsilon_1)}\Theta_{(a_2,\varepsilon_2)} \cdots \Theta_{(a_n,\varepsilon_n)}(\mathcal{U}),$$

where the sequence (a_n, ε_n) is obtained by the modified Jacobi-Perron algorithm. It will be recalled that $\Theta_{(a,\varepsilon)}$ is defined in Example 8.1.2 (see [189] for more details). Furthermore, quasi-periodic tilings generated by a non-negative integer matrix are discussed in [157].

8.7.2 The Markov partition of group automorphisms on $\mathbb{T}^3$

Let us consider the matrix $\mathbf{A}\,(= \mathbf{M}_\sigma)$ of a Pisot substitution satisfying the assumption of Theorem 8.4.1.

Over this assumption, let us define the sets $\overline{X}_i, i = 1, 2, 3$ of $\mathbb{R}^3$, by

$$\overline{X}_i = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \in X_i,\ \boldsymbol{y} = \lambda(\boldsymbol{e}_i - \pi\boldsymbol{e}_i),\ 0 \le \lambda < 1\}.$$

Then the fundamental domain $\triangle$:

$$\triangle := \bigcup_{i=1,2,3} \overline{X}_i$$

is the three-dimensional torus, that is,

(1) $\bigcup_{\boldsymbol{z}\in\mathbb{Z}^3} (\triangle + \boldsymbol{z}) = \mathbb{R}^3$,
(2) $\operatorname{int}(\triangle + \boldsymbol{z}) \bigcap \operatorname{int}(\triangle + \boldsymbol{z}') = \phi$ if $\boldsymbol{z} \ne \boldsymbol{z}'$.

Theorem 8.7.2. *Let ξ be the partition of the three-dimensional torus $\triangle(\simeq \mathbb{T}^3)$, that is,*

$$\xi = \left\{\overline{X}_i, i = 1, 2, 3\right\}.$$

Then the partition ξ is a Markov partition of the group automorphism

$$\begin{aligned} T_A : \mathbb{T}^3 &\longrightarrow \mathbb{T}^3 \\ \boldsymbol{x} &\longmapsto \mathbf{A}\boldsymbol{x} \quad (\ mod\ 1) \end{aligned}$$

with the structure matrix ${}^t\mathbf{A}$.

Remark. The existence of Markov partitions of group automorphisms on $\mathbb{T}^n$ is discussed in [3] and [332]. In [76], Bowen claims that the boundary of Markov partition of 3-dimensional group automorphisms cannot be smooth. Theorem 8.7.2 explains how we can construct the (non-smooth) Markov partition (an analogous discussion can be found in [51]).

The following question is reasonable. For any element of $\mathbf{A} \in SL(3, \mathbb{Z})$, does there exist a substitution σ such that its incidence matrix $\mathbf{M}_\sigma$ satisfies the assumption of Theorem 8.4.1? Is $\mathbf{M}_\sigma$ isomorphic to $\mathbf{A}$? We only know that for any $\mathbf{A}$ there exists $N > 0$ such that $\mathbf{A}^N$ satisfies the assumption of Theorem 8.4.1 (see [156]).

8.7.3 Diophantine approximation

Let $\langle 1, \alpha, \beta\rangle$ be the basis of the cubic field $\boldsymbol{Q}(\lambda)$ given by

$$\mathbf{A}\begin{bmatrix}1\\ \alpha\\ \beta\end{bmatrix} = \lambda\begin{bmatrix}1\\ \alpha\\ \beta\end{bmatrix}$$

for some $\mathbf{A} \in SL(3, \boldsymbol{Z})$, and let us assume that λ is a complex Pisot number.

In the above setting, let us consider the limit set of the points

$$\left\{\sqrt{q}\begin{pmatrix}q\alpha - p\\ q\beta - r\end{pmatrix} \ \middle|\ (q, p, r) \in \boldsymbol{Z}^3, q > 0\right\}.$$

Theorem 8.7.3. *The limit set of above points consists in a family of ellipses.*

Theorem 8.7.3 is proved by algebraic geometry in [1]. For a proof of this result using substitutions, see [153].

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