11.11.1999

Chapters already renumbered, 2b=3 etc

CONFORMAL FRACTALS, DIMENSIONS AND ERGODIC THEORY

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This book is an introduction to the theory of iteration of non-uniformly expanding holomorphic maps and topics in geometric measure theory of the underlying invariant fractal sets. Probability measures on these sets yield informations on Hausdorff and other fractal dimensions and properties. The book starts with a comprehensive chapter on abstract ergodic theory followed by chapters on uniform distance expanding maps and *thermodynamical formalism*. This material is applicable in many branches of dynamical systems and related fields, far beyond the applications in this book.

Popular examples of the fractal sets to be investigated are Julia sets for rational functions on the Riemann sphere. The theory which was initiated by Gaston Julia [J] and Pierre Fatou [F] became very popular since the time when Benoit Mandelbrot's book [M] with beautiful computer made pictures appeared. Then it became a field of spectacular achievements by top mathematicians during the last 20 years.

Consider for example the map $f(z) = z^2$ for complex numbers z. Then the unit circle $S^1 = \{|z| = 1\}$ is f-invariant, $f(S^1) = S^1 = f^{-1}(S^1)$. For $c \approx 0, c \neq 0$ and $f_c(z) = z^2 + c$, there still exists an f_c-invariant set $J(f_c)$ called the *Julia set* of f_c, close to S^1, homeomorphic to S^1 via a homeomorphism h satisfying equality $f \circ h = h \circ f_c$. However $J(f_c)$ has a fractal shape. For large c the curve $J(f_c)$ pinches at infinitely many points; it may pinch everywhere to become a dendrite, or even crumble to become a Cantor set.

These sets satisfy two main properties, standard attributes of "conformal fractal sets": 1. Their fractal dimensions are strictly larger than the topological dimension. 2. They are conformally "self-similar", namely arbitrarily small pieces have shapes similar to large pieces *via* conformal mappings, here *via* iteration of f.

To measure fractal sets invariant under holomorphic mappings one applies probability measures corresponding to equilibria in the thermodynamical formalism. This is a beautiful example of interlacing of ideas from mathematics and physics.

A *prototype lemma* [B, Lemma 1.1] at the roots of the thermodynamical formalism says that for given real numbers $a_1, ..., a_n$ the quantity

$$F(p_1, ...p_n) = \sum_{i=1}^{n} -p_i \log p_i + \sum_{i=1}^{n} p_i \phi_i$$

has maximum value $P = \log \sum_{i=1}^{n} e^{\phi_i}$ as $(p_1, ..., p_n)$ ranges over the simplex $\{(p_1, ..., p_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}$ and the maximum is assumed only at

$$\hat{p}_j = e^{\phi_j} \big(\sum_{i=1}^{n} e^{\phi_i}\big)^{-1}$$

We can read $\phi_i, p_i, i = 1, ..., n$ as a function (*potential*), resp. probability distribution, on the finite space $\{1, ..., n\}$. Let us further follow Bowen [B]: The quantity

$$S = \sum_{i=1}^{n} -p_i \log p_i$$

is called *entropy* of the distribution $(p_1, ..., p_n)$. The maximizing distribution $(\hat{p}_1, .., \hat{p}_n)$ is called *Gibbs* or *equilibrium state*. In statistical mechanics $\phi_i = -\beta E_i$, where $\beta = 1/kT$, T is a temperature of an external "heat source" and k a physical (Boltzmann) constant. The quantity $E = \sum_{i=1}^{n} p_i E_i$ is the average energy. The Gibbs distribution maximizes then the expression

$$S - \beta E = S - \frac{1}{kT} E$$

or equivalently minimizes the so-called *free energy* $E - kTS$. The nature prefers states with low energy and high entropy. It minimizes free energy.

The idea of Gibbs distribution as limit of distributions on finite spaces of configurations of states (spins for example) of interacting particles over increasing growing to ∞ bounded parts of the lattice $\boldsymbol{Z}^d$ introduced in statistical mechanics first by Bogolubov and Hacet [BH] and playing there a fundamental role was applied in dynamical systems to study Anosov flows and hyperbolic diffeomorphisms at the end of sixties by Ja. Sinai, D. Ruelle and R. Bowen. For more historical remarks see [Ru] or [Si]. This theory met the notion of entropy S borrowed from information theory and introduced by Kolmogorov as an invariant of a measure-theoretic dynamical system.

Later the usefulness of these notions to the geometric dimensions has become apparent. It was present already in [Billingsley] but crucial were papers by Bowen [Bo1] and McCluskey & Manning [McM].

In order to illustrate the idea consider the following example: Let $T_i : I \to I$, $i = 1, ..., n > 1$, where $I = [0, 1]$ is the unit interval, $T_i(x) = \lambda_i x + a_i$, where λ_i, a_i are real numbers chosen in such a way that all the sets $T_i(I)$ are pairwise disjoint and contained in I. Define the limit set Λ as follows

$$\Lambda = \bigcap_{k=0}^{\infty} \bigcup_{(i_0,...,i_k)} T_{i_0} \circ ... \circ T_{i_k}(I) = \bigcup_{(i_0,i_1...)} \lim_{k\to\infty} T_{i_0} \circ ... \circ T_{i_k},$$

the latter union taken over all infinite sequences $(i_0, i_1, ...)$, the previous over sequences of length $k + 1$.

It occurs that its Hausdorff dimension is equal to the only number α for which

$$|\lambda_1|^\alpha + ... + |\lambda_n|^\alpha = 1.$$

Λ is a Cantor set. It is self-similar with small pieces similar to large pieces with the use of linear (more precisely, affine) maps $(T_{i_0} \circ ... \circ T_{i_k})^{-1}$. We call such a Cantor set *linear*. We can distribute measure μ by setting $\mu(T_{i_0} \circ ... \circ T_{i_k}(I)) = \left(\lambda_{i_0} ... \lambda_{i_k}\right)^\alpha$. Then for each interval $J \subset I$ centered at a point of Λ its diameter raised to the power α is comparable to its measure μ (this is immediate for the intervals $T_{i_0} \circ ... \circ T_{i_k}(I)$). (A measure with this property for all small balls centered at a compact set, in a euclidean space of any dimension, is called a *geometric measure*.) Hence $\sum(\mathrm{diam} J)^\alpha$ is bounded away from 0 and ∞ for all economical (of multiplicity not exceeding 2) covers of Λ by intervals J.

Note that for each k μ restricted to the space of unions of $T_{i_0} \circ ... \circ T_{i_k}(I)$, each such interval viewed as one point, is the Gibbs distribution, where we set $\phi((i_0, ..., i_k)) = \phi_\alpha((i_0, ..., i_k)) = \sum_{l=0,...,k} \alpha \log \lambda_{i_l}$. The number α is the unique 0 of the *pressure function* $\mathrm{P}(\alpha) = \frac{1}{k+1} \log \sum_{(i_0,...,i_k)} e^{\phi_a((i_0,...,i_k))}$. In this special affine example this is independent of k. In general non-linear case to define pressure one passes with k to ∞.

The family T_i and compositions is an example of very popular in recent years *Iterated Function System* [Barnsley]. Note that on a neighbourhood of each $T_i(I)$ we can consider $\hat{T} := T_i^{-1}$. Then Λ is an invariant repeller for the distance expanding map $\hat{T}$.)

The relations between dynamics, dimension and geometric measure theory start in our book with the theorem that the Hausdorff dimension of an expanding repeller is the unique 0 of the adequate pressure function for sets built with the help of $C^{1+\varepsilon}$ usually non-linear maps in $I\!\!R$ or conformal maps in $I\!\!R^d$.

This theory was developed for non-uniformly hyperbolic maps or flows in the setting of smooth ergodic theory, see [HK], by Mañé [M], Lai-Sang-Young and Ledrappier [LY]; see [Pesin] for recent developments. The advanced chapters of our book are devoted to this theory, but we restrict ourselves to complex dimension 1. So the maps are non-uniformly expanding and the main technical difficulties are caused by critical points, where we have strong contraction since the derivative by definition is equal to 0 at critical points.

A direction not developed in this book are Conformal Iterated Function Systems with infinitely many generators T_i. They occur naturally as return maps in many important constructions, for example for rational maps with parabolic periodic points or in the *Induced Expansion* construction for polynomials [GS]. Beautiful examples are provided by infinitely generated Kleinian groups [.]. The systematic treatment of Iterated Function Systems with infinitely many generators can be found in [MU1], [MU2], [MU3], [MPU] and [U1] for example.

Below is a short description of the content of the book.

Chapter 1 is an introduction to abstract ergodic theory, here T is a probability measure preserving transformation. The reader will find proofs of the fundamental theorems: Birkhoff Ergodic Theorem and Shannon-McMillan-Breiman Theorem. We introduce entropy, measurable partitions and discuss canonical systems of conditional measures in Rohlin's Lebesgue space the notion of *natural extension* (inverse limit in the appropriate category). We follow here Rohlin's Theory [Ro], see also [FKS]. Next to prepare to applications for finite-to-one rational maps we sketch Rohlin's theory on countable-to-one endomorphisms and introduce the notion of Jacobian, see also [Parry]. Finally we discuss mixing properties (K-propery, exactness, Bernoulli) and probability laws (Central Limit Theorem, abbr. CLT, Law of Iterated Logarithm, LIL, Almost Sure Invariance Principle, ASIP) for the sequence of functions (random variables on our probability space) $\phi \circ T^n, n = 0, 1, ...$

Chapter 2 is devoted to ergodic theory and termodynamical formalism for general continuous maps on compact metric spaces. The main point here is the so called Variational Principle for pressure, compare the *prototype lemma* above. We apply also functional analysis in order to explain Legendre transform duality between entropy and pressure. We follow here [Israel] and [Ruelle]. This material is applicable in *large deviations* and *multifractal analysis*, and is directly related to the uniqueness of Gibbs states question.

In Chapters 1, 2 we often follow the beautiful book by Peter Walters [Wa].

In Ch 3. *distance expanding maps* are introduced. Analogously to Axiom A diffeomorphisms [Smale, Bowen] or endomorphisms [Przy] we outline a topological theory: spectral decomposition, specification, Markov partition, and start a "bounded distortion" play with Hölder continuous functions.

In Chapter 4 termodynamical formalism and mixing properties of Gibbs measures for open distance expanding maps T and Hölder continuous potentials ϕ are studied. To large extend we follow [Bo] and [Ru]. We prove the existence of Gibbs measures (states): m with Jacobian being $\exp -\phi$ up to a constant factor, and T-invariant $\mu = \mu_\phi$ equivalent to m. The idea is to use the *transfer operator* $\mathcal{L}_\phi(u)(x) = \sum_{y \in T^{-1}(x)} u(y) \exp \phi(y)$ on the Banach space of Hölder continuous functions u. We prove the exponential convergence $\xi^{-n}\mathcal{L}^n_\phi(u) \to (\int u\, dm)u_\phi$, where ξ is the eigenvalue of the largest absolute value and u_ϕ the corresponding eigenfunction. One obtains $u_\phi = dm/d\mu$. We deduce CLT, LIL and ASIP, and the Bernoulli property for the natural extension.

We provide three different proofs of the uniqueness of the invariant Gibbs measure. The first, simplest, follows [Keller???], the second relies on the *prototype lemma*, the third one on the differentiability of the pressure function in adequate function directions.

Finally we prove Ruelle's formula

$$d^2P(\phi+tu+sv)/dt\, ds|_{t=s=0} = \lim_{n\to\infty} \frac{1}{n} \int (\sum_{i=0}^{n-1}(u\circ T^i - \int u\, d\mu_\phi)\cdot(\sum_{i=0}^{n-1}(v\circ T^i - \int v\, d\mu_\phi)\, d\mu_\phi.$$

This expression for $u = v$ is equal to σ^2 in CLT for the sequence $u \circ T^n$ and measure μ_ϕ.

(In the book we use the letter T to denote a measure preserving transformation. Maps preserving an additional structure, continuous smooth or holomorphic for example, are usually denoted f or g.)

In Chapter 5 the metric space with the action of an open distance expanding map is embedded in a smooth manifold and it is assumed that the map smoothly extends to a neighbourhood. We call the space with the extended dynamics: *Smooth Expanding Repeller*, abbr. SER. We study smoothness of the density u_ϕ. Finally we provide in detail D. Sullivan's theory classifying line Cantor sets via *scaling function*, sketched in [Su] and discuss the realization problem [PT]. We also discuss applications for *solenoids* for *Feigenbaum maps.*

In Chapter 6 we provide definitions of various "fractal dimensions": Hausdorff, box and packing. We consider also Hausdorff measures with gauge functions difefrent from t^α. We prove "Volume Lemma" linking, roughly speaking, (global) dimension with local dimensions.

In Chapter 7 we finally introduce Conformal Expanding Repellers, abbr. CER, and relate pressure with Hausdorff dimension. We prove $C^{?-1}$ dependence of the dimension on the parameter if the dependence on the parameter of the expanding map is $C^?$. We deal with smooth repellers in $I\!R$ and conformal repellers in $\mathbb{C}$. Here $2 <? \leq \omega$, the real analytic case.

Next we follow the easy (uniform) part of [PUZ]. We prove that for CER (X, f) and Hölder continuous $\phi : X \to R$, for $\kappa = \mathrm{HD}(\mu_\phi)$, Hausdorff dimension of the Gibbs measure μ_ϕ (infimum of Hausdorff dimensions of sets of full measure), either $\mathrm{HD}(X) = \kappa$ the measure μ_ϕ is equivalent to Λ_κ, the Hausdorff measure in dimension κ, and is a *geometric measure*, or μ_ϕ is singular with respect to Λ_κ and the right gauge function for the Hausdorff measure to be compared to μ_ϕ is $\Phi(\kappa) = t^\kappa \exp(c\sqrt{\log 1/t \log\log\log 1/t})$. In the proof we use LIL. This theorem is used to prove a dichotomy for the harmonic measure on a Jordan curve ∂, bounding a domain Ω, which is a repeller for a conformal expanding map. Either ∂ is real analytic or harmonic measure is comparable to the Hausdorff measure with gauge function $\Phi(1)$. This yields an information about the lower and upper growth rates of $|R'(r\zeta)|$, for $r \nearrow 1$, for almost every ζ with $|\zeta| = 1$ and univalent function R from the unit disc $|z| < 1$ to Ω. This is a dynamical counterpart of Makarov's theory of boundary behaviour for general simply connected domains, [Makarov].

We prove in particular that for $f_c(z) = z^2 + c, c \neq 0, c \approx 0$ $1 < \mathrm{HD}(J(f_c)) < 2$.

We show how to express in the language of pressure another interesting function: $\int_{|\zeta|=1} |R'(r\zeta)|^t \, |d\zeta|$ for $r \nearrow 1$.

We also look closer at the Gibbs measures, discuss so called *multifractal analysis*, and study *large deviations*.

Finally we apply our theory to the boundary of von Koch "snowflake" and more general Carleson fractals.

Chapter 8 is devoted to Sullivan's rigidity theorem, saying that two non-linear expanding repellers $(X, f), (Y, g)$ that are Lipschitz conjugate (or more generally there exists a measurable conjugacy that transforms a geometric measure on X to a geometric measure on Y, then the conjugacy extends to a conformal one. This means that measures classify non-linear conformal repellers. This fact, annouced in [Su] only with a sketch of the proof, is proved here rigorously for the first time. We sketch also a generalization by E. Prado.

In Chapter 9 we start to deal with non-uniform expanding phenomena. A heart of this chapter is the proof of the formula $\mathrm{HD}(\mu) = \mathrm{h}_\mu(f)/\chi_\mu(f)$ for an arbitrary f-invariant ergodic measure μ of positive Laypunov exponent $\chi_\mu := \int \log |f'| \, d\mu$.

(The word non-uniform expanding is used just to say that we consider (typical points of) an ergodic measure with positive Lyapunov exponent. In higher dimension one uses the name non-uniform hyperbolic for measures with all Lyapunov exponents non-zero.)

It is so roughly because a small disc around z, whose n-th image is large, has diameter of order $|(f^n)'(z)|^{-1} \approx \exp -n\chi_\mu$ and measure $\exp -n\mathrm{h}_\mu(f)$ (Shannon-McMillan-Breiman theorem is involved here)

Chapter 10 is devoted to conformal measures, namely probability measures with Jacobian $\mathrm{Const} \exp -\phi$ or more specifically $|f'|^\alpha$ in a non-uniformly expanding situation, in particular for any rational mapping f on its Julia set J. It is proved that there exists a minimal exponent $\delta(f)$ for which such a measure exists and that $\delta(f)$ is equal to each of the following quantities:

Dynamical Dimension $\mathrm{DD}(J) := \sup\{\mathrm{HD}(\mu)\}$, where μ ranges over all ergodic f-

invariant measures on J of positive Lyapunov exponent.

Hyperbolic Dimension $\mathrm{HyD}(J) := \sup\{\mathrm{HD}(Y)\}$, where Y ranges over all Conformal Expanding Repellers in J, or CER's that are Cantor sets.

It is an open problem whether for every rational mapping $\mathrm{HyD}(J) = \mathrm{HD}(J) =$ box dimension of J, but for many nonuniformly expandig mappings these equalities hold. It is often easier to study the continuity of $\delta(f)$ with respect to a parameter, than directly Hausdorff dimension. So one obtains an information about the continuity of dimensions due to the above equalities.

Most of the book was written in the years 1990-1992 and was lectured to graduate students by each of us in Warsaw, Yale and Denton. We neglected finishing writing, but recently unexpectedly to us the methods in Chapter 10, relating hyperbolic dimension to minimal exponent of conformal measure, were used to study the dependence on ε of the dimension of Julia set for $z^2 + 1/4 + \varepsilon$, for $\varepsilon \to 0$ and other parabolic bifurcations, by A. Douady, P. Sentenac and M. Zinsmeister in [DSZ] and by C. McMullen in [McM]. So we decided to make a final effort. Meanwhile nice books appeared on some topics of our book, let us mention [Falconer], [Zinsmeister], [Gora,Boyarsky], [Viana], but a lot of important material in our book is new or was hardly accessible, or is written in an unconventional way.

[Barnsley] ....

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