CHAPTER 7. SULLIVAN'S CLASSIFICATION OF CONFORMAL EXPANDING REPELLERS.

Dijon, June, 1991

(This is a very preliminary version of one of chapters of a book by Przytycki and Urbański, in preparation, on conformal fractals. It relies on ideas of the proof of the rigidity theorem drafted by D. Sullivan in Proceedings of Berkeley's ICM in 1986.)

In Chapter 4.6 we proved that the scaling function for an expanding repeller in the line determines the $C^{1+\varepsilon}$-structure. In this chapter we will basically concentrate on nonlinear conformal expanding repellers, called CER's, proving that the class of equivalence of the geometric measure determines the conformal structure.

Section 1. Equivalent notions of linearity.

Definition 7.1.1. We call a CER (X, f) *linear* if one of the following conditions holds:

a) The Jacobian of f with respect to the Gibbs measure μ_X equivalent to a geometric measure m_X on X , Jf, is locally constant.

b) The function $\mathrm{HD}(X) \log |f'|$ is cohomologous to a locally constant function on X.

c) The conformal structure on X admits a conformal affine refinement so that f is affine (i.e., see Ch.4.3, there exists an atlas $\{\varphi_t\}$ that is a family of conformal injections $\varphi_t : U_t \to \mathbb{C}$ where $\bigcup_t U_t \supset X$ such that all the maps $\varphi_t \varphi_s^{-1}$ and $\varphi_t f \varphi_s^{-1}$ are affine)

Recall that as the conformal map f may change the orientation of $\mathbb{C}$ on some components of its domain we can write $|f'|$ but not f' unless f is holomorphic.

Proposition 7.1.2. The conditions a), b) and c) are equivalent.

Before we shall prove this proposition we distinguish among CER's *real- analytic* repellers:

Definition 7.1.3. We call (X, f) *real- analytic* if X is contained in the union of a finite family of real analytic open arcs and closed curves.

Lemma 7.1.4. If there exists a connected open domain U in $\mathbb{C}$ intersecting X for a CER (X, f) and if there exists a real analytic function k on it equal identically 0 on $U \cap X$ but not on U then (X, f) is real-analytic.

Proof. Pick an arbitrary $x \in U \cap X$. Then in a neighbourhood V of x the set $E = \{k = 0\}$ is a finite union of pairwise disjoint real- analytic curves and of the point x. This follows from the existence of a finite decomposition of the germ of E at x into irreducible germs and the form of each such germ, see for example Proposition 5.8 in the Malgrange book [Malgrange]. As the sets $f^n(X \cap V)$, $n \geq 0$ cover X, X is compact and f is open on X we conclude that X is contained in a finite union of real-analytic curves γ_j and a finite set of points A such that the closures of γ_j can intersect only in A.

Suppose that there exists a point $x \in X$ such that X is not contained in any real-analytic curve in every neighbourhood of x. Then the same is true for every point $z \in X \cap f^{-n}\{x\}$, $n \geq 0$, hence for an infinite number of points (because pre-images of x are dense in X by the topological exactness of f, see Ch.?). But we proved above that the number of such points is finite so we arived at a contradiction. We conclude that X is contained in a 1-dimensional real-analytic submanifold of $\mathbb{C}$.

Proof of Proposition 7.1.2.

a)$\Rightarrow$ b). Let u be the eigenfunction $\mathcal{L}u = u$ for the transfer operator $\mathcal{L} = \mathcal{L}_\varphi$ for the function $\varphi = -\kappa \log |f'|$, where $\kappa = \mathrm{HD}(X)$, as in Ch. 3.3.. Here the eigenvalue $\lambda = \exp P(f, \varphi)$ is equal to 1, see Ch.6.2..

For an arbitrary $z \in X$ we have in its neighbourhood in X

$$\text{Const} = \log Jf = \kappa \log |f'(x)| + \log u(f(x)) - \log u(x) \tag{7.1.1}$$

b)$\Rightarrow$ c). The function u extends to a real-analytic function $u_{\mathbb{C}}$ in a neighbourhood of X, see Ch.4.4, so the function $\log Jf$ extends to a real-analytic function $\log Jf_{\mathbb{C}}$ by the right hand side equality in the formula (7.1.1), for $u_{\mathbb{C}}$ instead of u. We have two cases: either $\log Jf_{\mathbb{C}}$ is not locally constant on every neighbourhood of X and then by Lemma 7.1.4 (X, f) is real-analytic or $\log Jf_{\mathbb{C}}$ is locally constant. Let us consider first the latter case.

Fix $z \in X$. Choose an arbitrary sequence of points $z_n \in X$, $n \geq 0$ such that $f(z_n) = z_{n-1}$ and choose branches f_ν^{-n} mapping z to z_n. Due to the expanding property of f they are all well defined on a common domain around z. For every x close to z denote $x_n = f_\nu^{-n}(x)$. We have $\mathrm{dist}(x_n, z_n) \to 0$ so by (7.1.1) for $\log Jf_{\mathbb{C}}$

$$\begin{aligned} &\sum_{n=1}^{\infty} \kappa(\log |f'(x_n)| - \log |f'(z_n)|) \\ &= \log u_{\mathbb{C}}(x) - \log u_{\mathbb{C}}(z) + \lim_{n\to\infty} (\log u_{\mathbb{C}}(z_n) - \log u_{\mathbb{C}}(x_n)) \\ &= \log u_{\mathbb{C}}(x) - \log u_{\mathbb{C}}(z) \end{aligned} \tag{7.1.2}$$

We conclude that $u_{\mathbb{C}}(x)$ is a harmonic function in a neighbourhood of z in $\mathbb{C}$ as the limit of a convergent series of harmonic functions; we use the fact that the compositions of harmonic functions with the conformal maps f_ν^{-n} are harmonic. Close to z we take a so-called harmonic conjugate function h so that $\log u(x) + ih(x)$ is holomorphic.

Write $F_z = \exp(\log u + ih)$ and denote by $\tilde{F}_z$ a primitive function for F_z in a neighbourhood of z. This is a chart because $F_z(z) \neq 0$. The *atlas* given by the charts $\tilde{F}_z$ is affine (conformal) by the construction. We have due to (7.1.1) for the extended u

$$|(\tilde{F}_{f(z)} \circ f \circ \tilde{F}_z^{-1})'(F_z(x))| = u_{\mathbb{C}}(f(x))|f'(x)|/u_{\mathbb{C}}(x) = \text{Const}$$

so the differential of f is locally constant in our atlas.

In the case (X, f) is real-analytic we consider just the charts φ_t being primitive functions of u on real-analytic curves containing X into $\mathbb{R}$ with unique complex extensions to neighbourhoods of these curves into a neighbourhood of $\mathbb{R}$ in $\mathbb{C}$. The equality $\log J f_{\mathbb{C}} = \text{Const}$ holds on these curves so the derivatives of $\varphi_t f \varphi_s^{-1}$ are locally constant.

c)⇒ a). Denote the maps $\varphi_t f \varphi_s^{-1}$ by $\tilde{f}_{t,s}$. In a neighbourhood (in X) of an arbitrary $z \in X$ we have

$$
\begin{aligned}
u(x) &= \lim_{n\to\infty} \mathcal{L}^n(1)(x) = \lim_{n\to\infty} \sum_{y\in f^{-n}(x)} |(f^n)'(y)|^{-\kappa} \\
&= \lim_{n\to\infty} |\varphi'(x)|^{\kappa} \sum_y |\varphi'(y)|^{-\kappa} |\tilde{f}'(y)|^{-n\kappa} \\
&= \text{Const} \lim_{n\to\infty} |\varphi'(x)|^{\kappa} \sum_y \tilde{f}'(y)|^{-n\kappa} = |\varphi'(x)|^{\kappa}\text{Const}
\end{aligned}
\tag{7.1.3}
$$

To simplify the notation we omitted the indices at φ and $\tilde{f}$ here, of course they depend on z and y's more precisely on the branches of f^{-n} on our neighbourhood of z mapping z to y's . Const also depends on z. We could omit the functions $\varphi'(y)$ in the last line of (7.1.3) because the diameters of the domains of $\varphi'(y)$ which were involved converged to 0 when $n \to \infty$ due to the expanding property of f, so these functions were almost constant.

Hence due to (7.1.3) in a neighbourhood of every $x \in X$ we get

$$
Jf(x) = \text{Const } u(f(x))|f'(x)|^{\kappa}/u(x) = \text{Const}|\tilde{f}'(x)|^{\kappa} = \text{Const}
$$

♣

Remark 7.1.5. In the b)⇒c) part of the proof of Proposition (7.1.2) as $-\kappa \log |f'|$ is harmonic we do not need to refer to Ch.4.4 for the real-analyticity of u . The formula (7.1.2) gives a harmonic extension of u to a neighbourhood of an arbitrary $z \in X$, depending on the choice of the sequence (z_n). If two extensions u_1, u_2 do not coincide on a neighbourhood of z then in a neighbourhood of z, $X \subset \{u_1 - u_2 = 0\}$. If the equation (7.1.1) does not extend to a neighbourhood of z then again $X \subset \{v = \text{Const}\}$ for a harmonic function v extending the right hand side of (7.1.1).

In each of the both cases (X, f) happens to be real-analytic and to prove it we do not need to refer to Malgrange's book as in the proof of Lemma 7.1.3. Indeed, for any non-constant harmonic function k on a neighbourhood of $x \in X$ such that $X \subset \{k = 0\}$ we consider a holomorphic function F such that $k = \text{Re}F$ and $F(x) = 0$. Then $E = \{k = 0\} = \{\text{Re}F = 0\}$. If F has a d-multiple zero at x then it is a standard fact that E is a union of d analytic curves intersecting at x within the angle $\frac{\pi}{d}$.

We end this Section with giving one more condition implying the linearity.

Lemma 7.1.6. Suppose for a CER (X, f) that there exists a Hölder continuous line field in the tangent bundle on a neighbourhood of X invariant under the differential of f. In other words there exists a complex valued nowhere zero Hölder continuous function α such that for every x in a neighbourhood of X

$$\text{(7.1.4)} \qquad \mathrm{Arg}\alpha(x) + \mathrm{Arg}f'(x) = \mathrm{Arg}\alpha(f(x)) + \varepsilon(x)\pi$$

where $\varepsilon(x)$ is a locally constant function equal 0 or 1. This is in the case f preserves the orientation at x, if it reverses the orientation we replace in (7.2.1) $\mathrm{Arg}f'$ by $-\mathrm{Arg}\bar{f}'$.

Then (X, f) is linear.

Proof. As in Proof of Proposition 7.1.2, the calculation (7.1.2), if f is holomorphic we have for x in a neighbourhood of $z \in X$ in $\mathbb{C}$

$$\mathrm{Arg}\alpha(z) - \mathrm{Arg}\alpha(x) = \sum_{n=1}^{\infty} (\mathrm{Arg}(f'(z_n)) - \mathrm{Arg}(f'(x_n))),$$

if we allow f to reverse the orientation then we replace $\mathrm{Arg}f'$ by $-\mathrm{Arg}\bar{f}'$ in the above formula for such n that f changes the orientation in a neighbourhood of x_n. So $\mathrm{Arg}\alpha(x)$ is a harmonic function. Close to z we find a conjugate harmonic function h so we get a family of holomorphic functions $F_z = \exp(-h + i\mathrm{Arg}\alpha$ which primitive functions give an atlas we have looked for.

Remark 7.1.7. The condition for (X, f) in Lemma 7.1.6 is stronger than the linearity property. Indeed we can define f on the union of the discs $D_1 = \{|z| < 1\}$ and $D_2 = \{|z-3| < 1\}$ by $f(z) = 5 \exp 2\pi\vartheta i$ on D_1 where ϑ is irrational, and $f(z) = 5(z-3)$ on D_2. This is an example of an iterated function system from Ch.4.5. We get a CER (X, f) where $X = \bigcap_{n=0}^{\infty} f^{-n}(\{|z| < 5\})$. It is linear because it satisfies the condition c). Meanwhile $0 \in X, f(0) = 0$ and $f'(0) = 5 \exp 2\pi\vartheta i$, so the equation (7.1.4) has no solution at $x = 0$ even for any iterate of f.

Remark 7.1.8. If we assume in place of (7.1.4) that $\mathrm{Arg}f'(x) - \mathrm{Arg}\alpha(f(x)) - \mathrm{Arg}\alpha(x)$ is locally constant, then we get the condition equivalent to the linearity.

Section 2. Rigidity of nonlinear CER's

In this section we shall prove the main theorem of Chapter 7:

Theorem 7.2.1. Let $(X, f), ((Y, g)$ be two non-linear conformal expanding repellers in $\mathbb{C}$. Let h be an invertible mapping from X onto Y preserving Borel σ-algebras and conjugating f to g, $h \circ f = g \circ h$. Suppose that one of the following assumption is satisfied:

1. h and h^{-1} are Lipschitz continuous.

2. h and h^{-1} are continuous and preserve so-called Lyapunov spectra, namely for every periodic $x \in X$ and integer n such that $f^n(x) = x$ we have $|(f^n)'(x)| = |(g^n)'(h(x))|$.

3. h_* maps a geometric measure m_X on X to a measure equivalent to a geometric measure m_Y on Y.

Then h extends from X (or from a set of full measure m_X in the case 3.) to a conformal homeomorphism on a neighbourhood of X.

We start the proof with a discussion of the assumptions. The equivalence of the conditions 1. and 2. has been proved in Ch.4.3. The condition 1. implies 3. by the definition of geometric measures 5.6.5. One of the steps of the proof of Theorem will assert that 3. implies 1. under the non-linearity assumption. Without this assumption the assertion may happen false. A positive result is that if h is continuous then for a constant $C > 0$ and every $x_1, x_2 \in X$

$$C < \frac{|h(x_1) - h(x_2)|^{\mathrm{HD}(Y)}}{|x_1 - x_2|^{\mathrm{HD}(X)}} < C^{-1}.$$

(We leave the proof to the reader.)

It may happen that $\mathrm{HD}(X) \neq \mathrm{HD}(Y)$ for example if X is a $1/3$ – Cantor set and for g we remove each time half of the interval from the middle.

A basic observation to prove Theorem 7.2.1 is that

$$Jg \circ h = Jf \text{ and moreover } Jg^j \circ h = Jf^j \tag{7.2.1}$$

for every integer $j > 0$. This follows from $g^j \circ h = h \circ f^j$ and $Jh \equiv 1$. We recall that we consider Jacobians with respect to the Gibbs measures equivalent to geometric measures.

Observe finally that (X,f) linear implies (Y,g) linear. Indeed, if (X, f) is linear then Jf hence Jg admit only a finite number of values in view of $Jg \circ h = Jf$. As Jg is continuous this implies that Jg is locally constant i.e. (Y, g) is linear.

Lemma 7.2.2. If a CER (X, f) is non-linear then there exists $x \in X$ such that $\mathrm{grad} Jf_{\mathbb{C}}(x) \neq 0$.

Proof. If $\mathrm{grad} Jf_{\mathbb{C}} \equiv 0$ on X then as $Jf_{\mathbb{C}}$ is real- analytic we have either $\mathrm{grad} Jf_{\mathbb{C}} \equiv 0$ ona neighbourhood of X in $\mathbb{C}$ or by Lemma 7.1.4 (X, f) is real-analytic and $\mathrm{grad} Jf_{\mathbb{C}} \equiv 0$ on real- analytic curves containing X. In both cases by integration we obtain Jf locally constant on X what contradicts the non-linearity assumption.

Now we can prove Theorem in the simplest case to show the reader the main idea working later also in the general case.

Proposition 7.2.3. The assertion of Theorem 7.2.1 holds if we suppose additionally that (X, f) and $Y, g)$ are real-analytic and the conjugacy h is continuous.

Proof. Let M, N be real analytic manifolds containing X, Y respectively. By the non-linearity of X and Lemma 7.2.2 there exists $x \in X$ and its neighbourhood U in M such that $F := Jf_{\mathbb{C}}|_U : U \to \mathbb{R}$ has a real-analytic inverse $F^{-1} : F(U) \to U$. Then in view of (7.2.1) $h^{-1} = F^{-1} \circ Jg_{\mathbb{C}}$ on $h(U \cap X)$ so h^{-1} on $h(U \cap X)$ extends to a real analytic map on a neighbourhood of $h(U \cap X)$ in N.

Now we use the assumption that h^{-1} is continuous so $h(U \cap X)$ contains an open set v in Y. There exists a positive integer n such that $g^n(V) = Y$ hence for every $y \in Y$

there exists a neighbourhood W of y in N such that a branch g_ν^{-n} of g^{-n} mapping y and even $W \cap Y$ into V is well defined. So we have $h^{-1} = f^n \circ h^{-1} \circ g_\nu^{-n}$ extended on W to a real-analytic map. This gives a real-analytic extension of h^{-1} on a neighbourhood of Y because two such extensions must coincide on the intersections of their domains by the real-analyticity and the fact that Y has no isolated points.

Similarly using the non-linearity of (Y,g) and the continuity of h we prove that h extends analytically. By the analyticity and again lack of isolated points in X and Y the extentions are inverse to each other, so h extends even to a biholomorphic map.

Now we pass to the general case.

Lemma 7.2.4. Suppose that there exists $x \in X$ such that $\mathrm{grad} Jf_{\mathbb{C}}(x) \neq 0$ in the case X is real-analytic, or there exists an integer $k \geq 1$ such that $\det(\mathrm{grad} Jf_{\mathbb{C}}, \mathrm{grad}(Jf_{\mathbb{C}} \circ f^k)) \neq 0$ in the other case .

(In other words we suppose that $Jf_{\mathbb{C}}$, respect. $(Jf_{\mathbb{C}}, Jf_{\mathbb{C}} \circ f^k)$, give a coordinate system on a real, respect. complex neighbourhood of x.)

Suppose the analogous property for (Y, g).

Let $h : X \to Y$ satisfy the property 3. assumed in Theorem 7.2.1. Then h extends from a set of full geometric measure in X to a bi-Lipschitz homeomorphism of X onto Y conjugating f with g.

Proof. We can suppose that $\mathrm{HD}(X) \geq \mathrm{HD}(Y)$, recall that HD denotes Hausdorff dimension . Pick x with the property assumed in the Lemma. Let U be its neighbourhood in M (as in Proof of Proposition 7.2.3) or in $\mathbb{C}$ if (X, f) is not real-analytic, so that $F := (Jf_{\mathbb{C}}, Jf_{\mathbb{C}} \circ f^k)$ is an embedding on U. Let $y \in Y$ be a density point of the set $h(U \cap X)$ with respect to the Gibbs measure μ_Y equivalent to the geometric measure m_Y. (Recall that we have proved that almost every point is a density point for an arbitrary probability measure on a euclidean space in Ch.5.2 relying on Besicovitch's Theorem.) . So if we denote $(Jg_{\mathbb{C}}, Jg_{\mathbb{C}} \circ g^k)$ in a neighbourhood (real or complex) of y by G, we have for every $\delta > 0$ such $\varepsilon_0 = \varepsilon_0(\delta) > 0$ that for every $0 < \varepsilon < \varepsilon_0$:

$$\frac{\mu_Y(B(y,\varepsilon) \cap h(U \cap X))}{\mu_Y(B(y,\varepsilon))} > 1 - \delta$$

and

$$h^{-1} = F^{-1} \circ G \qquad \text{on} \quad h(U \cap X).$$

(Observe that the last equality may happen false outside $h(U \cap X)$ even very close to y because h^{-1} may map such points to $(Jf_{\mathbb{C}}, Jf_{\mathbb{C}} \circ f^k)^{-1} \circ G$ with a branch of $(Jf_{\mathbb{C}}, Jf_{\mathbb{C}} \circ f^k)^{-1}$ different from F^{-1}.)

Now for every $\varepsilon > 0$ small enough there exists an integer n such that $\mathrm{diam} g^n B(y, \varepsilon)$ is greater than a positive constant , $g^n|_{B(y,\varepsilon)}$ is injective and the distortion of g^n on $B(y, \varepsilon)$ is bounded by a constant C, both constants depending only on (Y, g). Then if $\varepsilon < \varepsilon_0(\delta)$ we obtain for $Y_\delta := g^n(h(U \cap X) \cap B(y, \varepsilon))$,

$$\frac{\mu_Y(g^n(B(y,\varepsilon)) \setminus Y_\delta}{\mu_Y(g^n(B(y,\varepsilon)))} < C \frac{\mu_Y(B(y,\varepsilon) \setminus h(U \cap X))}{\mu_Y(B(y,\varepsilon))} < C\delta.$$

So

$$\frac{\mu_Y(Y_\delta)}{\mu_Y(g^n(B(y,\varepsilon)))} > 1 - C\delta \tag{7.2.2}$$

We have

$$|(f^n)'(h{-}1(y))|^{\mathrm{HD}(X)} \le \mathrm{Const} Jf(h{-}1(y)) = \mathrm{Const} Jg(y) \le \mathrm{Const}|(f^n)'(y)|^{\mathrm{HD}(Y)}.$$

As we assumed $\mathrm{HD}(X) \ge \mathrm{HD}(Y)$ we obtain

$$|(f^n)'(h{-}1(y))| \le \mathrm{Const}|(f^n)'(y)|^{\mathrm{HD}(Y)/\mathrm{HD}(X)} \le \mathrm{Const}|(f^n)'(y)| \tag{7.2.3}$$

Then due to the bounded distortion property for iteration of f and g we obtain that $h^{-1} = f^n h^{-1} g^{-1}$ is Lipschitz on Y_δ with Lipschitz constant independent of δ, more precisely bounded by $\mathrm{Const}\sup\|D(F^{-1}\circ G\|$, where $F^{-1}\circ G$ is considered on a real (complex) neighbourhood of y and Const is that from (7.2.3).

There exists an integer $K > 0$ such that for every n, $g^K g^n B(y,\varepsilon(n))$ covers Y. Because Jg is bounded, separated from 0, this gives h^{-1} on $g^K(Y_\delta)$ Lipschitz with a Lipschitz constant independent from δ and $\mu(g^K(Y_\delta)) > 1 - \mathrm{Const}\,\delta$ for δ arbitrarily small. We conclude that h^{-1} is Lipschitz on a set of full measure μ_Y so it has a Lipschitz extension to Y.

We conclude also that $\mathrm{HD}(X) = \mathrm{HD}(Y)$. Otherwise $\mathrm{diam} h^{-1}(Y_\delta) \to 0$, so because $\mathrm{supp}\mu_X = X$ we would get $\mathrm{diam} X = 0$. So we can replace above the roles of (X,f) and (Y,g) and prove that h is Lipschitz.

The next step will assert that for non-linear repellers the assumptions of Lemma 7.2.4 about the existence of coordinate systems are satisfied.

Lemma 7.2.5. If (X,f) is a non-linear CER then there exists $x \in X$ such that either $\mathrm{grad} Jf_{\mathbb{C}}(x) \neq 0$ in the case X is real-analytic, or there exists an integer $k \ge 1$ such that $\det(\mathrm{grad} Jf_{\mathbb{C}}, \mathrm{grad}(Jf_{\mathbb{C}} \circ f^k)) \neq 0$ in the case (X,f) is not real-analytic.

Proof. We know already from Lemma 7.2.2 that there exists $\hat{x} \in X$ such that $\mathrm{grad} Jf_{\mathbb{C}}(z) \neq 0$ so we may restrict our considerations to the case (X,f) is not real-analytic.

Suppose Lemma is false. Then for all $k > 0$ the functions

$$\Phi_k := \det(\mathrm{grad} Jf_{\mathbb{C}}, \mathrm{grad}(Jf_{\mathbb{C}} \circ f^k))$$

are identically equal to 0 on X. Let W be a neighbourhood of $\hat{x}$ in $\mathbb{C}$ where $\mathrm{grad} Jf_{\mathbb{C}} \neq 0$.

Let us consider on W the line field $\mathcal{V}$ orthogonal to $\mathrm{grad} Jf_{\mathbb{C}}$. Due to the topological exactness of f on X for every $x \in X$ there exists $y \in W \cap X$ and $n \ge 0$ such that $f^n(y) = x$.

Thus define at x

$$\mathcal{V}_x := Df^n(\mathcal{V}_y) \tag{7.2.4}$$

We shall prove now that if $x = f^k(y) = f^l(z)$ for some $y, z \in W \cap X,\ k, l \geq 0$, then

$$Df^k(\mathcal{V}_y) = Df^l(\mathcal{V}_z). \tag{7.2.5}$$

If (7.2.5) is false, then close to x there exist $x' \in X$ and $m \geq 0$ such that $f^m(x') \in W$ (we again refer to the topological exactness of f) and $Df^k(\mathcal{V}_{y'}) \neq Df^l(\mathcal{V}_{z'})$, where $f^k(y') = f^l(z') = x'$, $y' \in X$ is close to y and $z' \in X$ is close to z. We obtain $Df^{k+m}(\mathcal{V}_{y'}) \neq Df^{l+m}(\mathcal{V}_{z'})$ so either $Df^{k+m}(\mathcal{V}_{y'}) \neq \mathcal{V}_{f^m(x')}$ or $Df^{l+m}(\mathcal{V}_{z'}) \neq \mathcal{V}_{f^m(x')}$. Consider the first case (the second is of course similar). We obtain that Jf and $Jf \circ f^{k+m}$ give a coordinate system in a neighbourhood of y' i.e. $\Phi_{k+m}(y') \neq 0$ contrary to the supposition.

Thus the formula (7.2.4) defines a line field at all points of X which is Df-invariant. Observe however that the same formula defines a real-analytic extension of the line field to a neighbourhood of x in $\mathbb{C}$ because $\mathcal{V}$ is real-analytic on a neighbourhood of $y \in W$ and f is analytic. Each two such germs of extensions related to two different pre-images of x must coincide because they coincide on X, otherwise (X, f) would be real-analytic. Now we can choose a finite cover $B_j = B(x_j, \delta_j)$ of a neighbourhood of X with discs, $x_j \in X$ so that for the respective F_j-branches of f^{-n_j} leading x_j into W, we have $F_j(3B_j) \subset W$ where $3B_j := B(x_j, 3\delta_j)$. Hence the formula (7.2.4) defines $\mathcal{V}$ on $3B_j$. So if $B_i \cap B_j \neq \emptyset$, then we have $3B_i \subset B_j$ or vice versa. So $3B_i \cap 3B_j \cap X \neq 0$ hence the extensions of $\mathcal{V}$ on $3B_i$ and on $3B_j$, in particular on B_i and on B_j, coincide on the intersection. This is so because they coincide on the intersection with X and $(X, f$ is not real-analytic.
(We made the trick with 3δ because it can happen that $B_i \cap B_j \neq \emptyset$ but $B_i \cap B_j \cap X = \emptyset$.)

Thus $\mathcal{V}$ extends real-analytically to a neighbourhood of X. This field is Df-invariant on a neighbourhood of X because we can define it in a neighbourhood of $x \in X$ and $f(x)$ by (7.2.4) taking the same $y \in W \cap X$ where $f^n(y) = x$, $f^{n+1}(y) = f(x)$. So by Lemma 7.1.7 (X, f) is linear what contradicts the assumption that (X, f) is non-linear.

Corollary 7.2.6. If for $(X, f), (Y, g)$ the assumptions of Theorem 7.2.1 are satisfied and if (y, g) is real-analytic then (X, f) is real-analytic too.

Proof. Due to Lemma 7.2.5 the assumptions of Lemma 7.2.4 are satisfied. So $h^{-1} = F^{-1} \circ G$ on a neighbourhood of $y \in Y$ by the continuity of h^{-1}, (see the notation in Proof of Lemma 7.2.4). Denote a real-analytic manifold Y is contained in by N. Then $Jg_{\mathbb{C}} \neq$ Const on any neighbourhood of y in N. Otherwise h^{-1} would be constant, but y is not isolated in Y so h^{-1} would not be injective.

Remind that we can consider $F^{-1} \circ G$ as a real analytic extension of h^{-1} to a neighbourhood V of y in N. So the differential of $F^{-1}G$ is 0 at most at isolated points, so different from 0 at a point $y' \in V \cap Y$. We conclude due to the continuity of h that in a neighbourhood of $h^{-1}(y')$, X is contained in a real-analytic curve. So (X, f) is a real-analytic repeller.

Now we shall collect together what we have done and make a decidive step in proving Theorem 7.2.1, namely we shall prove that the conjugacy extends to a real-analytic diffeomorphism.

Proof of Theorem 7.2.1. If both (X, f) and (Y, g) are real- analytic then the conjugacy extends real-analytically to a real-analytic manifold so complex analytically to its neighbourhood by Proposition 7.2.3. Its assumptions hold by Lemmas 7.2.4 and 7.2.2. If both (X, f) and (Y, g) are not real-analytic (a mixed situation is excluded by Corollary 7.2.6), then by Lemma 7.2.4 which assumptions hold due to Lemma 7.2.5 we can assume the conjugacy h is a homeomorphism of X onto Y. But h^{-1} extends to a neighbourhood of $y \in Y$ in $\mathbb{C}$ to a real-analytic map. We use here again the notation of Proposition 7.2.4 and proceed precisely like in Proposition 7.2.3, Proposition 7.2.4 and Corollary 7.2.6 by writing $h^{-1} = F^{-1} \circ G$. This gives a real-analytic extension of h^{-1} to a neighbourhood of an arbitrary $y \in Y$ by the formula $f^n \circ h^{-1} \circ g_\nu^{-1}$ precisely as in Proof of Proposition 7.2.3.

For two different branches F_1, F_2 of g^{-n_1}, g^{-n_2} respectively, mapping y into the domain of $F^{-1} \circ G$ germs of the extensions must coincide because they coincide on the intersection with Y, see Lemma 7.1.4.

Now we build a real-analytic extension of h^{-1} to a neighbourhood of Y similarly as we extended $\mathcal{V}$ in Proof of Lemma 7.2.5, again using the assumption (Y, g) is not real-analytic.Similarly we extend h.

Denote the extensions by $\tilde{h}, \tilde{h^{-1}}$. We have $\tilde{h^{-1}} \circ \tilde{h}$ and $\tilde{h} \circ \tilde{h^{-1}}$ equal to the identity on X, Y respectively. The these compositions extend to the identities to neighbourhoods, otherwise (X, f) or (Y, g) would be real-analytic. We conclude that $\tilde{h}$ is a real-analytic diffeomorphism. Finally observe that $g\tilde{h} = \tilde{h}f$ on a neighbourhood of X because this equality holds on X itself and our functions are real-analytic, otherwise (X, f) would be real-analytic.

The only thing we should still prove is the following

Lemma 7.2.7. If (X, f) is a non-linear CER, not real-analytic , and there is a real-analytic diffeomorphism h on a neighbourhood of X to a neighbourhood of Y for another CER (Y, g) such that $h(X) = Y$ and h conjugates f with g in a neighbourhood of X then h is conformal.

Proof. Suppose for the simplification that f, g and h preserve the orientation of $\mathbb{C}$, we will comment the general case at the end.

For any orientation preserving diffeomorphism Φ of a domain in $\mathbb{C}$ into $\mathbb{C}$ denote the complex dilatation function by ω_Φ . We recall that $\omega_\Phi := \frac{d\Phi}{d\bar{z}} / \frac{d\Phi}{dz}$. (The reader not familiar with the complex dilatation and its properties is advised to read the first 10 pages of the classical Ahlfors book [Ahlfors].) The geometric meaning of the argument of $\omega_\Phi(z)$ may be explained by the equality $\frac{1}{2}\omega_\Phi = \alpha$ where α corresponds to the the direction in which the differential $D\Phi$ at z attains its maximum. In another words it is the direction of the smaller axis of the ellipse in the tangent space at z which is mapped by $D\Phi$ to the unit circle. Of course this makes sense if $\omega(z) \neq 0$. Observe finally that $\omega(z) = 0$ iff $\frac{d\Phi}{d\bar{z}} = 0$. Let go back now to our concrete maps.

If $\frac{dh}{d\bar{z}} \equiv 0$ on X then as $\frac{dh}{d\bar{z}}$ is a real-analytic function we have $\frac{dh}{d\bar{z}} \equiv 0$ on a neighbourhood of X, otherwise (X, f) would be real-analytic. But this means that h is holomorphic what proves our Lemma. It rests to prove that the case $\frac{dh}{d\bar{z}} \not\equiv 0$ on X is impossible.

Observe that if $\frac{dh}{d\bar{z}}(x) = 0$ then $\frac{dh}{d\bar{z}}(f(x)) = 0$ because $h = ghf_\nu^{-1}$ on a neighbourhood of $f(x)$ for the branch f_ν^{-1} of f^{-1} mapping $f(x)$ to x and because g and f_ν^{-1} are conformal. So if there exists $x \in X$ such that $\frac{dh}{d\bar{z}}(x) \neq 0$ then this holds also for all x's from a neighbourhood and as a consequence of the topological exactness of f for all x in a neighbourhood of X. Thus we have a complex-valued function ω_h nowhere zero on a neighbourhood of X.

Recall now that for any two orientation preserving diffeomorphisms Φ and Ψ, if Ψ is holomorphic then

$$\omega_{\Psi\circ\Phi} = \omega_\Phi$$

and if Φ is conformal then

$$\omega_\Psi \circ \Phi = (\frac{\Phi'}{|\Phi'|})^2 \omega_{\Psi\circ\Phi} = \omega_\Phi$$

Applying it to the equation $h \circ f = g \circ h$ we obtain

$$\omega_h \circ f = (\frac{f'}{|f'|})^2 \omega_{h\circ f} = (\frac{f'}{|f'|})^2 \omega_{g\circ h} = (\frac{f'}{|f'|})^2 \omega h.$$

Thus $\alpha(x) := \frac{1}{2}\omega_h(x)$ satisfies the equation (7.1.4) and by Lemma (7.1.6) (X, f) happens linear what contradics our assumption that it is non- linear.

In the case a diffeomorphism reverses the orientation we write everywhere above $\omega_{\bar{\Phi}}$ instead of ω_Φ and if Φ is conformal reversing orientation we write $\overline{\bar{\Phi}'}$ instead of Φ'. Additionally some omegas should be conjugated in the formulas above. We also arive at (7.1.4). (In this situation the complex notation is not confortable. Everythig gets trivial if we act with differentials on line fields. We leave writing this down to the reader.)