7. Spectral theory and geometric representation of substitutions

There are two categories of particularly simple dynamical systems one can find within symbolic dynamics. The first category consists of the ***subshifts of finite type***, the entropy of which is strictly positive. Some very important literature has been devoted to their many properties (see for instance the references in [223]). As these systems are obtained with a finite number of constraints, one can associate in a natural way to each subshift of finite type some algebraic integer. Subshifts of finite type provide a good description of topological dynamical systems, thanks to the notion of *Markov partitions*. These partitions can be seen as a generalization of the binary system, that is the symbolic representation by the shift on $\{0,1\}^{\mathbb{N}}$ of the multiplication by 2 modulo 1, which is at the origin of the dyadic expansion of real numbers.

The second category of symbolic dynamical systems is that of the ***substitutive dynamical systems***. Let us recall that a substitution is a combinatorial object which associates to any letter of a finite alphabet a nonempty finite word over the alphabet. If the substitution mixes the letters sufficiently (the condition of primitivity), one can associate with the substitution a minimal and uniquely ergodic dynamical system. Substitutive dynamical systems are deterministic, i.e., of zero entropy, so that they are totally different from a subshift of finite type. Nevertheless, like subshifts of finite type, substitutive dynamical systems are defined with the help of a finite type of constraints, so that algebraic integers again naturally appear in the structure of these systems.

The first substitution which has been studied is the Prouhet-Thue-Morse substitution, defined by $\sigma(1) = 12$ and $\sigma(2) = 21$. In 1907, Thue studied the combinatoric of the fixed point of this substitution, within the framework of symbolic sequences without squares. In 1929, M. Morse used this substitution to build nonclosed recurrent geodesics on surfaces with negative curvature. Later, it was proved that the symbolic dynamical system associated with this substitution is a two-point extension of the dyadic odometer, that is the group $\mathbb{Z}_2$ of 2-adic integers. This substitution is studied in detail in Chap. 2 and Chap. 3.

Another significant example is the Fibonacci substitution, defined by $\sigma(1) = 12$ and $\sigma(2) = 1$. This substitution generates a system which, on the one hand, is isomorphic with a translation over the one-dimensional torus, and on the other hand, contains words the combinatorial complexity of which is minimal among nonultimately periodic sequences. Properties of this substitution are given in Chap. 2, 3, 6, 8, and Chap. 9.

The three-letter equivalent of the Fibonacci substitution is the Tribonacci substitution, defined by $\sigma(1) = 12$, $\sigma(2) = 13$ and $\sigma(3) = 1$, also called the Rauzy substitution. G. Rauzy, with methods taken from number theory, proved in 1981 that the symbolic dynamical system associated to this substitution is isomorphic on one hand, with a translation on the two-dimensional torus, and on the other hand, with a domain exchange on a self-similar compact subset of $\mathbb{R}^2$, with fractal boundary [292].

Considering these examples, one can naturally ask whether substitutive dynamical systems are standard or totally new dynamical systems. More precisely, it is interesting to know which substitutive dynamical systems are isomorphic to a rotation on a compact group, and more generally to find their equicontinuous factor.

A precise answer was obtained for substitutions of constant length during the seventies, in particular by T. Kamae [191, 192], J. C. Martin [237] and F. M. Dekking [113]. In particular, we know that the maximal equicontinuous factor of substitutive systems of constant length l is a translation under the direct product of the adic group $\mathbb{Z}_l$ and a finite group. Moreover, there exists a measure-theoretic isomorphism between a substitutive system and its maximal equicontinuous factor, if and only if, the substitution verifies a combinatorial condition called the *coincidence condition*.

It is natural but more difficult to work on substitutions of nonconstant length.

G. Rauzy first tackled this question with a complete study of the system associated with the Tribonacci substitution [292]. Then, B. Host made a significant advance by proving that all eigenfunctions of primitive substitutive dynamical systems are continuous [181]. Thus, very happily, the two main dynamical classifications (up to measure-theoretic and topological isomorphisms) are equivalent for primitive substitutive systems. In the continuation of this, S. Ferenczi, C. Mauduit and A. Nogueira proved that the spectrum of a substitutive system can be divided into two parts. The first part has an arithmetical origin, and depends only on the incidence matrix of the substitution. The second part has a combinatorial origin, and is linked with the return words associated with the fixed point of the substitution [147].

B. Host [181, 182], A. N. Livshits [224, 225, 226, 362], B. and M. Solomyak [336, 337, 338], and M. Hollander [172] gave conditions for a substitutive dynamical system to have discrete spectrum. Some are necessary conditions, the others are sufficient conditions. In particular, B. Host and independently A. N. Livshits, whose work was restated and generalized by M. Hollander, defined a combinatorial condition on two-letter substitutions called *coincidence condition*, which generalizes the condition of F. M. Dekking for constant length substitutions. This condition is equivalent with the fact that a two-letter primitive substitutive system has a discrete spectrum.

Except for the Tribonacci substitution [292] and some examples developed by B. Solomyak [336, 337], these works do not give an explicit realization of the maximal equicontinuous factor. For every unimodular substitution of Pisot type on d letters, P. Arnoux and S. Ito built explicitly an auto-similar compact subset of $\mathbb{R}^{d-1}$ called *Rauzy fractal associated with the substitution* [37]. They generalized the coincidence condition to all substitution of nonconstant length substitutions, and proved that this condition is sufficient for the system on the one hand to be measure-theoretically isomorphic with a domain exchange on the Rauzy fractal of the substitution, on the other hand to admit as a topological factor a minimal translation under the $(d-1)$-dimensional torus. The techniques they used do not provide results for a measure-theoretical isomorphism between the substitutive system and its toral topological translation factor, as was the case for a two-letter alphabet.

An alternative construction of the Rauzy fractal associated with a substitution of Pisot type can be obtained by generalizing the techniques used by G. Rauzy [292, 295], as developed by V. Canterini and A. Siegel [86, 87]. This construction is based on the use of formal power series, and provides new proofs for the results of P. Arnoux and S. Ito in [37]. Moreover, this point of view allows us to give a combinatorial necessary and sufficient condition for a substitutive unimodular system of Pisot type to be isomorphic with its toral topological translation factor [329]. This has consequences for the construction of explicit Markov partitions for toral automorphisms, the main eigenvalue of which is a Pisot number [328]. Those Markov partitions have the interest that their topological and geometrical properties can be studied, such as connectedness [85], fractal boundary [243, 244, 313, 174] or simple connectedness [188].

The aim of this chapter is to provide a more precise description of the above results. However, a good understanding of spectral properties of substitutive dynamical systems requires us to introduce a number of tools and concepts. Thus, we have no claim to being exhaustive in our exposition since each of the notions we refer to requires a consequent material to be exposed in detail. In each case, we refer the interested reader to the literature and to the classical references on these subjects. We hope that even a superficial understanding of these notions will help the reader to replace them in their context.

The chapter is organized as follows. The aim of Secs. 7.1 7.2 and 7.3 is to provide a brief introduction to the classical notion of topological dynamical systems, and to the main systems we will focus on in the rest of the chapter, that is, translations on abelian compact groups (mainly torus and p-adic groups), substitutive dynamical systems (and in particular Pisot type systems) and subshifts of finite type, linked with Markov partitions and exchanges of domains.

In Sec. 7.4 are recalled classical notions about measure-theoretic dynamical systems, already exposed in Chap. 3. We focus on the relationship between the spectrum of such a system and its measure-theoretic factors.

The aim of Sec. 7.5 is to expose the different known results about the spectrum of substitutions of constant length. Sec. 7.6 is devoted to the fundamental notion of recognizability, developed by B. Host, M. Queffelec and B. Mossé [258, 259]. This notion is quite natural but needed a long time to be correctly defined. The lack of results about this notion before B. Mossé's work was one of the reasons for the difficulty of getting results about substitutions of nonconstant length.

An important part of the spectral properties of substitutive dynamical systems with nonconstant length comes from the properties of adic systems. In Sec. 7.7, we explore the deep connections between substitutive systems and adic transformations.

Section 7.8 presents an overview of the general spectral theory of substitutive systems of nonconstant length.

Substitutions of Pisot type constitute an important and extensively studied class. In Sec. 7.9 the main attention is devoted to the consequences of Sec. 7.8 on the spectral properties of systems associated with these substitutions. In Sec. 7.10 we explore the different tools used to get results about the problem of geometrical representation of unimodular systems of Pisot type. Finally, the aim of Sec. 7.11 is to explain how geometrical representation of substitutions of Pisot type provides explicit Markov partition for some Pisot toral automorphisms.

7.1 Topological dynamical systems : examples

Let us recall some basic notions in topological dynamics. For more details, see [103].

A *topological dynamical system* is a continuous map $T : X \to X$ from a compact metric space X onto itself. The dynamical system is said to be *minimal* if X does not contain a nontrivial closed subset which is invariant under the action of T. This system is *uniquely ergodic* if their exists one and only one probability measure on the Borel sets of X which is invariant under T.

Definition 7.1.1. *Two dynamical systems (X_1, T_1) and (X_2, T_2) are said to be* topologically isomorphic *or* topologically conjugate *if and only if their exists an homeomorphism f from X_1 onto X_2 with conjugates T_1 and T_2, that is:*

$$T_1 \circ f = f \circ T_2.$$

A topological dynamical system (Y, T_1) is a topological factor *of (X, T) if their exists a continuous map from X onto Y which conjugates the maps T and T_1.*

7.1.1 Rotations on compact abelian groups

A rotation with irrational angle on the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is minimal and uniquely ergodic, the invariant measure being the Lebesgue measure. More generally, we have:

Proposition 7.1.1 (see [196]). *The rotation by the vector* $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$ *on the d-dimensional torus* $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ *is minimal if and only if* $t_1, \ldots, t_n$ *and 1 are rationally independent.*

Fundamental examples of rotation over a compact group are given by additions in p-adic groups (see Sec. 7.1.2), and of course by additions on finite groups.

Let us stress that any translation on a compact abelian group is a translation over a direct product of torus, p-adic integer groups and finite groups (see [326]). This product can be finite or infinite.

7.1.2 p-adic groups

As translations on p-adic groups play here a decisive role, let us recall some definitions and properties on p-adic groups. One can refer to [29, 71, 162, 309] for more precisions on the following definitions. See also Chap. 4 and Chap. 3.

The *p-adic absolute value* is defined on $\mathbb{Q}$ by the formula $|z|_p = p^{-v_p(z)}$, where $v_p(z)$ is the largest power of p which appears in the decomposition of p in prime factors. This absolute value is *non-Archimedean* (see Chap. 4).

The *field of p-adic numbers* $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ for the p-adic absolute value. The *ring of p-adic integers* $\mathbb{Z}_p$ is the ring of integers of $\mathbb{Q}_p$. The p-adic absolute value extends to $\mathbb{Q}_p$ and is also denoted $|\ |_p$. As a consequence of the fact that v_p is non-Archimedean, a power series converges in $\mathbb{Q}_p$ if and only if its generic term tends to zero.

Topologically, $\mathbb{Z}_p$ is the closed unit ball of $\mathbb{Q}_p$. A basis for the topology of $\mathbb{Q}_p$ consists in the clopen balls $z + p^k\mathbb{Z}_p$, where $k \in \mathbb{N}$ and $z \in \mathbb{Q}_p$. Moreover, $\mathbb{Z}_p$ is the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$.

The set $\mathbb{Z}_p$ of p-adic integers can also be obtained as the *projective limit* of the finite groups $\mathbb{Z}/p^n\mathbb{Z}$. More precisely, for $i \leq j$, let p^i_j denote the canonical projection from $\mathbb{Z}/p^j\mathbb{Z}$ onto $\mathbb{Z}/p^i\mathbb{Z}$. Then we have

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z} = \{(a_i)_{i\in\mathbb{N}};\ a_i \in \mathbb{Z}/p^i\mathbb{Z},\ \ \forall j \geq i,\ p^i_j(a_j) = a_i\}.$$

That way, we get an expansion of the elements of $\mathbb{Q}_p$ with respect to the reference sequence $(p^i)_i$, which tends to zero in $\mathbb{Q}_p$. This expansion, similar to the decimal expansion in $\mathbb{R}$, is called *Hensel expansion*:

$$\forall\, z \in \mathbb{Q}_p,\ \exists\, n_0 \in \mathbb{Z},\ \exists\, (a_i)_{i\geq n_0} \in \{0, \ldots, p-1\},\ \ z = \sum_{i\geq n_0} a_i\, p^i.$$

The coefficient n_0 is nonnegative if and only if $x \in \mathbb{Z}_p$.

From a symbolic point of view, the Hensel expansion means that the addition of 1 on $\mathbb{Z}_p$ is topologically isomorphic with the *p-odometer* (see [163] and Chap. 3), that is, the following *successor map* defined the symbolic space $\{0,\ldots,p-1\}^{\mathbb{N}}$: succ : $\{0,\ldots,p-1\}^{\mathbb{N}} \to \{0,\ldots,p-1\}^{\mathbb{N}}$ Let $w = w_0w_1\cdots \in \{0,\ldots,p-1\}^{\mathbb{N}}$. The successor *succ*$(w)$ of w is the following sequence $v = \mathrm{succ}(w) = v_0v_1\ldots$ defined as:

$$\begin{array}{llll} \text{if } \forall\, i \geq 0,\, w_i = p-1, & \text{then } v_i = 0, \forall\, i \geq 0; \\ \text{if } \begin{cases} \forall\, i < i_0,\, w_i = p-1 \\ w_{i_0} \neq p-1, \end{cases} & \text{then } \begin{cases} \forall\, i < i_0,\, v_i = 0 \\ v_{n_0} = w_{i_0} + 1, \\ \forall\, i > i_0,\, v_i = w_i. \end{cases} \end{array}$$

In particular, on can note that the Hensel map sends the clopen sets $z + p^k\mathbb{Z}_p$ in $\mathbb{Z}_p$ onto the cylinders in $\{0,\ldots,p-1\}^{\mathbb{N}}$. Over this space, the Bernoulli measure is invariant under the action of the odometer. A consequence is:

Proposition 7.1.2. *The addition of 1 on the group $\mathbb{Z}_p$ of p-adic integers is a minimal and uniquely ergodic topological dynamical system.*

The unique probability measure defined over the Borelian sets of $\mathbb{Z}_p$ which is invariant under the addition of 1 is defined on the clopen sets by:

$$\forall\, z \in \widetilde{\mathbb{Z}_p}, \quad \forall\, n \geq 0, \qquad \mu_p(z + p^n\,\mathbb{Z}_p) = 1/p^n.$$

7.2 Substitutions

7.2.1 Substitutive dynamical systems

We recall here some fundamental concepts on substitutions and fix the terminology we use in this chapter. See also Chap. 1 and 3.

Let $\mathcal{A}$ be a finite alphabet, $\mathcal{A}^\star$ the set of finite words in $\mathcal{A}$ and $\mathcal{A}^+$ the subset of $\mathcal{A}^\star$ of nonempty words. The empty word is denoted by ε.

A two-sided *sequence* in $\mathcal{A}$, also called bi-infinite *word*, is denoted $w = \ldots w_{-2}w_{-1}.w_0w_1\ldots$. The topology of the set of bi-infinite words $\mathcal{A}^{\mathbb{Z}}$ is the product topology of the discrete topology over $\mathcal{A}$. This topology is metrizable. A *cylinder* of $\mathcal{A}^{\mathbb{Z}}$ is a clopen set of the form:

$$[U_1.U_2] = \{(w_i)_i \in \mathcal{A}^{\mathbb{Z}} | w_{-|U_1|}\ldots w_{-1}w_0\ldots w_{|U_2|-1} = U_1U_2\},$$

where $U_1, U_2 \in \mathcal{A}^\star$ (if U_1 is empty, the cylinder is denoted $[U_2]$); the cylinders form a basis of the topology on $\mathcal{A}^{\mathbb{Z}}$.

A *substitution* σ is defined in this chapter as an endomorphism of the free-monoid $\mathcal{A}^\star$ which satisfies the following two assumptions:

- the image of each letter of $\mathcal{A}$ is not equal to the nonempty word;

- for every letter $a \in \mathcal{A}$, we have $\lim_{n\to\infty} |\sigma^n(a)| = +\infty$.

A substitution extends naturally to the set of bi-infinite words $\mathcal{A}^{\mathbb{Z}}$:

$$\sigma(\dots w_{-2}w_{-1}.w_0w_1\dots) = \dots\sigma(w_{-2})\sigma(w_{-1}).\sigma(w_0)\sigma(w_1)\dots$$

A bi-infinite word $u = (u_i)_{i\in\mathbb{Z}}$ in $\mathcal{A}^{\mathbb{Z}}$ such that there exists a positive integer k with $\sigma^k(u) = u$ is called a *periodic point* of σ. If $\sigma(u) = u$, then u is a *fixed point* of σ. Every substitution has at least one periodic point (see [285]).

Denote by S the *shift map* on $\mathcal{A}^{\mathbb{Z}}$:

$$S(\,(w_i)_{i\in\mathbb{Z}}\,) = (w_{i+1})_{i\in\mathbb{Z}}.$$

A word w in $\mathcal{A}^{\mathbb{Z}}$ such that there exists a positive integer h with $S^h(w) = w$ is called *S-periodic*. A substitution is said to be *S-periodic* when there exists a periodic point of σ which is also S-periodic. It is *nonperiodic* if the substitution is not S-periodic.

The *language* $\mathcal{L}(w)$ of a bi-infinite word w is the set of finite words which appear in w. The *symbolic dynamical system* generated by a word u is the pair (X_u, S), with $X_u = \{w \in \mathcal{A}^{\mathbb{Z}}, \mathcal{L}(w) \subset \mathcal{L}(u)\}$. Note that X_u is the closure in $\mathcal{A}^{\mathbb{Z}}$ of the shift orbit of u, and that X_u is finite if and only if u is S-periodic. This set is a compact subset of $\mathcal{A}^{\mathbb{Z}}$, and S is an homeomorphism on X_u. See also Chap. 3.

We call a substitution *primitive* if there exists k such that $\sigma^k(a)$ contains at least one occurrence of b for every pair of letters (a, b), k being independent of the letters. If σ is primitive and u is a periodic point of σ, then X_u does not depend on u and we denote by (X_σ, S) the *symbolic dynamical system generated by σ*.

If the substitution is primitive, then the system (X_σ, S) is *minimal*. This system is *uniquely ergodic*: there exists a unique S-invariant probability measure on X_σ, denoted by μ_{X_σ}. The measure of the cylinder $[U_1.U_2]$ is the frequency of the finite word U_1U_2 in any element of X_σ (it will be recalled that the frequencies exist in this case).

In the primitive case, one can note that the system (X_σ, S) is finite if and only the substitution is S-periodic. From here on, all substitutions are assumed to be primitive and non S-periodic.

7.2.2 Abelianization of substitution

If the alphabet $\mathcal{A}$ is $\mathcal{A} = \{1, \dots, d\}$, let $\mathbf{l} : \mathcal{A}^\star \mapsto \mathbb{Z}^d$ be the *canonical homomorphism* associated with σ, obtained by abelianization of the free monoid: if $|W|_a$ denotes the number of occurrences of the letter $a \in \mathcal{A}$ in a finite word W, then we have $\mathbf{l}(W) = (|W|_i)_{i=1,\dots,d} \in \mathbb{N}^d$.

Note that d-tuples are considered as column vectors in this chapter.

To each substitution σ defined over $\mathcal{A}$ is canonically associated a linear map called the *abelianized map* or *abelianization* of the substitution σ, the matrix of which (in the canonical basis of $\mathbb{Z}^3$) $\mathbf{M}_\sigma = (m_{i,j})_{1\le i,j\le d}$ (called the *incidence matrix of* σ) is defined by $m_{i,j} = |\sigma(j)|_i$. We thus have:

$$\forall\, W \in \mathcal{A}^\star \qquad \mathbf{l}(\sigma(W)) = \mathbf{M}_\sigma \mathbf{l}(W). \tag{7.1}$$

7.2.3 Substitutions of Pisot type

Let us recall that an algebraic integer is a *Pisot-Vijayaraghavan number* or a *Pisot number* if all its algebraic conjugates β satisfy $|\beta| < 1$. This class of number has been intensively studied and has some special diophantine properties. If the incidence matrix of the substitution admits a Pisot number as dominant eigenvalue, we are going to see that the substitution verifies some special properties.

Indeed, a substitution σ is *of Pisot type* if its characteristic polynomial P_σ is irreducible over $\mathbb{Q}$ and the dominant eigenvalue α is a Pisot number.

A substitution of Pisot type σ is *unimodular* if $det\, \mathbf{M}_\sigma = \pm 1$.

The next proposition states useful information about the eigenvalues of $\mathbf{M}_\sigma$. A proof of it can be found in [87].

Proposition 7.2.1. *Let σ be a substitution and P_σ be the characteristic polynomial of the incidence matrix $\mathbf{M}_\sigma$.*

The substitution σ is of Pisot type if and only if every nondominant root of P_σ satisfies $0 < |\beta| < 1$.

If σ is of Pisot type, then σ cannot be of constant length, the roots of P_σ are all simple, and the matrix $\mathbf{M}_\sigma$ is diagonalizable with simple eigenvalues.

Proof. From Gauss Lemma, P_σ is irreducible over $\mathbb{Q}$ if and only if it is irreducible over $\mathbb{Z}$. If σ is not, there exists two polynomials Q and R with integer coefficients such that $P_\sigma = QR$, then Q and R each have at least one root (in $\mathbb{C}$) whose modulus is greater than 1, so that P_σ has at least two roots of modulus greater than 1. Two of those roots are different since the dominant eigenvalue is simple.

Consequently, if every non dominant root of P_σ verifies $0 < |\beta| < 1$, then P_σ is irreducible over $\mathbb{Q}$ and σ is of Pisot type. Conversely, if σ is of Pisot type, P_σ is irreducible over $\mathbb{Q}$ so that 0 is not a root of P_σ and every non dominant root of P_σ verifies $0 < |\beta| < 1$.

If σ is of constant length l, that is, the image of each letter contains exactly l letters, then l is an eigenvalue for the eigenvector $(1, \ldots, 1)$ and P_σ is reducible over $\mathbb{Q}$. ∎

In [174], C. Holton and L. Zamboni prove that a substitution of Pisot type cannot be S-periodic:

Proposition 7.2.2 (see [174]). *If σ is a primitive substitution the matrix of which has an eigenvalue of modulus strictly smaller than 1, then any fixed point u of σ is not S-periodic.*

Finally, the theory of matrices tells us that a matrix with an irreducible characteristic polynomial which admits a Pisot number as a root is primitive. Thus we have

Proposition 7.2.3 (V. Canterini). *Any Pisot type substitution is primitive.*

Thus, all the usual hypotheses (primitivity, nonperiodicity) are satisfied by Pisot type substitutions. In the three last sections of this chapter, we will see that Pisot type substitutive systems have very special spectral and geometrical properties.

7.3 Subshifts of finite type, Markov partitions and autosimilarity

The most well-known class of symbolic dynamical systems is the class of subshifts of finite type. They are briefly defined in the first subsection. In Sec. 7.7 we expose a combinatorial relationship between subshifts of finite type and substitutive dynamical systems which allows one to transpose some of the properties of subshifts of finite type to substitutive dynamical systems.

Geometrically, subshifts of finite type are very much linked with Markov partition, which are the subject of subsections 7.3.2 and 7.3.3. We will see in the last section how a geometrical representation of substitutive systems of Pisot type provides a Markov partition for the incidence matrix of the substitution, the Markov partition being described by the combinatorial representation in terms of subshifts exposed in Sec. 7.7.

Finally, Markov Partitions and subshift of finite type have connections with the notion of autosimilarity, notions about which are exposed in subsection 7.3.4.

7.3.1 Subshifts of finite type

Subshifts of finite type are described by a finite set of forbidden words. They have a simple representation using finite graph. For more details, see [223].

A *graph* $\mathcal{G}$ consists of a finite set of *vertices* or *states* $\mathcal{V}$ together with a finite set of *edges* or *transition maps* $\mathcal{E}$, i.e., maps in $\mathcal{V} \times \mathcal{V}$. If (i,t) is an edge, i is the *initial vertex* and t is the *terminal vertex.* Two vertices can be connected by two distinct edges.

The *adjacency matrix* of a graph $\mathcal{G}$ has size the cardinality of $\mathcal{V}$. If $i, t \in \mathcal{V}^2$, its coefficient of index (i,t) is equal to the number of edges in $\mathcal{G}$ with initial state i and terminal state t. A graph is thus uniquely determined by its adjacency matrix (up to graph isomorphism).

Let $\mathbf{A}$ be a matrix with coefficients in $\{0,1\}$; the *bilateral subshift* associated with the graph $\mathcal{G}$ of adjacency matrix $\mathbf{A}$ is defined as the following set of bi-infinite paths in the graph $\mathcal{G}$:

$$X_{\mathcal{G}} = X_{\mathbf{A}} = \{(s_i)_{i\in\mathbb{Z}} \in \mathcal{V}^{\mathbb{Z}};\ \forall\, i \in \mathbb{Z},\, (s_i, s_{i+1}) \in \mathcal{E}\}.$$

One can easily check that this set is closed and hence compact, and invariant under the shift in $\mathcal{V}^{\mathbb{Z}}$. This definition is nonambiguous since two vertices cannot be connected by two distinct edges.

The *unilateral subshift* is defined as the set of right paths in $\mathcal{G}$, i.e., to the set of paths indexed by $\mathbb{N}$.

Definition 7.3.1. *Let $\mathcal{B}$ be a finite set. A subset X of $\mathcal{B}^{\mathbb{Z}}$ is a* bilateral subshift of finite type *if and only if there exists a finite set $\mathcal{M} \subset \mathcal{B}^{\star}$ of finite words over $\mathcal{B}$ such that X is the set of bi-infinite words in $\mathcal{B}^{\mathbb{Z}}$ which have no factor in $\mathcal{M}$.*

Equipped with the shift and the topology on $\mathcal{B}^{\mathbb{Z}}$, a subshift of finite type is a symbolic dynamical system, since it is compact and invariant under the shift. One defines similarly the notion of *unilateral subshift of finite type.*

The subshifts of finite type for which the forbidden words have length two are exactly the subshifts associated with a finite graph, the matrix of adjacency of which has coefficients in $\{0,1\}$. More generally, one can always describe a subshift of finite type by means of a graph together with a matrix. From now on, we identify subshifts of finite type and subshifts associated with a graph.

An *automaton* $(\mathcal{G}, t)$ (also called *labelled graph* by the set $\mathcal{B}$) consists of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ together with a labeling map of the edges $t : \mathcal{E} \to \mathcal{B}$.

A subset of $\mathcal{B}^{\mathbb{Z}}$ or $\mathcal{B}^{\mathbb{N}}$ is called a (unilateral or bilateral) *sofic system* if it is defined as the labeling of infinite (unilateral or bilateral) paths of an automaton. Endowed with the shift together with the topology over $\mathcal{B}^{\mathbb{Z}}$ or $\mathcal{B}^{\mathbb{N}}$, such a system is a still a symbolic dynamical system.

These two notions are fundamentally connected. Indeed one shows that sofic systems are exactly the topological factors of subshifts of finite type (see [223]).

Subshifts of finite type provide a vast class of non-uniquely ergodic systems: there exists an infinite number of shift invariant probability measures. Some of them are obtained for instance by means of *stochastic matrices* $\mathbf{P} = (p_{i,j})$ together with a left eigenvector (p_i) of $\mathbf{P}$ for the eigenvalue 1 satisfying: $\sum p_i = 1$. Indeed the probability measure $\mu_{(\mathbf{P},(p_i))}$ defined as follows over the cylinders of the subshift of finite type is easily seen to be shift invariant:

$$\mu_{(\mathbf{P},(p_i))}([a_0, a_1, \ldots, a_n]) = p_{a_0} p_{a_0,a_1} p_{a_1,a_2} \cdots p_{a_{n-1},a_n}.$$

One of these measures plays a particular role when the matrix $\mathbf{A} = (a_{i,j})$ of the subshift is primitive. Namely this matrix thus admits a dominant eigenvalue α, and right and left eigenvectors $\mathbf{u} = (u_i)_i$ $\mathbf{v} = (v_i)_i$ respectively, with positive coefficients. One can normalize these vectors so that $< \mathbf{v}, \mathbf{u} >= 1$. One sets $p_i = v_i u_i$ and $p_{i,j} = a_{i,j} u_j / (\alpha u_i)$. The matrix $\mathbf{P}$ defined by $\mathbf{P} = (p_{i,j})$ is thus stochastic, and (p_i) is a right eigenvector associated with the eigenvalue 1. The associated measure has thus *maximal entropy* (see for instance [367]).

7.3.2 Markov Partitions, autosimilary and domain exchanges

We refer the reader to [202, 326] for more details of the concepts introduced in this section.

Markov partitions allow a good combinatorial understanding of symbolic dynamical systems. These are partitions of a space X which describe by means of a subshift of finite type, the dynamics of a map acting on X, this description preserving properties such as periodicity or transitivity. The existence of Markov partitions allows one to deduce from the properties of the subshift of finite type a "quasi-immediate" description of certain dynamical properties of the system.

Hence, Adler and Weiss show [4] that the topological entropy is a complete invariant for the measure-theoretic equivalence of continuous ergodic automorphisms of the torus $\mathbb{T}^2$. Their proof is based on the construction of an explicit Markov partition of the torus $\mathbb{T}^2$ with respect to the automorphism.

More precisely, a matrix $\mathbf{M}$ with integer coefficients induces in a natural way an endomorphism of the torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, which is called *hyperbolic* if and only if the matrix $\mathbf{M}$ has no eigenvalue of modulus 1. The endomorphism is an *automorphism* if and only if the determinant of the matrix $\mathbf{M}$ equals ± 1. This is a particular case of the general notion of *hyperbolic diffeomorphisms* on which we will not insist here.

In every point z of the torus, one defines in a natural way a map $f_z : \mathbb{R}^{n_u} \times \mathbb{R}^{n_s} \mapsto \mathbb{T}^n$, locally injective, where n_s et n_u denote respectively the dimensions of the stable and unstable subspaces of the matrix (we thus have $n_u + n_s = n$). One thus gets local coordinates around the point z with respect to the stable and unstable directions.

A *rectangle* is a set $f_z(B_u \times B_s)$ where B_u and B_s are closed balls of $\mathbb{R}^{n_u}$ and $\mathbb{R}^{n_s}$. The *stable boundary* of such a rectangle is the set $f_z(\partial B_u \times B_s)$, its *unstable boundary* is $f_z(B_u \times \partial B_s)$.

A *Markov partition* of an hyperbolic automorphism f of the torus $\mathbb{T}^n$ is a finite partition $\Lambda = (C_i)_{i \leq k}$ of $\mathbb{T}^n$ into rectangles with disjoint interiors, such that

- the image under f of the intersection of a cylinder C_i with an affine subspace directed with the stable direction of f is included in a intersection of the same type,
- the image under f^{-1} of the intersection of a cylinder C_i with an affine subspace directed with the unstable direction of f is included in a intersection of the same type.

The *transition matrix* of the Markov partition is the matrix of size k whose (i, j) coefficient is equal to 1 if and only if $f(C_i)$ intersects the interior of C_j.

R. Bowen defines more generally in [75] the Markov partitions associated with hyperbolic diffeomorhisms; he shows that for every hyperbolic diffeomorphism of a manifold X, there exists a Markov partition of X, a two-sided subshift of finite type Y and a continuous map which is almost everywhere injective from Y onto X, which conjugates f to the shift over Y. If f is a noninjective but *expansive* map, then the conjugacy map is replaced by a semi-conjugacy map defined over a one-sided subshift.

The most simple example of Markov partition is the binary system on the circle: the torus $\mathbf{T} = \mathbb{R}/\mathbb{Z}$ is provided with the action $f(z) = 2z \bmod 1$. the mapping which maps a sequence $z \in \{0,1\}^{\mathbb{N}}$ on the point of $[0,1[$ whose binary expansion is z, realizes a Markov partition of X. More generally, Markov partitions can be seen as generalizations of the binary system (see [203]).

One will note that, except for the one and two dimensional cases, Markov partition for toral automorphisms are made of sets that cannot be simple parallelograms. Indeed, R. Bowen proved in [76] that for hyperbolic automorphisms of $\mathbb{T}^3$, the the boundary of the elements which constitutes a Markov partition has a Haussdorf dimension which is not an integer.

7.3.3 Explicit Markov partitions for toral hyperbolic automorphisms

Markov partitions remained abstract objects for a long time: except for linear automorphisms of the two-dimensional torus and some automorphisms of $\mathbb{T}^3$ [30, 51, 243, 244], explicit constructions of Markov partitions for toral hyperbolic automorphisms were obtained only recently. For precise details about results on this subject, see [203].

Thanks to β-developments defined by A. Bertrand-Mathis, B. Praggastis [281] builds tilings which induce Markov partitions for toral hyperbolic automorphisms whose dominant eigenvalue is a unimodular Pisot number. These constructions are particularly simple and explicit for the automorphisms associated with the companion matrix of a polynomial of the type $x^n - d_1x^{n-1} - \cdots - d_n$ with $d_1 \geq d_2 \geq \cdots \geq d_n = 1$. Indeed, the β-development associated with the Pisot root of such polynomials admits very special properties [152] that are intensively used by B. Praggastis.

More generally, S. Leborgne [219] on one hand, R. Kenyon et A. Vershik [204] on the other hand, study the case of hyperbolic automorphisms of $\mathbb{T}^n$, using algebraic properties of the eigenvalues of the automorphism to code the action of the automorphism. Just as with B. Praggastis, those codings are based on numeration system with non integral basis.

Thus, with an equivalent of β-developments, S. Leborgne represents hyperbolic automorphism by sofic systems. Independently, choosing particular digits in expansions with non integral basis, R. Kenyon et A. Vershik construct a representation by a subshift of finite type. This representation is not totally explicit, since the digits are chosen by chance.

7.3.4 Autosimilarity

A collection Λ of compact subsets of $\mathbb{R}^n$ is a *tiling* of a set X which is the closure of its interior if and only if:

- $X = \cup_{C\in\Lambda} C$,
- every element of Λ is the closure of its interior,
- every compact subset of X intersects a finite number of elements of Λ,
- elements of Λ have disjoint interior.

The tiling is said to be a *periodic tiling* of $\mathbb{R}^n$ modulo $\mathcal{L}$, where $\mathcal{L}$ is a net, if and only if

- $\mathbb{R}^n = \cup_{C\in\Lambda,\, \mathbf{x}\in\mathcal{L}} (C+\mathbf{x})$,
- for all $\mathbf{x}\in\mathcal{L}$, $(C+\mathbf{x})\cap \mathrm{Int}\, C' \neq \emptyset$ implies $\mathbf{x}=\mathbf{0}$ and $C = C'$.

A tiling is said to be *autosimilar* if it is invariant under the action of a given expanding diagonal map in $\mathbb{R}^n$, the image of each tile under the mapping being the exact union of original tiles. More precise and general definitions can be found in [202] and [353].

An autosimilar tiling is nothing else but a Markov partition, and can be seen as a generalization of the decimal system, through the theory of representations of number in non-standard basis (for more details, see [203]).

A compact subset X of $\mathbb{R}^n$ is said to be *autosimilar* if there exists a finite partition of X whose translates under a given lattice generate an autosimilar and periodic tiling of $\mathbb{R}^n$.

A *pseudo-exchange of domains* in a compact set X is given by a finite collection Λ of subsets of X such that $X = \cup_{C\in\Lambda} C$, and by translations $t_C : C \mapsto X$, $C\in\Lambda$.

If Λ is a partition of X, or at least if the elements of Λ have disjoint interior, and if the sets $t_C(C)$ constitute a measure-theoretic partition of X, the pseudo-exchange of domains is said to be an *exchange of domains*. It is one-to-one except on a subset of X of measure zero.

It will be noted that if the subsets which constitute Λ are not disjoint, one cannot define a dynamical system on X thanks to Λ. One can only define partial mapping which generates a pseudo-group of translations.

7.4 Measure-theoretic dynamical systems: spectral theory

Many of the notions defined in this section have already been defined in Chap. 3. We recall them to fix the notations and to help the reader. We also present connections between the spectrum of a measure-theoretic system and translations on abelian compact groups which are measure-theoretic factors of the system.

7.4.1 Measure-theoretic dynamical systems

For precisions about all of the notions defined in this section, one can refer to [103, 164, 367].

Definition 7.4.1. *A* measure-theoretic dynamical system *is defined as a system (X, T, μ), where μ is a probability measure defined on a σ-algebra over the set X, and $T : X \to X$ is a measurable map which preserves the measure μ.*

In particular, a uniquely ergodic topological dynamical system (X, T) is a measure-theoretic dynamical system with respect to the unique probability measure which is invariant under T on the σ-algebra of Borel sets of X. For instance, a minimal rotation on a compact abelian group or a primitive substitutive dynamical system are measure-theoretic dynamical systems.

Definition 7.4.2. *Two measure-theoretic dynamical systems (X_1, T_1, μ_1) and (X_2, T_2, μ_2) are said to be* measure-theoretically isomorphic *if there exists a measurable map $f : X_1 \to X_2$ called* conjugacy map*, bijective in measure, the reciprocal map of which is measurable, which conjugates T_1 and T_2, and such that μ_2 is the* induced measure $f_*\mu_1$ *of μ_1 with respect to f, that is*

$$\forall B \in \mathcal{A}_2,\ \mu_1(f^{-1}(B)) = \mu_2(B).$$

If f and f^{-1} are continuous on $X_1 \setminus Y_1$ and $X_2 \setminus Y_2$, where Y_1 and Y_2 are at most countable, then the isomorphism is said to be a *semi-topological isomorphism*.

Given a measure-theoretic dynamical system (X, T, μ), the *unitary operator* is the following map U_T:

$$\begin{aligned} U_T : L^2(X,\mu) &\to L^2(X,\mu) \\ f &\mapsto f \circ T. \end{aligned} \tag{7.2}$$

The map U_T is an isometry, since T preserves the measure. The *eigenvalues* of (X, T, μ) are defined as being those of the map U_T. The *spectrum* of the dynamical system (X, T, μ) is defined as being the spectrum of U_T. It is a subset of the unit circle.

The *eigenfunctions* of (X, T, μ) are defined to be the eigenvectors of U_T.

For instance, the spectrum of the addition of 1 on the p-adic group $\mathbb{Z}_p$ is $\{e^{2i\pi\, k/p^n};\, n \in \mathbb{N},\, k \in \mathbb{Z}\}$. The spectrum of the rotation on the one-dimensional torus $\mathbb{T}$ with irrational angle α is $\{e^{2i\pi\, k\alpha};\, k \in \mathbb{Z}\}$. More generally, the spectrum of the rotation of angle $(t_1, \dots t_n)$ on the d-dimensional torus $\mathbb{T}^d$ is $\{e^{2i\pi\, \sum k_j t_j};\, k_j \in \mathbb{Z}\}$ [103].

The spectrum is said to be *discrete* if it is pure point, that is if $L^2(X, \mu)$ admits an Hilbertian basis of eigenfunctions, which is equivalent to the fact that the eigenfunctions generate a vectorial space which is dense in $L^2(X, \mu)$. If the spectrum contains only the eigenvalue 1, with multiplicity 1, the system is said to be *weakly mixing* or to have a *continuous spectrum*. We say that the system has a *partially continuous spectrum* if the system neither has discrete spectrum, nor is weakly mixing.

The spectrum is said to be *irrational* and respectively *rational* if it is included in $exp(2i\pi\mathbb{R} \setminus \mathbb{Q})$, respectively if it is included in $exp(2i\pi\mathbb{Q})$.

A measure-theoretic dynamical system is said to be *ergodic* if every invariant measurable subset has zero measure or full measure. This is equivalent to the fact that the unitary operator U_T admits 1 as a simple eigenvalue. Under this hypothesis, the eigenvalues are all simple and the spectrum is a subgroup of the unit circle. Moreover, the eigenfunctions are of constant modulus [164].

After a theorem due to Von Neumann, two invertible and ergodic transformations with identical discrete spectrum are measure-theoretically isomorphic. More precisely, an ergodic system the spectrum of which is discrete is measure-theoretically isomorphic with a rotation on a compact abelian group equiped with the Haar measure [164]. We have already recalled that these rotations are products of toral rotations, additions on p-adic groups and additions on finite groups.

7.4.2 Measure-theoretic factors

A way of understanding the structure of a dynamical system is to look for the different dynamical systems that the system contains. More precisely:

Definition 7.4.3. *The measure-theoretic system (X_1, T_1, μ_1) is said to be a* measure-theoretic factor *of the measure-theoretic system (X, T, μ) if their exists a measure-theoretic onto map $f : X \to X_1$, defined almost-everywhere on X, which conjugates the maps T and T_1, and such that μ_1 is the induced measure $f_*\mu$ of μ with respect to f.*

If one knows a measure-theoretic factor of a dynamical system, one can deduce precisions about the spectrum of the system:

Lemma 7.4.1. *The spectrum of a measure-theoretic dynamical system contains the spectrum of every of all of its measure-theoretic factors.*

Proof. Let (X_1, T_1, μ_1) be a factor of (X, T, μ). Let g_1 be an eigenfunction of (X_1, T_1, μ_1) for the eigenvalue λ. We have $g_1 \circ T_1 = \lambda\, g_1$. Let $g = g_1 \circ f$. Then $g \circ T = g_1 \circ f \circ T = g_1 \circ T_1 \circ f = \lambda\, g_1 \circ f = \lambda\, g$. Thus, g is an eigenfunction of (X, T, μ) for λ. ∎

On the contrary, the following lemmas state that giving the spectrum of a dynamical system allows one to deduce some of its factors:

Lemma 7.4.2. *A rotation of irrational α on the one-dimensional torus $\mathbb{T}$ is a measure-theoretic factor of the ergodic dynamical system (X, T, μ) if and only if $e^{2i\pi\,\alpha}$ is an eigenvalue of (X, T, μ).*

Proof. The necessary condition is a consequence of Lemma 7.4.1. Let g be an eigenfunction of (X, T, μ) for the eigenvalue $e^{2i\pi\,\alpha}$. By ergodicity, g is of constant modulus, which can be chosen equal to 1. Thus, there exists a unique measurable function $f : X \to \mathbb{T}$ such that $g = \exp(2i\pi\, f)$. Since $g \circ T = e^{2i\pi\,\alpha} g$, we have $f \circ T = \alpha + T = R_\alpha \circ f$.

As a consequence, we have $\mu(f^{-1} f(X)) = \mu(X) = 1 \neq 0$ and the measure $\mu * f$ on $\mathbb{T}$ is nonzero and invariant under R_α. By unique ergodicity of R_α, this measure is nothing else that the Lebesgue measure ν. Since $f(X)$ is invariant under R_α and of nonzero measure, we get $f(X) = \mathbb{T}$ by ergodicity of R_α, and R_α is a measure-theoretic factor of (X, T, μ). ∎

Similar results are obtained for the addition of 1 on the finite group $\mathbb{Z}/p^n\mathbb{Z}$, this map being uniquely ergodic.

Lemma 7.4.3. *The addition of 1 on the finite group $\mathbb{Z}/p^n\mathbb{Z}$ is a measure-theoretic factor of the ergodic dynamical system (X, T, μ) if and only if $e^{2i\pi/p^n}$ is an eigenvalue of (X, T, μ).*

Proof. The necessary condition comes from Lemma 7.4.1.

Suppose that g is an eigenfunction of (X, T, μ) for $\exp(2i\pi/p^n)$, normalized so that it is of modulus 1. By ergodicity, there exists $z \in \mathbb{C}$ such that $g(X \setminus X_0)$ is a subset of $\{ze^{2i\pi k/p^n}, k \in \mathbb{Z}\}$, where X_0 is a set of measure zero.

This implies that there exists a unique function $f : X \to \mathbb{Z}/p^n\mathbb{Z}$ such that $g = z \exp(2i\pi/p^n\, f)$ in $L^2(X, \mu)$. This map satisfies $f \circ T = 1 + f$. By unique ergodicity of $\mathbb{Z}/p^n\mathbb{Z}$, we prove just as in the preceding lemma that f preserves the measure and is onto. ∎

We finally get the same result for the last class of translations on a compact abelian group.

Lemma 7.4.4. *The addition of 1 on the p-adic group $\mathbb{Z}_p$ is a measure-theoretic factor of the ergodic dynamical system (X, T, μ) if and only if, for every $n \in \mathbb{N}$, $e^{2i\pi/p^n}$ is an eigenvalue of (X, T, μ).*

Proof. The necessary condition comes from Lemma 7.4.1. From Lemma 7.4.3, if $e^{2i\pi/p^n}$ is an eigenvalue of (X, T, μ), there exists an eigenfunction $g_n : X \mapsto \mathbb{C}$ for the eigenvalue $e^{2i\pi/p^n}$, such that $g_n = z_n \exp(2i\pi/p^n\, f_n)$, with $f_n : X \to \mathbb{Z}/p^n\mathbb{Z}$ and $f_n \circ T = 1 + f_n$.

Since g_{n+1} is an eigenfunction for $e^{2i\pi/p^{n+1}}$, the map $(g_{n+1})^p$ is also an eigenfunction for $e^{2i\pi/p^n}$. By ergodicity of (X, T, μ) , the eigenvalues are simple, and $(g_{n+1})^p$ are g_n proportional. So that we get $z_n \exp(2i\pi/p^n\, f_n) = {z_{n+1}}^p \exp(2i\pi/p^n\, f_{n+1})$. In particular, $f_n - f_{n+1}$ is constant modulo p^n.

It will be recalled that p_i^j is the canonical projection of $\mathbb{Z}/p^j\mathbb{Z}$ onto $\mathbb{Z}/p^i\mathbb{Z}$. Modifying the map f_{n+1} by subtracting the constant, we obtain a sequence of maps $f_n : X \mapsto \mathbb{Z}/p^n\mathbb{Z}$ such that $f_n \circ T = 1 + f_n$ and $p_{n+1}^n \circ f_{n+1} = f_n$, for every integer n.

Let $f : X \to \mathbb{Z}^n$ be the map defined by $f(x) = (f_n(x))_{n\in\mathbb{N}}$. According to what precedes, we have for all $j \geq i$, $p_j^i(f_j(x)) = f_i(x)$. Thus, the map f maps on the projective limit of $\mathbb{Z}/p^n\mathbb{Z}$, that is $\mathbb{Z}_p$. Moreover, we have $f \circ T = 1 + f$. By unique ergodicity of the addition of 1 on $\mathbb{Z}_p$, we conclude just as in Lemma 7.4.2 that $\mathbb{Z}_p$ is a measure-theoretic factor of (X, T, μ). ∎

7.4.3 Maximal equicontinuous factor

A topological dynamical system (X, T) is said to be *equicontinuous* if the set of maps $\{T^n,\, n \in \mathbb{Z}\}$ is uniformly equicontinuous, that is:

$$(\forall\, \epsilon > 0)\, (\exists\, \alpha > 0)\, (\forall\, x, y \in X)\ (\, d(x, y) < \alpha \implies (\forall\, n \in \mathbb{Z}),\, d(T^n(x), T^n(y)) < \epsilon\,).$$

Every topological dynamical system (X, T) admits a largest equicontinuous factor, (Y, T_1), in the sense that any topological equicontinuous factor of (X, T) is a topological factor of (Y, T_1). This largest equicontinuous factor is called *maximal equicontinuous factor* [136].

There exists a one-to-one correspondence between the continuous characters of the maximal equicontinuous factor and the measure-theoretic eigenfunctions of a dynamical system. In particular, the maximal equicontinuous factor is nothing else but a rotation on a compact abelian group, about which we have already explained that they are obtained as products of toral rotations, p-adic translations and finite groups translations.

If the system is supposed ergodic, then the measure-theoretic eigenfunctions are simple. It is thus equivalent to look for the maximal equicontinuous factor of a system and to determine the discrete part of its spectrum.

In this chapter, both aspects will be considered: we will first survey general results about the spectrum of substitutive dynamical systems. Then, in the Pisot case, we will see how to realize this spectrum by constructing in an explicit way the maximal equicontinuous factor.

7.5 Spectral theory of constant length substitution

Let us recall that a substitution is said to be of *constant length* if the image of any letter of the alphabet under the substitution contains the same number of letters.

Substitutive dynamical systems were introduced by W. H. Gottschalk [161], as examples of symbolic dynamical systems, the study of which was initialized in [255, 256]. The attention first focused on systems associated with a substitution of *constant length*.

The first topological and ergodic properties of these systems were obtained at the beginning of the seventies: T. Kamae proves in [191] that the condition of primitivity is sufficient for the minimality of the system. Then he gives a sufficient condition for the system to have discrete spectrum [192]. B. G. Klein shows that primitive substitutive systems of constant length are uniquely ergodic and that their topological entropy equals zero [207].

In [104], E. M. Coven and M. S. Keane focus on a class of constant length substitutions over a two-letter alphabet which generate a minimal system, linked with Toeplitz substitutions. They prove that the maximal equicontinuous factor of these systems is the n adic group $\mathbb{Z}_n$, where n is the length of the substitution.

J. C. Martin generalizes this to substitutions of constant length n over a finite alphabet [237]: if the substitution is one-to-one on the letters, the maximal equicontinuous factor of the substitutive dynamical system is the addition of $(1,1)$ on $\mathbb{Z}_n \times \mathbb{Z}/m\mathbb{Z}$, where m in an integer depending on the substitution. J. C. Martin then gives some conditions for the factor to be isomorphic to the system: some are necessary conditions, the others are sufficient.

Finally, M. F. Dekking generalizes these results by introducing new methods:

Theorem 7.5.1 (F. M. Dekking [113]). *Let σ be a nonperiodic substitution of constant length n. Let u be a periodic point for σ. We call* height *of the substitution the greatest integer m which is coprime with n and divides all the strictly positive ranks of occurrence of the letter u_0 in u. The height is less that the cardinal of the alphabet.*

The maximal equicontinuous factor of the substitutive dynamical system associated with σ is the addition of $(1,1)$ on the abelian group $\mathbb{Z}_n \times \mathbb{Z}/m\mathbb{Z}$.

M. F. Dekking gives an algorithmic method for computing the height of a substitution. For instance, the Morse substitution has a height of 1.

M. F. Dekking also introduces the combinatorial condition which later will be called *coincidence condition*: a constant length substitution σ satisfies this condition if their exists an integer k such that the images of any letter of the alphabet under σ^k has the same k-th letter. Generalizing [192], F.

M. Dekking proves that a substitution satisfies this condition if and only if the associated substitutive dynamical system is isomorphic with its maximal equicontinuous factor, that is if the system has a discrete spectrum:

Theorem 7.5.2 (F. M. Dekking [113]). *Let σ be a substitution of constant length. The substitutive dynamical system associated with σ has a discrete spectrum if and only if the substitution σ verifies the condition of coincidence.*

The Morse substitution does not verify this condition. Consequently, the associated substitutive dynamical system is not isomorphic with a translation on $\mathbb{Z}_2$, as shown in Chap. 3.

7.6 Recognizability

One of the main problems raised by the generalization of the above results to substitutions of nonconstant length was to find a general way to decide the places where any factor of the fixed point appears in the fixed point, that is to desubstitute the substitutive system. The fundamental notion of recognizability deals with this problem. As we are going to see, it took quite a long time to be correctly defined and to get general results about it.

Let σ be a primitive substitution over the alphabet $\mathcal{A}$, and let u be a fixed point of σ. We thus have $u = \sigma(u)$. Let E_1 be the following set of lengths

$$E_1 = \{0\} \cup \{|\sigma(u_0 \dots u_{p-1})|, |\sigma(u_{-p} \dots u_{-1})|;\, p > 0\}.$$

We thus get $n \in E_1$ if and only if there exists a letter $a \in \mathcal{A}$ such that u_n is the first letter of $\sigma(a)$.

Since $u = \sigma(u)$, for every factor $W = u_i \dots u_{i+|W|-1}$ of u, there exists a rank j, a length l, a suffix S of $\sigma(x_j)$ and a prefix P of $\sigma(x_{j+l+1})$ such that

$$W = S\sigma(u_{j+1}) \dots \sigma(u_{j+l})P,$$

and such that $E_1 \cap \{i, \dots, i + |W| - 1\} = (i - k) + E_1 \cap \{k, \dots, k + |W| - 1\}$, with $k = |\sigma(u_0 \dots u_j)| - |S|$, which means that it is equivalent to cut u with respect to σ at the ranks i and k.

We say that $[S, \sigma(u_{j+1}), \dots, \sigma(u_{j+l}), P]$ is a *1-cutting at the rank i* of W, and that W comes from the word $u_j \dots u_{j+l+1}$, which will be called the *ancestor word* of W.

By minimality, the factor W appears infinitely many times in the sequence u. The question of the unicity of the 1-cutting of W is thus natural. Let us briefly survey the existing literature on this question.

In [238], J. C. Martin calls *rank one determined* substitutions for which every long enough factor of a fixed point admits a 1-cutting and a coming-out word independent from the rank of apparition of the factor. The author claims that any substitution on a two-letter alphabet which is not S-periodic is rank one determined. His proof is not convincing.

Later, B. Host [181] and M. Queffelec [285] introduce the notion of *recognizable substitution*, that is substitutions for which the 1-cutting of any long enough factor is independent of the rank of occurrence of the factor, except maybe for a suffix of the word, the length of this suffix being bounded. More precisely, a substitution σ is said to be *unilaterally recognizable* if there exists $L > 0$ such that $u_i \dots u_{i+L-1} = u_j \dots u_{j+L-1}$ with $i \in E_1$ implies $j \in E_1$.

B. Host proves in [181] that this property is equivalent to the fact that the image $\sigma(X_\sigma)$ under σ of the substitutive dynamical system X_σ associated with σ is an open set. G. Rauzy announced that he had a short proof of the fact that a constant length substitution which is one-to-one on letter which is not S-periodic is unilaterally recognizable [285]. Nobody could check this proof. Concerning substitutions of nonconstant length, the question remained unsettled.

In [258], B. Mossé studies the question of unilaterally recognizable substitutions. This characterization implies that a substitution is not necessarily unilaterally recognizable: a sufficient condition for a substitution not to be unilaterally recognizable is that for every couple of distinct letters (a, b), then $\sigma(a)$ is a strict suffix of $\sigma(b)$, or conversely. For instance, the substitution defined by $\sigma(1) = 1112$ and $\sigma(2) = 12$ is not unilaterally recognizable.

To get a general result of recognizability, B. Mossé introduces a new notion of recognizable substitution, for which the 1-cutting of any long enough factor is independent from the rank of apparition of the factor, except maybe for a suffix and a prefix of the factor, the length of those prefixes and suffixes being bounded. More precisely, a substitution σ is said to be *bilaterally recognizable* if there exists $L > 0$ such that $u_{i-L} \dots u_{i+L} = u_{j-L} \dots u_{j+L}$ with $i \in E_1$ implies $j \in E_1$.

The following theorem shows that this notion of recognizability is the right one:

Theorem 7.6.1 (B. Mossé [258, 259]). *Let σ be a primitive substitution with a nonperiodic fixed point u. Then the substitution σ is bilaterally recognizable, and the ancestor word of every factor of u is unique except on the end of the factor: there exists $L > 0$ such that if $u_{i-L} \dots u_{j+L} = u_{i'-L} \dots u_{j'+L}$, then $u_i \dots u_j$ and $u_{i'} \dots u_{j'}$ have the same 1-cutting and the same coming-out word at ranks i and i'.*

Thus, if a substitution is primitive, then one can always *desubstitute* any factor of the fixed point, except possibly on the ends of the factor.

If we just suppose from now that σ is a primitive substitution which is not S-periodic, σ has not necessarily a fixed point but has at least a periodic point. Consequently, a power of σ is bilaterally recognizable. Since the dynamical systems associated either with a substitution or with its iterations are identical, we deduce from Theorem 7.6.1 that any word of the bilateral system symbolic dynamical system X_σ associated with the substitution σ can be desubstituted in a unique bi-infinite word:

Corollary 7.6.1. *Let σ be a primitive substitution which is not S-periodic. Let X_σ be the substitutive dynamical system associated with σ. Then we have*

$$\forall w \in X_\sigma,\ \exists\, v \in X_\sigma,\ v \ \textit{unique};\, w = S^k\sigma(v), \qquad \textit{and} \qquad 0 \le k < |\sigma(v_0)| \tag{7.3}$$

This corollary means that any word w in X_σ can by cut or desubstituted in a unique way under the following form:

$$w = \ldots \mid \underbrace{\ldots}_{\sigma(v_{-1})} \mid \underbrace{w_{-k}\ldots w_{-1}.w_0\ldots w_l}_{\sigma(v_0)} \mid \underbrace{\ldots}_{\sigma(v_1)} \mid \underbrace{\ldots}_{\sigma(v_2)} \mid \ldots \tag{7.4}$$

where the word $v = \ldots v_{-n}\ldots v_{-1}v_0v_1\ldots v_n\ldots$ belongs to X_σ.

Note that this property is satisfied only by two-sided substitutive dynamical systems, and is not satisfied by a one-sided substitutive dynamical system. A counter-example is again given by the substitution defined by $\sigma(1) = 1112$ and $\sigma(2) = 12$.

This notion of recognizability is the fundamental hypothesis used by every author to get results about the spectrum of substitutive systems of nonconstant length. Before B. Mossé's work, most authors did suppose that the substitutions they considered were recognizable. After B. Mossé's work, this hypothesis was replaced by the hypothesis of primitivity.

7.7 Nonconstant length substitutions and adic transformations

At the end of the seventies, the spectral theory of constant length substitutive dynamical systems was quite well known, and people started to focus on spectral theory of nonconstant length substitutive dynamical systems. We will give in the next section the main results about this theory. An important part of them was obtained using a relationship between substitutive dynamical systems and the other main class of symbolic dynamical systems, that is, subshifts of finite type. The object of this section is to focus on this relationship.

7.7.1 Adic transformations

At the beginning of the eighties, A. M. Vershik defined a new type of dynamical systems, called *adic systems* [361] on a Markov compactum. In short, a *Markov compactum* is the set of paths in an infinite labeled graded graph.

One can define a partial order between the labels of the edges of the graph, and deduce a partial order between the elements of the Markov compactum. The choice of the successor with respect to this partial order is a continuous map called *adic transformation*.

A motivation for the introduction of this new family of systems is the question of the approximation of ergodic systems. Indeed, every automorphism of a Lebesgue space with an ergodic invariant continuous measure is measure-theoretically isomorphic with an adic transformation. One can refer to [361] for more details about these systems.

As a particular case of Markov compactum one can define the notion of *stationary Markov compactum*. A Markov compactum is said to be stationary if each level of the tree has the same structure. This implies that a stationary Markov compactum can be defined as a subshift of finite type. More precisely, let $\mathbf{M}$ be a $d \times d$ matrix, with entries 0 or 1. The stationary Markov compactum associated with $\mathbf{M}$ is the set of the infinite paths in the graded graph with levels indexed by $0, 1, 2 \ldots$, and having m vertices on each level. The edges connect vertices of the i-th level with vertices of the $i+1$-th level according to the adjacency matrix $\mathbf{M}$. Thus, the Markov compactum is the following set:

$$\mathcal{E}_{\mathbf{M}} = \{(x_i)_{i \geq 0} \in \{1 \ldots m\}^{\mathbb{N}};\ \ \forall i \geq 0,\ \mathbf{M}_{x_i, x_{i+1}} = 1\}.$$

The Markov compactum is said to be *proper* if the first level has only one vertex.

There exists a canonical map which acts on this compactum: the one-sided shift map defined by $T((x_i)_{i \geq 0}) = (x_i)_{i \geq 1}$. This map is a dynamical system with positive entropy the properties of which are quite well known.

From a totally different point of view, one can remark that the edges issued from a vertex i are ordered by the number of the exit vertex of the edge. From this partial order on vertices can be deduced the following partial order on the paths $\mathcal{E}_{\mathbf{M}}$:

$$(x_i)_{i \geq 0} \prec (y_i)_{i \geq 0} \text{ if there exists } i_0 \text{ such that } \begin{cases} x_i = y_i & \forall\, i > i_0 \\ x_n = y_n. \end{cases}$$

The *adic transformation* τ on $\mathcal{E}_{\mathbf{M}}$ is defined as follows: it maps $(x_i)_{i \geq 0} \in \mathcal{E}_{\mathbf{M}}$ on the minimal element $(y_i)_{i \geq 0} \in \mathcal{E}_{\mathbf{M}}$ such that $(x_i)_{i \geq 0} \prec (y_i)_{i \geq 0}$.

This transformation is defined on $\mathcal{E}_{\mathbf{M}}$ except on the set of the maximal elements which is finite, when $\mathbf{M}$ is primitive.

Theorem 7.7.1 ([361, 362]). *The adic transformation associated with a primitive matrix is minimal and uniquely ergodic. The unique invariant measure is equivalent to the probability measure on $\mathcal{E}_{\mathbf{M}}$ which is of maximal entropy for the shift map.*

7.7.2 Adic transformations and substitutive dynamical systems

A. N. Livshits proves in [225] that essentially the stationary adic transformation associated with a $d \times d$ matrix $\mathbf{M}$ is measure-theoretically isomorphic to the dynamical system associated with the substitution defined on $\{1, \ldots, d\}$ by $\sigma(i) = 1^{\mathbf{M}_{1,i}} 2^{\mathbf{M}_{2,i}} \ldots d^{\mathbf{M}_{d,i}}$. More generally, it is more or less proved in [362] that all primitive substitutive dynamical systems are isomorphic to a generalized adic transformation.

These results were completed by A. Forrest [148], who proved the following result in terms of Bratteli diagrams.

Theorem 7.7.2 (A. Forrest [148, 132]). *Each adic transformation on a stationary proper Markov compactum is topologically isomorphic either with a primitive substitutive dynamical system or with an odometer. Conversely, each primitive substitutive dynamical system or odometer is isomorphic with an adic transformation on a stationary proper Markov compactum.*

The proof of A. Forrest is not constructive. It was completed and restated in the above terms by F. Durand, B. Host and C. F. Skat in [132]. In these papers, the authors give a new proof using substitutions, return words, and more precisely results about recognizability stated in Sec. 7.6. (For a definition of return words, see Chap. 4). Their approach provides an algorithm which computes for any substitutive dynamical system an adic transformation on a Markov compactum which is isomorphic with a substitutive system. One can also refer to [183] for more details.

Nevertheless, the computation of an adic transformation isomorphic to a given substitutive dynamical system in [132] is not very natural and not very easy. Using the properties of recognizability, G. Rauzy initiated in [292] and [295] the idea of representing a substitutive dynamical system directly thanks to (7.3) and its interpretation given in (7.4). Let us illustrate this approach in more detail.

Indeed, it will be recalled that a consequence of the works of B. Mossé is that every word in X_σ can be expanded in the following form:

$$w = \ldots \mid \underbrace{\ldots}_{\sigma(y_{-1})} \mid \underbrace{w_{-k} \ldots w_{-1} . w_0 \ldots w_l}_{\sigma(y_0)} \mid \underbrace{\ldots}_{\sigma(y_1)} \mid \underbrace{\ldots}_{\sigma(y_2)} \mid \ldots$$

with $\ldots y_{-n} \ldots y_{-1} y_0 y_1 \ldots y_n \ldots$ in X_σ.

Let $p = w_{-k} \ldots w_{-1}$ denote the prefix of $\sigma(y_0)$ of length k and $s = w_1 \ldots w_l$ the suffix of length l. The word w is completely determined by the word y and the decomposition of $\sigma(y_0)$ as $\sigma(y_0) = p w_0 s$. Let $\mathcal{P}$ be the finite set of all such decompositions:

$$\mathcal{P} = \{(p, a, s) \in \mathcal{A}^\star \times \mathcal{A} \times \mathcal{A}^\star;\ \exists\, b \in \mathcal{A} \text{ and } \sigma(b) = pas\}\,.$$

The work of B. Mossé implies that we can define on one hand a *desubstitution map* θ on X_σ (which sends w to y), and on the other hand, a partition map γ from X_σ to $\mathcal{P}$, corresponding to the decomposition of $\sigma(y_0)$:

$$\begin{aligned} \theta : X_\sigma \to X_\sigma,\ w \mapsto y \quad & \text{such that } w = S^k\sigma(y) \text{ and } 0 \leq k < |\sigma(y_0)|, \\ \gamma : X_\sigma \to \mathcal{P},\ w \mapsto (p, w_0, s) \quad & \text{such that } \sigma(y_0) = pw_0s \text{ and } k = |p|\,. \end{aligned}$$

Let us note that these two maps are continuous.

In [86] and independently in [176], the itineraries of the points of X_σ under the desubstitution according to the partition defined by γ are studied. More precisely, the *prefix-suffix expansion* is the map $\Gamma : X_\sigma \to \mathcal{P}^{\mathbb{N}}$ defined by:

$$\Gamma(w) = (\gamma\,(\theta^i w)\,)_{i\geq 0} = (p_i, a_i, s_i)_{i\geq 0} \in \mathcal{P}^{\mathbb{N}}\,.$$

The main questions are the identification of the image of Γ :, the injectivity of Γ, and the action of the shift S on X_σ through Γ.

The definition of Γ leads us to define the subshift of finite type described by the automaton, called *prefix-suffix automaton* of the substitution σ, which has $\mathcal{A}$ as a set of vertices and $\mathcal{P}$ as a set of label edges: there is an edge labeled by $e = (p, a, s)$ from a to b if and only if $pas = \sigma(b)$.

The set of labels of infinite walks in this automaton is denoted by $\mathcal{D}$. It is the support of a subshift of finite type. It is proved in [86] and [176] that Γ is a measure-theoretic isomorphism between X_σ and $\mathcal{D}$.

We explained in the preceding section that a stationary Markov compactum is nothing other than the support of a subshift of finite type, provided with a partial order deduced from a partial order on the labels of the edges of the subshift. However, the substitution σ implies that there exists a very natural partial order on the set $\mathcal{P}$ of labels of $\mathcal{D}$: two labels (p, a, s) and (q, b, r) are comparable for this order if they are issued from the same letter, i.e., $pas = qbr = \sigma(c)$ for a letter c. The order of comparison then comes from the length of the prefix p. More precisely:

$$(p, a, s) \prec (q, b, r) \iff (\exists\, c \in \mathcal{A});\ \ (\,pas = qbr = \sigma(c)\,)\ \text{ and }\ (\,|p| < |q|\,). \tag{7.5}$$

As mentioned in the preceding section, this order on $\mathcal{P}$ induces an order on the Markov compactum $\mathcal{D}$, obtained as a reverse lexicographical order:

$$\begin{aligned} \forall (e_i)_{i\geq 0}, (f_i)_{i\geq 0} \in \mathcal{D}, \quad (e_i)_{i\geq 0} \prec (f_i)_{i\geq 0} \iff & \\ (\exists\, i_0 \geq 0);\ (\forall\, i > i_0,\ e_i = f_i)\ \text{ and }\ (e_{i_0} \prec f_{i_0}). & \end{aligned} \tag{7.6}$$

Thus, the map on the Markov compactum $\mathcal{D}$ which maps any nonmaximal element of $\mathcal{D}$ on the smallest of its upperbound is an adic transformation defined everywhere except on a finite number of points.

Theorem 7.7.3 ([86, 176]). *Let σ be a primitive substitution which is not S-periodic and (X_σ, S) the dynamical system generated by σ. The prefix-suffix expansion defined above is a continuous mapping onto the subshift of finite type $\mathcal{D}$. This map is one-to-one except on the orbit of periodic points of σ.*

This prefix-suffix expansion conjugates the shift map S over X_σ with an adic transformation on $\mathcal{D}$, considered as a Markov compactum when the set $\mathcal{P}$ of edges is provided with the natural partial ordering coming from the substitution described on formulas 7.5 and 7.6.

This gives a new proof of the fact that any primitive substitutive dynamical system is measure-theoretically isomorphic with an adic transformation on a Markov compactum. The conjugacy map is not a topological conjugacy map, contrary to the one obtained in [132]. But this conjugacy map can be deduced in a simple, natural and explicit way from the substitution.

It can be note that the prefix-suffix automaton which has allowed us to define the Markov compactum $\mathcal{D}$ is the most complete form of an important class of automata appearing in literature.

It was introduced in [86] in order to extend and to make more precise the prefix automaton, and its dual suffix automaton, defined by G. Rauzy in the seminal papers [292, 295] with the final goal to represent substitutive dynamical systems of Pisot type by domain exchanges in the Euclidean space. The difference between the prefix automaton and the prefix-suffix automaton is that the subshift generated by the second one is of finite type, while the one generated by the first automaton is only sofic. This has fundamental consequences for the injectivity of geometric representations of substitutive dynamical systems defined thanks to the subshifts (see [292, 295, 87] and Sec. 7.10.2).

Moreover, the prefix-suffix automaton can be found in a coded form among the number theory works which followed [292]: first among work by J.-M. Dumont and A. Thomas [127] about numeration scales associated with certain substitutions; then among their development by V. Sirvent [335] who proves some properties of the Rauzy fractal (see also Sec. 7.10.2). We can also mention the definition by T. Kamae [193] of colored tilings associated with a weighted substitution, the layout of the tiles following rules being close to the transition rules of the prefix automaton, but in a nonexplicit way.

To finish this brief survey, the prefix-suffix automaton also appears in the context of language theory: the prefix-suffix automaton projects on the one hand onto the automaton defined by A. Maes [232] in his works about decidability of arithmetical theory, and on the other hand, onto the automaton defined by P. Narbel [261] to study the boundary of a language thanks to the description with trees of the set of paths in his automaton.

7.8 Spectrum of substitutions of nonconstant length

The first partial results about the spectrum of substitutions of nonconstant length were obtained by studying a class of nonconstant length substitutive systems which are isomorphic with constant length substitutive systems [104, 113]. Later, F. M. Dekking et M. S. Keane proved that strongly mixing substitutive dynamical systems do not exist [117]. P. Michel proved the fact already claimed in Sec. 7.2, namely that the dynamical system associated with a primitive substitution of nonconstant length is uniquely ergodic [245]. M. Queffelec studies in details the spectral type of substitutive systems in [285]. Most of the results cited above appear in this last reference.

Those general topological and ergodic properties having been studied, authors focused on the explicit description of the spectrum of a substitutive dynamical system. They also interested themselves in obtaining conditions for a substitutive system to have a discrete spectrum, a partially continuous spectrum, or a continuous spectrum. The description of the known results is the aim of this section.

J. C. Martin obtains a partial result about the spectrum of nonconstant length substitutions on a two-letter alphabet in [238]. His work suffers from too weak a definition of the property of recognizability (see Sec. 7.6). If we take into account the results about recognizability proved later, his results become:

Theorem 7.8.1 (J.C. Martin [238]). *Let σ be a primitive substitution on a two-letter alphabet which is not S-periodic. Let $X^2 - TX + d$ be the characteristic polynomial of the incidence matrix of σ and $r \in \mathbb{Q}$. Let w be the positive integer whose prime factors are the primes dividing both $|\sigma(1)|$ and $|\sigma(2)|$ but not dividing r, the exponent of each prime factor being its exponent in the decomposition of $|\sigma(1)| - |\sigma(2)|$.*

Then $\exp(2\pi i r)$ *is an eigenvalue of the dynamical system associated with σ if and only if there exists integers k, m, n such that $r = k/w + m/(pgcd(d,T))^n$.*

A consequence of this theorem is the characterization of p-adic measure-theoretic factors of substitutive systems over a two-letter alphabet:

Corollary 7.8.1. *Let σ be a primitive substitution over a two-letter alphabet which is not S-periodic. Let p be a prime number. The addition of 1 on $\mathbb{Z}_p$ is a measure-theoretic factor of the substitutive dynamical system associated with σ if and only if p divides the determinant and the trace of the characteristic polynomial of the incidence matrix of σ.*

This corollary has been generalized recently in [327] by studying the ramifications of the determinant of the incident matrix of σ in the Galois extension of its dominant eigenvalue. Taking into account Host's results (see Theorem 7.8.3), we have:

Theorem 7.8.2 (A. Siegel [327]). *Let σ be a primitive substitution, the incidence matrix $\mathbf{M}_\sigma$ of which has an irreducible characteristic polynomial $\chi_{\mathbf{M}_\sigma}$. Let p be a prime number. The addition of 1 over $\mathbb{Z}_p$ is a measure-theoretic factor of the substitutive dynamical system associated with σ if and only if p divides all the coefficients of $\chi_{\mathbf{M}_\sigma}$ except the dominant one, that is, if and only if $\mathbf{M}_\sigma$ is nilpotent modulo p.*

In particular, there exist non-unimodular substitutive systems which do not admit any p-adic factor. This is the case for the substitution defined by $\sigma(1) = 11222$ and $\sigma(2) = 1222$. Such a behavior is totally different from that of constant length substitutions.

In [181], B. Host introduces a good notion of recognizability and thanks to this notion proves the following fundamental result.

Theorem 7.8.3 (B. Host [181]). *Let σ be a primitive substitution which is not S-periodic. Each eigenfunction of the substitutive dynamical system associated with σ is almost everywhere equal to a continuous eigenfunction.*

Corollary 7.8.2. *Let σ be a primitive substitution which is not S-periodic. A minimal toral rotation or the addition of* 1 *in the p-adic integer set $\mathbb{Z}_p$ is a topological factor of the substitutive dynamical system associated with σ if and only if it is a measure-theoretic factor of this system.*

Proof. Let $T : Y \to Y$ denote either a minimal toral rotation or the addition of 1 in $\mathbb{Z}_p$. Let (X_σ, S) be the substitutive dynamical system associated with σ. Suppose that T is a measure-theoretic factor of S. From Theorem 7.8.3, each eigenfunction appearing in the proofs of Lemma 7.4.2 and 7.4.4 are continuous. Consequently, the map $f : X_\sigma \mapsto Y$ which conjugates S and T is continuous, which implies that the measure-theoretic factor is a continuous factor.

Conversely, suppose that (Y, T) is a topological factor of (X_σ, S). Let $f : X_\sigma \to Y$ be a continuous and onto mapping such that $f \circ S = R \circ f$. The induced measure with respect to f of the unique ergodic measure of (X_σ, S) is a probability measure on Y invariant under T. But (Y, T) is uniquely ergodic since Y is either a torus or a p-adic set. Consequently, the induced measure is the unique ergodic measure of (Y, T), and (Y, T) is a measure-theoretic factor of (X_σ, S). ∎

B. Host deduces from Theorem 7.8.3 a complete description of the spectrum of the nonconstant length substitutions, which are recognizable, and one-to-one on the letters . This description is a generalization of [113] et [238].

Theorem 7.8.4 (B. Host [181]). *Let σ be a primitive substitution defined over the alphabet $\mathcal{A}$, which is not S-periodic. A* coboundary *of σ is defined as a map h from $\mathcal{A}$ into the unit circle $\mathbb{U}$ of $\mathbb{C}$, such that there exists a map*

$f : \mathcal{A} \to \mathbb{U}$ with $f(b) = f(a)h(a)$, for every word ab of length 2 which belongs to the language of the substitution.

A complex number λ of modulus 1 is an eigenvalue of (X_σ, S) if and only if there exists $p > 0$ such that for every $a \in \mathcal{A}$, the limit $h(a) = \lim_{n\to\infty} \lambda^{|\sigma^{pn}(a)|}$ is well defined, and h is a coboundary of σ.

For instance, this theorem implies that the system generated by the substitution defined by $\sigma(1) = 12121$ and $\sigma(2) = 122$ has no eigenvalue. Consequently, this system is weakly mixing.

In [147], S. Ferenczi, C. Mauduit et A. Nogueira give a new simplified version of the proofs of Theorems 7.8.3 and 7.8.4. They deduce from these new proofs a more explicit characterization of the spectrum of primitive substitutive dynamical systems (see Theorem 7.8.9).

The parallel between substitutive systems and adic systems exposed in the preceding section, and the ideas developed by B. Host in [181], led many authors to give some necessary and sufficient conditions for a substitutive dynamical system to have a discrete spectrum, a partially continuous spectrum or to be weakly mixing (see the survey in [362]).

For instance, A. N. Livshits gives the following characterization of substitutive systems with discrete spectrum. (This characterization was announced in [224] and [362] and proved in [226].)

Theorem 7.8.5 (A. N. Livshits [224]). *Let σ be a primitive substitution defined over the alphabet $\mathcal{A}$ which is not S-periodic.*

The alphabet $\mathcal{A}$ is divided into classes with respect to the length of the images of each letter. Given a class C, a C-block is defined as an element $a_1 \dots a_r$ of the language of σ such that $a_1, a_r \in C$ and $a_2, \dots a_{r-1} \notin C$. Denote by Δ_C the set of pairs $(a_1 \dots a_{r-1}, a_2 \dots a_r)$, where $a_1 \dots a_r$ is a C-block.

Suppose that there exists a finite set of words $\{W_i\}$ and a finite set Δ of pairs (W_{i_k}, W_{j_k}) satisfying $\mathbf{l}(W_{i_k}) = \mathbf{l}(W_{j_k})$, such that for every pair $(V_1, V_2) \in \Delta \cup \Delta_C$, there exists a natural integer m for which one has: $\sigma^m(V_1) = W_{l_1} \dots W_{l_s}$ and $\sigma^m(V_2) = W_{j_1} \dots W_{j_s}$, with $(W_{l_r}, W_{j_r}) \in \Delta \cup \Delta_C$ for all r and $W_{l_r} = W_{j_r}$ for at least one r.

Then the substitutive dynamical system associated with σ has discrete spectrum.

Examples of application of this theorem can be found among substitutions of Pisot type, which will be treated in detail in the next section (see in particular Theorem 7.9.2).

The converse of Theorem 7.8.5 can be expressed under the following form.

Theorem 7.8.6 (A. N. Livshits [224]). *Let σ be a primitive substitution which is not S-periodic. Let c, d, a be two letters such that $c \neq d$ and ac, ad belong to the language of the substitution.*

Suppose that there exists a finite set of words $\{W_i\}$ *and an increasing sequence of integers* m_i *such that for all* i*, we have:* $\sigma^{m_i}(c) = W_{l_1} \dots W_{l_s} A$ *et* $\sigma^{m}(d) = W_{k_1} \dots W_{k_s} B$*, with nonempty* A *or* B*, and for all* t*,* $\mathbf{l}(W_{l_t}) = \mathbf{l}(W_{k_t})$ *and* $W_{l_t} \neq W_{k_t}$*.*

Then the substitutive dynamical system associated with σ *has a partially continuous spectrum.*

For instance the system associated with the substitution defined by $\sigma(1) = 23$, $\sigma(2) = 12$ and $\sigma(3) = 23$ admits a continuous spectral component. However, the substitution is of constant length, so that the system admits $\mathbb{Z}_2$ as a factor (Theorem 7.5.1) and is not weakly mixing.

Another consequence of this theorem is that the Morse substitution system has a partially continuous spectrum, which is known to be a Riesz product (see the bibliography of [285]). This implies that the substitutive system associated with the Morse substitution is isomorphic with a two-point extensions of its maximal equicontinuous factor, that is the addition of 1 on $\mathbb{Z}_2$ (see also Chap. 3).

Finally, A. N. Livshits gives a sufficient condition for a substitutive dynamical system to be weakly mixing.

Theorem 7.8.7 (A. N. Livshits [225]). *Let* σ *be a primitive substitution which is not* S*-periodic.*

If every eigenvalue of the incidence matrix of σ *has a modulus greater than or equal to* 1*, then the substitutive dynamical system associated with* σ *does not have any irrational eigenvalue.*

If there exist an element of the language of σ*, denoted* $w = a_1 \dots a_k = a_1 v$ *with* $a_1 = a_k$*, such that for all integer* n*, the lengths* $|\sigma^n(v)|$ *are coprime to each other for* $n > N$*, then the substitutive dynamical system associated with* σ *does not have any rational eigenvalue.*

For instance, the substitution introduced in [117] and defined by $\sigma(1) = 112$ and $\sigma(2) = 12222$ is weakly mixing [224].

B. Solomyak makes the above theorem somewhat more precise by giving a characterization of irrational eigenvalues of substitutive systems in a more explicit way than Host's.

Theorem 7.8.8 (B. Solomyak [336]). *Let* σ *be a primitive substitution such that its incidence matrix* $\mathbf{M}_\sigma$ *admits an irreducible characteristic polynomial.*

The substitutive system associated with σ *admits* $\exp(2\pi i\beta)$ *as an eigenvalue, with irrational* β*, if and only is there exists a polynomial* $P \in \mathbb{Z}[X]$ *such that* $P(\alpha) = \beta$ *for every eigenvalue* $\alpha \geq 1$ *of* $\mathbf{M}_\sigma$*.*

For instance, the spectrum of the system associated with the substitution defined by $\sigma(1) = 1244$, $\sigma(2) = 23$, $\sigma(3) = 4$ and $\sigma(4) = 1$ contains the set $\exp(2\pi i\mathbb{Z}\sqrt{2})$.

The last theorem produces various sufficient conditions for a substitutive system to have rational spectrum (see [336]). For instance, if s denotes the smallest prime divisor of the cardinality d of the alphabet, then the fact that the number of eigenvalues of $\mathbf{M}_\sigma$ greater than 1 is strictly greater that d/s is sufficient to ensure that the substitutive dynamical system associated with σ does not have irrational spectrum.

Using techniques from the study of substitutive normal sets [239], S. Ferenczi, C. Mauduit and A. Nogueira give a more explicit and constructive version of the proofs and results of B. Host [181], in terms of return words.

Theorem 7.8.9 (Ferenczi, Mauduit, Nogueira [147]). *Let σ be a primitive substitution which is not S-periodic and (X_σ, S) be the substitutive dynamical system associated with σ.*

We call return word *any word $W = a_1 \ldots a_{k-1}$ in the language $\mathcal{L}(X_\sigma)$ of the substitution, such that W can be extended into a word $a_1 \ldots a_k \in \mathcal{L}(X_\sigma)$ which verifies for every large enough n: $\sigma^n(a_k) = \sigma^n(a_1)$ et $\sigma^n(a_j) \neq \sigma^n(a_1)$, $\forall\, 1 < j < k$. The associated return time is defined to be $r_n(W) = |\sigma^n(a_1 \ldots a_{k-1})|$.*

A complex number λ with modulus 1 is an eigenvalue of the substitutive dynamical system (X_σ, S) if and only if $\lim_{n\to\infty} \lambda^{r_n(W)} = 1$ for every return word w.

For instance, the spectrum of the system associated with the substitution already considered above defined by $\sigma(1) = 1244$, $\sigma(2) = 23$, $\sigma(3) = 4$ and $\sigma(4) = 1$, is exactly the set $\exp(2\pi i \mathbb{Z}\sqrt{2})$.

The authors deduce from this expression in terms of return words a characterization of eigenvalues which generalizes Theorem 7.8.8. This characterization is too long to be cited here. A consequence of it is the following:

Corollary 7.8.3 (Ferenczi, Mauduit, Nogueira [147]). *Let σ be a primitive substitution which is not S-periodic. If λ is a complex number with modulus 1 such that $\lim_n \lambda^{|\sigma^n(a)|} = 1$ for every letter a of the alphabet, then λ is an eigenvalue of the substitutive dynamical system associated with σ.*

For every $\lambda = \exp(2\pi i\beta)$ which satisfies this condition with irrational β, there exists a polynomial $P \in \mathbb{Z}[X]$ such that $P(\alpha) = \beta$ for every eigenvalue $\alpha \geq 1$ of the incidence matrix of σ.

Thus, the spectrum of a primitive dynamical system can be divided into two parts. The first part is the set of complex numbers which satisfy the assumptions of Corollary 7.8.3: these eigenvalues depend only on the incidence matrix of the substitution, and, more precisely, for the irrational eigenvalues, on the matrix eigenvalues which are greater than 1. In this sense, this class of eigenvalues is quite natural. In a more exotic and surprising way, the substitutive system can have some eigenvalues which do not satisfy the assumptions of the corollary. These eigenvalues depend heavily on the return times and more generally on the combinatorics of the substitution, which in other words gives them a non-commutative aspect.

7.9 Substitutions of Pisot type: spectral theory

Some of the general results of Sec. 7.8 have special consequences for the spectral theory of Pisot type substitutive systems.

In [182], B. Host claims that his work (Theorems 7.8.3 and 7.8.4) implies that any unimodular substitution of Pisot type with 1 as unique coboundary generates a substitutive system whose group of eigenvalues is generated by the frequencies of letters in any word of the system, i.e., by the coordinates of a right normalized eigenvector associated with the dominant eigenvalue of the incidence matrix of the substitution.

Being less precise, but more general (without any condition about coboundaries), a consequence of Theorem 7.8.8 is the following result, since the characteristic polynomial of a Pisot type substitution is always irreducible over $\mathbb{Q}$:

Proposition 7.9.1. *If α denotes the dominant eigenvalue of the incidence matrix of a Pisot type substitution, then the spectrum of the substitutive dynamical system associated with σ is the set $\exp(2\pi i\mathbb{Z}[\alpha])$.*

In particular, Pisot type substitutive dynamical systems are never weakly mixing.

This means that Pisot type substitutions are quite close to constant length substitutions in respect of the fact that both are never weakly mixing: constant length substitutive dynamical systems always have rational eigenvalues whereas Pisot type substitutive dynamical systems always have irrational eigenvalues. But if a substitution is neither of Pisot type, nor of constant length, then completely different behaviors can occur, as the absence of spectrum for which we gave conditions sufficient in Theorem 7.8.7, and an example of which is given by $\sigma(1) = 112$, $\sigma(2) = 12222$.

Since $\mathbb{Z}[\alpha]$ is of rank $d-1$, we deduce from Corollary 7.8.2 that:

Corollary 7.9.1. *Any Pisot type substitutive dynamical system on d letters admits as a topological factor a minimal translation on the torus $\mathbb{T}^{d-1}$.*

Moreover, in [338] , M. Solomyak focuses on the rational part of unimodular Pisot type substitutive dynamical systems:

Theorem 7.9.1 (M. Solomyak [338]). *The rational spectrum of a Pisot type and unimodular substitutive dynamical system is finite.*

Note that this is not true in general for non-unimodular substitutions, as stated in Theorem 7.8.2.

Examples have been treated by B. Solomyak. Indeed, in [337, 336], this author develops the results of B. Host in [181]. He proves in that way that for every d, the substitutive dynamical system associated with the unimodular and Pisot type substitution defined by $\sigma(1) = 123\ldots d$, $\sigma(2) = 1$, $\sigma(3) = 2$, ..., $\sigma(d) = d-1$ has a discrete spectrum, and is measure-theoretically isomorphic with the $(d-1)$-dimensional torus rotation by a normalized right eigenvector associated with the dominant eigenvalue of the incidence matrix of σ.

Moreover, thanks to properties of the β-expansion, B. Solomyak proves in [337] that Pisot type substitutions among those which are defined by $\sigma(1) = 1^{a_1}2$, $\sigma(2) = 1^{a_2}3$, $\ldots$, $\sigma(d-1) = 1^{a_{d-1}}d$ and $\sigma(d) = 1^{a_d}$, with $a_1 \geq a_2 \geq \cdots \geq a_d \geq 1$, have a discrete spectrum.

To finish, M. Hollander restated in [172] the proof of the results of A. N. Livshits and applies them to the case of Pisot type substitutions on a two-letter alphabet.

He proves that the systems associated with such substitutions have a discrete spectrum if the substitution satisfies a combinatorial condition which generalizes the coincidence condition introduced by M. F. Dekking for substitutions of constant length (see Sec. 7.5). Indeed, a two-letter substitution is said to satisfy the *coincidence condition* if there exist two integers k, n such that $\sigma^n(0)$ and $\sigma^n(1)$ have the same k-th letter, and the prefixes of length $k-1$ of $\sigma^n(0)$ and $\sigma^n(1)$ have the same image under the abelianization map.

This definition appeared in literature before [172] (see [285]). In particular, B. Host did notice the relationship between this condition and the spectrum of substitutive dynamical systems (see Theorem 7.10.1 in the next section). However, the following results, although if conjectured, were not proved before Hollander's work.

Theorem 7.9.2 (Hollander [172]). *Let σ be a primitive substitution over a two-letter alphabet which satisfies the coincidence condition. Then the substitutive dynamical system associated with σ has discrete spectrum.*

Note that we still do not know at the moment if there exists a Pisot type substitution which does not satisfy the coincidence condition (for more details, see Chap. 12).

7.10 Geometric realization of substitutions

According to the spectral theory of Pisot type substitutive dynamical systems exposed in Sec. 7.9, it is natural to imagine that unimodular Pisot type substitutive systems are good candidates to have discrete spectrum, that is to be measure-theoretically isomorphic with a toral rotation, this rotation being obtained with the frequencies of apparition of letters in any word of the system. Since topological factors and measure-theoretic factors are the same (Corollary 7.8.2), a possibility of testing this hypothesis is to realize geometrically theses systems as a toral translation, that is to map continuously unimodular Pisot type substitutive systems onto torus, so that the mapping realizes a a commutation between the shift map and a rotation. This brings us to the following general notion of geometric realization.

7.10.1 Geometric realization

Symbolic dynamical systems were introduced by coding trajectories of points in a geometrical space, with respect to a given partition, under the action of a map with a non-trivial dynamic, in the aim to understand this dynamic better. For instance, the Morse sequence is fundamentally connected with the existence of reccurent and nonclosed geodesics on surfaces with negative curvature [254] (see Chap. 1).

Another significant example is the substitutive dynamical system associated with the Fibonacci substitution, defined by $\sigma(a) = ab$, $\sigma(b) = a$: this system is the set of the codings of points of the one-dimensional torus $\mathbb{T}$, with respect either to the partition $\mathcal{T} = [0, 1-\alpha[\cup[1-\alpha, 1[$, or to $\mathcal{T}' =]0, 1-\alpha]\cup]1-\alpha, 1]$, under the action of the addition of α, where $\alpha = (\sqrt{5}-1)/2$. Moreover, the coding map is 2-to-1 and almost everywhere 1-to-1 [256] (for more details, see Chap. 6). More generally, codings of toral rotations generate sequences called *Sturmian*, the combinatorial, arithmetical and geometrical properties of which are particularly well known (see for instance [5, 256, 57] and Chap. 6).

Thus, it is quite natural to look for a geometrical interpretation of those abstract systems that are symbolic dynamical systems. M. Queffélec defines in [285] (pp. 140) the notion of geometric representation:

Definition 7.10.1. *A* geometric representation *of a symbolic dynamical system (X_u, S) is a continuous map φ from X_u* onto *a geometrical dynamical system $(\mathcal{X}, T)$, on which there exists a partition indexed by the alphabet $\mathcal{A}$, such that every word in X_u is the itinerary of a point of $(\mathcal{X}, T)$ with respect to the partition.*

It is thus natural to ask which systems admit such a geometric representation, and, if so, how precise is the representation, that is what is the degree of injectivity of the representation. The representation is considered to be precise if the map φ is a semi-topologically isomorphism.

There have been many partial answers given to Queffélec's question: in [42], G. Rauzy and P. Arnoux introduce the so-called *Arnoux-Rauzy sequences*: these sequences form a class of uniformly recurrent sequences of combinatorial complexity $2n+1$ satisfying a certain combinatorial criterion (the so-called * condition of Arnoux-Rauzy); they show that each such sequence codes the orbit of a point under an exchange of six intervals on the circle. In [74], M. Boshernitzan and I. Kornfeld give explicitly a conjugacy map between the dynamics of the restriction to the limit of a piecewise translation and the dynamics of a substitutive system. In [139], S. Ferenczi obtains a geometric realization of the Chacon sequence as exduction of a triadic rotation (see also Chap 3). Recently, J. Cassaigne, S. Ferenczi and L.

Q. Zamboni have exhibited an example of a nonsubstitutive Arnoux-Rauzy sequence whose properties are very far from the properties of substitutive Arnoux-Rauzy sequences: in particular, their example cannot be geometrically realized as a natural coding of a toral translation [93] (we will come back to this topic in Chap. 12).

The idea of trying to represent geometrically substitutive systems of Pisot type was developed first by G. Rauzy [292] and then by many authors. Developments of this aspect are the aim of the following section.

7.10.2 Geometric realization of substitutions of Pisot type: Rauzy fractals

The first example of geometrical realization of Pisot type substitution was given in [292] by G. Rauzy. This paper is at the origin of all the developments which are exposed here.

In [292], G. Rauzy proves that the dynamical system generated by the Tribonacci substitution or Rauzy substitution $\sigma(1) = 12, \sigma(2) = 13\,, \sigma(3) = 1$, is measure-theoretically isomorphic with an exchange of domains in a compact subset $\mathcal{R}$ of the complex plane called *Rauzy fractal* (Fig. 7.1). This compact subset has a self-similar structure: using methods introduced by F. M. Dekking in [115], S. Ito and M. Kimura obtain in [186] an alternative construction of $\mathcal{R}$ and prove that each of the exchanged domains has a fractal boundary. In the same paper they also study the Hausdorff dimension of the boundary of $\mathcal{R}$. A. Messaoudi obtains in [243, 244] additional properties of the fractal $\mathcal{R}$ including a boundary parametrisation.

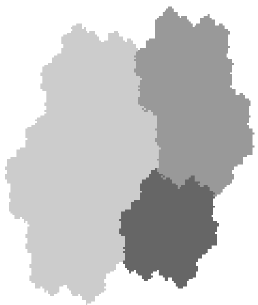

Fig. 7.1. Rauzy fractal for the Tribonnacci substitution

By studying the numeration system of the *Tribonacci number* (defined as the unique root of the polynomial $x^3 - x^2 - x - 1$ of modulus smaller than one), G. Rauzy proves that $\mathcal{R}$ projects onto the torus in a measure-theoretically one-to-one way. This implies that the compact set $\mathcal{R}$ generates a periodic tiling of the plane with fractal boundary. The self-similarity of $\mathcal{R}$ implies that the induced tiling is quasi-periodic. This is analogous to the well understood connection between Sturmian sequences, tilings of the line and quasi-crystals (see [33]), and more generally to the theory of quasi-crystals: through methods similar to those used by G. Rauzy in [292], E. Bombieri and J. Taylor use substitutions to exhibit a connection between quasi-crystals and number theory in [44, 70, 203, 321, 354].

From the point of view of ergodic theory, the result of G. Rauzy [292] means that the subshift generated by the substitution σ is measure-theoretically isomorphic with a rotation on the two-dimensional torus.

This example being quite well understood, the question of its generalization to a more important class of substitutions is thus natural.

B. Host, in an unfortunately unpublished work [182], focuses on the geometric realization of unimodular Pisot type substitutive systems over a two-letter alphabet. Let X_σ denote such a system. He proves that there exists a continuous mapping from X_σ onto the one-dimensional torus $\mathbb{T}$ such that the shift map S commutes with a pseudo-exchange of domains of a compact subset of the line, the domains being the images of cylinders of X_σ through the mapping:

$$\exists\, f : X \to \mathbb{R},\ f \text{ continuous},\ \exists\, a_0, a_1 \in \mathbb{R};\quad \forall w \in X_\sigma,\ f(Sw) = f(w) + a_{w_0}.$$

B. Host proves that the mapping is measure-theoretically one-to-one if and only if its projection onto the torus $\mathbb{R}/(a_1 - a_0)\mathbb{Z}$ is also 1-to-1. This is also equivalent to the fact that the substitution σ satisfies the coincidence condition. We hence get the following result:

Theorem 7.10.1 (Host [182]). *Let σ be a unimodular Pisot type substitution over a two-letter alphabet, which satisfies the coincidence condition. The substitutive system associated with σ is measure-theoretically isomorphic on the one hand with an interval exchange, and on the other hand, with a one-dimensional toral rotation, these realizations being explicit and continuous.*

This result was obtained much before Hollander's [172]. It is less general, since it depends on the unimodular condition, but it has the advantage of being constructive.

Thus, the work of B. Host give a complete generalization of [292] for substitutions over two letters. For substitutions over more than two letters, one can remark that the definition of the Rauzy fractal is easy to extend to any unimodular Pisot type substitution. It was formally done in [176].

Definition 7.10.2. *Let σ be a unimodular Pisot type substitution over d letters. Let $\alpha_1 > 1, \ldots, \alpha_r$ be the real eigenvalues of the incidence matrix of σ, $\alpha_{r+1}, \overline{\alpha_{r+1}}, \ldots, \alpha_{r+s}, \overline{\alpha_{r+s}}$ be its complex eigenvalues. Let $\mathbf{v}_1, \ldots \mathbf{v}_{r+s} \in \mathbb{R}^d$ be the right eigenvectors of the incidence matrix.*

For any finite word W, we define the following vector in $\mathbb{R}^{r-1} \times \mathbb{D}^s$:

$$\delta(W) = \sum_{k=1}^{|W|} ((\mathbf{v}_2)_{W_i}, \ldots, ((\mathbf{v}_{r+s})_{W_i} \in \mathbb{R}^{r-1} \times \mathbb{D}^s .$$

Let $u = (u_i)_{i\in\mathbb{Z}}$ be a periodic point for σ. Then the Rauzy fractal *associated with the substitution σ is the following compact set:*

$$\mathcal{F} = \overline{\{\delta(u_0 \ldots u_i);\ i \in \mathbb{N}\}} \subset \mathbb{R}^{r-1} \times \mathbb{D}^s .$$

Since $r + 2s = d$, $\mathbb{R}^{r-1} \times \mathbb{D}^s$ is a $\mathbb{R}$-vectorial space of dimension $d - 1$, so that the Rauzy fractal associated with a unimodular substitution of Pisot type can be considered as a part of $\mathbb{R}^{d-1}$.

Topological properties of Rauzy fractals have been studied by many authors:

- V. F. Sirvent and Y. Wang prove in [332] that Rauzy fractals have nonempty interior and are the closure of their interior.
- Conditions for simple connectedness are given by par S. Ito and M. Ohtsuki in [188].
- Conditions for connectedness are given by V. Canterini in [85], and restated in a more algorithmic way in [329].

Moreover, C. Holton and L. Q. Zamboni define in [174, 176, 177] real and complex representations associated with every substitution the incidence matrix of which has a nonzero eigenvalue of modulus less than one. Their work gives explicit bounds for the Haussdorf dimension of the projections on axes of the Rauzy fractal associated with a substitution. More precisely, for any $k = 2, \ldots, r + s$, they prove that the projection of the Rauzy fractal on the axis associated with the eigenvalue α_k, that is the k-coordinate of the Rauzy fractal in $\mathbb{R}^{r-1} \times \mathbb{D}^s$ (this axis can be $\mathbb{R}$ or $\mathbb{C}$), admits a Haussdorf dimension which is less than $-\log(\alpha_1)/\log(\alpha_k)$.

Thus, for any $k = 2, \ldots, r + s$, the projection is a Cantor set. If $k = s + 1, \ldots, s + d$ and $|\alpha_k|^2\alpha_1 < 1$, then the projection has nonempty interior.

The construction of the Rauzy fractal being generalized, the question is to generalize [292] by obtaining conditions for a substitutive system to be semi-topologically isomorphic with an exchange of domains on the Rauzy fractal associated with the substitution.

An answer to this question was developed by P. Arnoux and S. Ito by giving an alternative construction of the Rauzy fractal thanks to induction and successive approximation. Their main results are exposed in the next subsection, and are also stated in Chap. 8.

Another approach to this problem was made by V. Canterini and A. Siegel by formalizing and generalizing the ideas of G. Rauzy in [295]. This approach is based on recognizability and formal power series. It is exposed in subsection 7.10.4.

7.10.3 Rauzy fractal, successive approximations and induction

P. Arnoux and S. Ito generalize in [37] the alternative construction given by S. Ito and M. Kimura [186] of the Rauzy fractal associated with the Rauzy substitution. Indeed, they construct by successive approximation the Rauzy fractal for any unimodular Pisot type substitution. The construction is described precisely in Chap. 8.

Furthermore, the authors introduce in [37] a condition of coincidence which generalizes to every substitution the condition already given for constant length substitutions and two-letter substitutions (Theorems 7.5.2 and 7.9.2) .

Definition 7.10.3 (Arnoux, Ito [37]). *A substitution σ satisfies the* coincidence condition *on prefixes (respectively suffixes) if for every couple of letters (j,k), there exists a constant n such that $\sigma^n(j)$ and $\sigma^n(k)$ can be decomposed in the following way:*

$$\sigma^n(b_1) = p_1 a s_1 \quad \textit{and} \quad \sigma^n(b_2) = p_2 a s_2, \quad \textit{with} \quad \mathbf{l}(p_1) = \mathbf{l}(p_2)$$
$$(\textit{respectively} \quad \mathbf{l}(s_1) = \mathbf{l}(s_2)) \, .$$

We still do not know whether there exists a Pisot type substitution which does not satisfy this condition. B. Host has a partial result which can lead to such a substitution, but with an important number of letters. For 2 or 3 letters, it is conjectured that there does not exist a Pisot type substitution without coincidence.

Under this condition, P. Arnoux and S. Ito prove that the Rauzy fractal associated with a substitution over d letters can be canonically divided into d parts which are disjoint in measure, so that the Rauzy fractal has an auto-similar structure.

Theorem 7.10.2 (Arnoux, Ito [37], see Chap. 8). *Let σ be a unimodular Pisot type substitution over a d-letter alphabet which satisfies the condition of coincidence. Then the substitutive dynamical system associated with σ is measure-theoretically isomorphic with the exchange of d domains defined almost everywhere on the auto-similar Rauzy fractal of σ.*

Furthermore, $\mathbb{R}^{d-1}$ projects onto $\mathbb{T}^{d-1}$ so that the domain exchange can be factorized by an irrational translation. Thus, they get a more geometrical version of Corollary 7.9.1:

Proposition 7.10.1 (Arnoux, Ito [37]). *Any unimodular Pisot type substitutive dynamical system which satisfies the coincidence condition admits as a continuous factor an explicit irrational translation on the torus tore $\mathbb{T}^{d-1}$, the fibers being finite almost everywhere.*

Unfortunately, this projection on the torus does not preserve the injectivity; the same problem occured for substitutions over 2 letters. The authors give a sufficient condition for the rotation factor to be isomorphic with the substitutive system: if there exists a set with nonempty interior which is included in all the domains which appear in the construction by successive approximation of the Rauzy fractal of the substitution, then we get an isomorphism (see Chap. 8).

Such an approach gives results about the boundary of Rauzy fractals (see Chap. 8).

7.10.4 Rauzy fractal and formal power series

Thus, the work of P. Arnoux and S. Ito gives a generalization of [292] for a big class of substitutions, but two fundamental questions remain unsettled: first, the mapping from the substitutive dynamical system onto the Rauzy fractal which conjugates the shift map and the domain exchange is not explicit, and the same for the mapping onto the torus. Secondly, there is no result about the injectivity of the mapping onto the torus. Both questions were solved by G. Rauzy in [292] for the Tribonacci substitution. He gave in [295] ideas to generalize his work for any Pisot type substitution. This approach has been partially realized in [85] and [330], to obtain explicit representation mappings and conditions for injectivity of the toral mapping.

Indeed, in [86], the prefix-suffix expansion defined in Sec. 7.7.2 is used to produce a measure-theoretical isomorphism between the shift map on a substitutive system with an adding machine over the subshift of finite type, considered as a Markov compactum in which the order is naturally defined by the substitution. We already explained that this provides a new proof of A. N. Vershik results, originally proved thanks to a stationary Markov compactum with a partial ordering on the labels of the edges. This ordering was arbitrary, so that the representation by the Markov compactum was not explicit. On the contrary, the definition of labels thanks to prefixes of images of letters under the substitution induces a natural ordering between the edges and provides a well determined combinatorial representation.

If the substitution is of Pisot type over a d-letter alphabet, then this prefix-suffix expansion allows one to realize explicitly any element of X_σ as a point in $\mathbb{R}^{d-1}$ expressed in terms of power series. On each coordinate, this mapping coincides with the real or complex representations of C. Holton and L. Zamboni in [174, 177]. Gathering all the one-dimensional realizations together is natural: indeed each of them is a different expression of the same formal numeration system (the one generated by the characteristic polynomial of the incidence matrix of the substitution).

We prove in Sec. [87] that this representation map projects onto the $(d-1)$-dimensional torus and satisfies a commutation relationship with a translation. This gives a new proof of Corollary 7.9.1 which makes the representation totally explicit.

Proposition 7.10.2 (V. Canterini, A. Siegel [87]). *Every dynamical system generated by a primitive substitution of Pisot type over d letters admits as a topological factor a minimal translation on the torus $\mathbb{T}^{d-1}$. The conjugacy map is explicit.*

Moreover, the self-similar compact subset of $\mathbb{R}^{d-1}$ which is the image of the geometrical representation in [87] is exactly the Rauzy fractal associated with the substitution. This gives a new proof of Theorem 7.10.1. The interest of this method lies in the fact the Rauzy fractal is obtained as the image of an explicit map, so that it makes it possible to derive various topological properties of the fractal, such as a criterion for the connectedness of the fractal [85], or a description of the boundary of the Rauzy fractal [329].

The main interest of such a construction is that the explicit expression of the mapping from the substitutive dynamical system onto the Rauzy fractal on one side, and the torus one the other side, gives an efficient approach for a detailed study of the injectivity of the realizations. Indeed, in [329], the question of injectivity is not treated as usual from a measurable point of view [182, 37] but from a formal and algebraic point of view, based first on the substitutive system's combinatorial structure deduced from the prefix-suffix automaton, secondly on the algebraic structure of the system, coming from the incidence matrix of the substitution.

More precisely, it is proved that the representation map into the Rauzy fractal of a Pisot type substitutive system is just obtained by gathering the set of finite values which can be taken for any Archimedean topology by a formal power series with coefficient into the splitting field of the characteristic polynomial of the incidence matrix of the substitution. In that way, the study of the injectivity of the representation map is reduced to the study of sequences of digits which make a formal power series tend to zero for all the Archimedean metrics for which the power series has a limit. This implies that the sequences of digits are labels of paths in a finite automaton. The understanding of the structure of the automaton is fundamentally connected with the injectivity of the geometrical (onto the Rauzy fractal) and toral realizations.

For instance, the injectivity in measure of the geometrical representation of a unimodular Pisot type substitutive system can be characterized in terms of explicit combinatorial conditions on graphs. Contrary to the coincidence condition, it is a necessary and sufficient condition, but it is quite difficult to satisfy this condition quickly. This condition is furthermore satisfied by substitutions with coincidence, but we do not know whether the equivalence holds.

The main interest of this approach is that, contrary to what happens with the coincidence condition, those methods also give results about the injectivity of the toral realization.

Theorem 7.10.3 (A. Siegel [329]). *The injectivity in measure of the toral representation of a unimodular Pisot type substitutive dynamical system de-*

duced from the Rauzy fractal construction is equivalent to an explicit combinatorial condition which depends only on the definition of the substitution which generates the system.

Thus one can test whether the spectrum of a given unimodular Pisot type substitutive dynamical system is discrete.

Corollary 7.10.1 (A. Siegel [329]). *There exists an explicit and effective combinatorial condition for a unimodular Pisot type substitutive dynamical system to be isomorphic with a toral translation, that is to have a discrete spectrum.*

For instance, systems generated by the substitutions $\sigma(1) = 12, \sigma(2) = 31, \sigma(3) = 1$ et $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 132$ have purely discrete spectrum: both are measure-theoretically isomorphic with a translation on the two-dimensional torus, both conjugacy maps being explicit semi-topologically isomorphisms.

In more geometrical terms, the injectivity of the toral realization has the following interpretation in terms of tilings:

Corollary 7.10.2 (A. Siegel [329]). *There exists an explicit and effective combinatorial condition for the Rauzy fractal associated with a unimodular Pisot type substitution over d letters to generate a regular tiling of $\mathbb{R}^{d-1}$.*

The condition is too long to be exposed here. The difficulty in checking whether this condition holds prevents one from obtaining general results about the injectivity of the toral realization. However, even if the condition is too long to be checked by hand, it can be checked in a finite time for any explicit example, thanks to the effective algorithm given in [329].

The most important remaining question at the moment is to obtain a general sufficient condition, for instance in terms of coincidence, for a unimodular Pisot type substitutive system to have discrete spectrum, or equivalently, to be measure-theoretically isomorphic with a toral rotation.

7.11 Substitutions and Markov partitions

When this criterion is satisfied, the results and methods contained in this paper can be used to obtain explicit fractal tilings of the plane and produce Markov partitions for Pisot toral automorphisms, generalizing the work of A. Messaoudi in [243] about the Rauzy substitution. The existence of such Markov partitions is established by R. Kenyon and A. Vershik in [204], although their proof is not constructive. The formalism of substitution induces a natural order on the Markov compactum and yields a Markov partition which is explicit, so that its topological properties can be studied (boundary, connectedness, simple connectedness).