CHAPTER 9

CONFORMAL MEASURES

§9.1. GENERAL NOTION OF CONFORMAL MEASURES.

Let $T : X \to X$ be a continuous map of a compact metric space (X, ρ) and let $g : X \to I\!\!R$ be a non–negative measurable function. A Borel probability measure m on X is said to be g–conformal for $T : X \to X$ if

$$m(T(A)) = \int_A g\, dm \tag{9.1.1}$$

for any Borel set $A \subset X$ such that $T|_A$ is injective and $T(A)$ is measurable. Sets with this property will be called special. There is a close relation between conformal measures and Perron-Frobenius type operators. In order to describe it notice first that

$$\int_{T(A)} \phi\, dm = \int_A (\phi \circ T) g\, dm \tag{9.1.2}$$

for any Borel function ϕ and any special set A. Assume now (only till Proposition 9.1.1) additionally that T is bounded–to–one (i.e. the numbers of preimages of points are uniformly bounded) and that g takes values in $I\!\!R_+$. Define then the Perron–Frobenius operator $\mathcal{L}_g$, associated to T and g, putting for a measurable function $\phi : X \to I\!\!R_+$

$$\mathcal{L}_g\phi(x) = \sum_{T(y)=x} \frac{\phi(y)}{g(y)}$$

$\mathcal{L}_g\phi$ is a well defined measurable function. We shall prove the following.

Proposition 9.1.1. Assume that there exists a finite partition of X into special sets X_i $(1 \le i \le s)$, such that all the maps $T : X_i \to T(X_i)$ are measurable isomorphisms. Then m is g–conformal if and only if $\mathcal{L}_g$ acts on $L^1(m)$ and $\mathcal{L}_g^* m = m$, where $\mathcal{L}_g^*$ is the operator conjugate with $\mathcal{L}_g$.

Proof. Let m be g–conformal and let $\phi \in L^1(m)$. By (9.1.2)

$$\int_{T(X_i)} \frac{\phi}{g} \circ \left(T|_{X_i}\right)^{-1} dm = \int_{X_i} \phi\, dm$$

for every $i = 1, \ldots, s$. Thus, summing over all i yields

$$\int_X \mathcal{L}_g \phi \, dm = \int_X \phi \, dm$$

Conversely, assume that $\mathcal{L}_g$ acts on $L^1(m)$ and that m is a fixed point of $\mathcal{L}_g^*$. Let A be a special set. Then, by the definition of the Perron–Frobenius operator

$$\int_A g \, dm = \int_A f \, d(\mathcal{L}_g^* m) = \int_X \mathcal{L}_g(1_A g) \, dm = \int_X \sum_{T(y)=x} 1_A(y) \, m(dx) = m(T(A))$$

Thus m is f–conformal. ♣

Now we shall provide a general method of constructing conformal measures. The construction will make use of the following simple analytical fact. For a sequence $\{a_n : n \geq 1\}$ of reals the number

$$c = \limsup_{n \to \infty} \frac{a_n}{n} \tag{9.1.3}$$

will be called the transition parameter of $\{a_n : n \geq 1\}$. It is uniquely determined by the property that

$$\sum_{n \geq 1} \exp(a_n - ns)$$

converges for $s > c$ and diverges for $s < c$. For $s = c$ the sum may converge or diverge. By a simple argument one obtains the following.

Lemma 9.1.2. There exits a sequence $\{b_n : n \geq 1\}$ of positive reals such that

$$\sum_{n=1}^{\infty} b_n \exp(a_n - ns) \begin{cases} < \infty & s > c \\ = \infty & s \leq c \end{cases}$$

and $\lim_{n \to \infty} \frac{b_n}{b_{n+1}} = 1$.

Proof. If $\sum \exp(a_n - nc) = \infty$, put $b_n = 1$ for every $n \geq 1$. If $\sum \exp(a_n - nc) < \infty$, choose a sequence $\{n_k : k \geq 1\}$ of positive integers such that $\lim_{k \to \infty} n_k n_{k+1}^{-1} = 0$ and $\varepsilon_k := a_{n_k} n_k^{-1} - c \to 0$. Setting

$$b_n = \exp\bigl(n\bigl(\frac{n_k - n}{n_k - n_{k-1}} \varepsilon_{k-1} + \frac{n - n_{k-1}}{n_k - n_{k-1}} \varepsilon_k\bigr)\bigr) \quad \text{ for } n_{k-1} \leq n < n_k,$$

it is easy to check that the lemma follows. ♣

Getting back to dynamics let $\{E_n\}_{n=1}^{\infty}$ be a sequence of finite subsets of X such that

$$T^{-1}(E_n) \subset E_{n+1} \qquad \text{for every} \quad n \geq 1 \tag{9.1.4}$$

and let

$$a_n = \log\big(\sum_{x\in E_n} \exp(S_n g(x))\big)$$

where $S_n g = \sum_{0\le k<n} g\circ T^k$. Denote by c the transition parameter of this sequence. Choose a sequence $\{b_n : n \ge 1\}$ of positive reals as in lemma 9.1.2 for the sequence $\{a_n : n \ge 1\}$. For $s > c$ define

$$M_s = \sum_{n=1}^{\infty} b_n \exp(a_n - ns) \tag{9.1.5}$$

and the normalized measure

$$m_s = \frac{1}{M_s}\sum_{n=1}^{\infty}\sum_{x\in E_n} b_n \exp(S_n g(x) - ns)\delta_x, \tag{9.1.6}$$

where δ_x denotes the unit mass at the point $x \in X$. Let A be a special set. Using (9.1.4) and (9.1.6) it follows that

$$\begin{aligned}
m_s(T(A)) &= \frac{1}{M_s}\sum_{n=1}^{\infty}\sum_{x\in E_n\cap T(A)} b_n \exp(S_n g(x) - ns)\\
&= \frac{1}{M_s}\sum_{n=1}^{\infty}\sum_{x\in A\cap T^{-1}E_n} b_n \exp(S_n g(T(x)) - ns)\\
&= \frac{1}{M_s}\sum_{n=1}^{\infty}\sum_{x\in A\cap E_{n+1}} b_n \exp[S_{n+1} g(x) - (n+1)s]\exp(s - g(x))\\
&\quad - \frac{1}{M_s}\sum_{n=1}^{\infty}\sum_{x\in A\cap(E_{n+1}\setminus T^{-1}E_n)} b_n \exp(S_n g(T(x)) - ns).
\end{aligned} \tag{9.1.7}$$

Set

$$\begin{aligned}
&\Delta_A(s)\\
&= \Big| \frac{1}{M_s}\sum_{n=1}^{\infty}\sum_{x\in A\cap E_{n+1}} b_n \exp[S_{n+1} g(x) - (n+1)s]\exp(s - g(x)) - \int_A \exp(c - g)\, dm_s \Big|
\end{aligned}$$

and observe that

$$\begin{aligned}
&\Delta_A(s) \\
&= \frac{1}{M_s}\Big|\sum_{n=1}^{\infty}\sum_{x\in A\cap E_{n+1}}\exp[S_{n+1}g(x)-(n+1)s]\exp(-g(x))\big[b_ne^s-b_{n+1}e^c\big] \\
&\quad - b_1\sum_{x\in A\cap E_1}e^{c-s}\Big| \\
&\le \frac{1}{M_s}\sum_{n=1}^{\infty}\sum_{x\in A\cap E_{n+1}}\Big|\frac{b_n}{b_{n+1}}-e^{c-s}\Big|\,b_{n+1}\exp(s-g(x))\exp[S_{n+1}g(x)-(n+1)s] \\
&\quad + \frac{1}{M_s}b_1\exp(c-s)\,\sharp(A\cap E_1) \\
&\le \frac{1}{M_s}\sum_{n=1}^{\infty}\sum_{x\in E_{n+1}}\Big|\frac{b_n}{b_{n+1}}-e^{c-s}\Big|\,b_{n+1}\exp(s-g(x))\exp[S_{n+1}g(x)-(n+1)s] \\
&\quad + \frac{1}{M_s}b_1\exp(c-s)\,\sharp E_1.
\end{aligned}$$

By lemma 9.1.2 we have $\lim_{n\to\infty} b_{n+1}/b_n = 1$ and $\lim_{s\downarrow c} M_s = \infty$. Therefore

$$\lim_{s\downarrow c}\Delta_A(s) = 0 \tag{9.1.8}$$

uniformly for all special sets A.

Any weak accumulation point, when $s\downarrow c$, of the measures $\{m_s : s > c\}$ defined by (9.1.6) will be called a limit measure (associated to the function g and the sequence $\{E_n : n \ge 1\}$).

In order to find conformal measures among the limit measures, it is necessary to examine (9.1.7) in greater detail. To beginn with, for a Borel set $D \subset X$, consider the following condition

$$\lim_{s\downarrow c}\frac{1}{M_s}\sum_{n=1}^{\infty}\sum_{x\in D\cap(E_{n+1}\setminus T^{-1}E_n)} b_n\exp[S_ng(T(x))-ns] = 0. \tag{9.1.9}$$

We will need the following definitions.

A point $x \in X$ is said to be singular for T if at least one of the following two conditions is satisfied:

(9.1.10) There is no open neighbourhood U of x such that $T|_U$ is injective.

(9.1.11) $\forall_{\varepsilon>0}\exists_{0<r<\varepsilon} T(B(x,r))$ is not an open subset of X.

The set of all singular points is denoted by $\mathrm{Sing}(T)$, the set of all points satisfying condition (9.1.10) is denoted by $\mathrm{Crit}(T)$ and the set of all points satisfying condition (9.1.11) is denoted by $X_0(T)$.

It is easy to give examples where $X_0 \cap \mathrm{Crit}(T) \neq \emptyset$. If $T : X \to X$ is an open map, no point satisfies condition (9.1.11) that is $X_0(T) = \emptyset$.

In spite of what was assumed in [ECM] and similarly as in [Sul] the set $\mathrm{Sing}(T)$ is not required to be finite. Let us prove the following.

Lemma 9.1.3. Let m be a Borel probability measure on X and let Γ be a compact set containing $\mathrm{Sing}(T)$. If (9.1.1) holds for every special set A whose closure is disjoint from Γ and such that $m(\partial A) = m(\partial T(A)) = 0$, then (9.1.1) continues to hold for every special set A disjoint from Γ.

Proof. Let A be a special set disjoint from Γ. Fix $\varepsilon > 0$. Since on the complement of Γ the map T is open, for each point $x \in A$ there exists an open neighbourhood $U(x)$ of x such that $T|_{U(x)}$ is a homeomorphism, $m(\partial U(x)) = m(\partial T(U(x))) = 0$, $\overline{U(x)} \cap \Gamma = \emptyset$ and such that

$$\int_{\cup U(x) \setminus A} g \, dm < \varepsilon$$

Choose a countable family $\{U_k\}$ from $\{U(x)\}$ which covers A and define recursively $A_1 = U_1$ and $A_n = U_n \setminus \bigcup_{k<n} U_k$. By the assumption of the lemma, each set A_k satisfies (9.1.1) and hence

$$\begin{aligned} m(T(A)) &= m\Big(\bigcup_{k=1}^{\infty} T(A \cap A_k)\Big) \leq \sum_{k=1}^{\infty} m(T(A_k)) \\ &= \sum_{k=1}^{\infty} \int_{A_k} g \, dm = \int_A g \, dm + \sum_{k=1}^{\infty} \int_{A_k \setminus A} g \, dm \\ &\leq \int_A g \, dm + \varepsilon. \end{aligned}$$

If $\varepsilon \to 0$, it follows that

$$m(T(B)) \leq \int_B g \, dm$$

for any special set B disjoint from Γ. Using this fact, the lower bound for $m(T(A))$ is obtained from the following estimate, if $\varepsilon \to 0$:

$$\begin{aligned} m(T(A)) &= m\Big(\bigcup_{k=1}^{\infty} T(A \cap A_k)\Big) = \sum_{k=1}^{\infty} m(T(A \cap A_k)) \\ &= \sum_{k=1}^{\infty} \big(m(T(A_k)) - m(T(A_k \setminus A))\big) \geq \sum_{k=1}^{\infty} \int_{A_k} g \, dm - \int_{A_k \setminus A} g \, dm \\ &= \int_{\cup_{k \geq 1} A_k} g \, dm - \int_{\cup_{k \geq 1} A_k \setminus A} g \, dm \geq \int_A g \, dm - \varepsilon. \end{aligned}$$

This proves the lemma. ♣

Lemma 9.1.4. Let m be a limit measure and let Γ be a compact set containing $\mathrm{Sing}(T)$. Assume that every special set $D \subset X$ with $m(\partial D) = m(\partial T(D)) = 0$ and $\bar{D} \cap \Gamma = \emptyset$ satisfies condition (9.1.9). Then $m(T(A)) = \int_A \exp(c-f)\,dm$ for every special set A disjoint from Γ.

Proof. Let $D \subset X$ be a special set such that $\bar{D} \cap \Gamma = \emptyset$ and $m(\partial D) = m(\partial T(D)) = 0$. It follows immediately from (9.1.7)–(9.1.9) that $m(T(D)) = \int_D \exp(c-f)\,dm$. Applying now Lemma 9.1.3 completes the proof. ♣

Lemma 9.1.5 Let m be a limit measure. If condition (9.1.9) is satisfied for $D = X$, then $m(T(A)) \geq \int_A \exp(c-f)\,dm$ for every special set A disjoint from $\mathrm{Crit}(T)$.

Proof. Suppose first that A is compact and $m(\partial A) = 0$. From (9.1.7), (9.1.8) and the assumption one obtains

$$\lim_{s \in J} \mid m_s(T(A)) - \int_A \exp(c-f)\,dm_s \mid = 0$$

where J denotes the subsequence along which m_s converges to m. Since $T(A)$ is compact, this implies

$$m(T(A)) \geq \liminf_{s \in J} m_s(T(A)) = \lim_{s \in J} \int_A \exp(c-f)\,dm_s = \int_A \exp(c-f)\,dm$$

Now, drop the assumption $m(\partial A) = 0$ but keep A compact and assume additionally that for some $\varepsilon > 0$ the ball $B(A, \varepsilon)$ is also special. Choose a descending sequence A_n of compact subsets of $B(A, \varepsilon)$ whose intersection equals A and $m(\partial A_n) = 0$ for every $n \geq 0$. By what has been already proved

$$m(T(A)) = \lim_{n \to \infty} m(T(A_n)) \geq \int_{A_n} \exp(c-f)\,dm = \int_A \exp(c-f)\,dml$$

The next step is to prove the lemma for A, an arbitrary open special set disjoint from $\mathrm{Crit}(T)$ by partitioning it by countably many compact sets. Then one approximates from above special sets of sufficiently small diameters by special open sets and the last step is to partition an arbitrary special set disjoint from $\mathrm{Crit}(T)$ by sets of so small diameters that the lemma holds. ♣

Lemma 9.1.6 Let Γ be a compact subset of X containing $\mathrm{Sing}(T)$. Suppose that for every integer $n \geq 1$ there are a continuous function $g_n : X \to X$ and a measure m_n on X satisfying (9.1.1) for $g = g_n$ and for every special set $A \subset X$ with

(a) $$\overline{A} \cap \Gamma = \emptyset$$

and satisfying

$$m_n(B) \geq \int_B g_n\,dm_n$$

for any special set $B \subset X$ such that $B \cap \text{Crit}(T) = \emptyset$.

Suppose, moreover, that the sequence $\{g_n\}_{n=1}^{\infty}$ converges uniformly to a continuous function $g : X \to I\!R$. Then for any weak accumulation point m of the sequence $\{m_n\}_{n=1}^{\infty}$ we have

$$\text{(b)} \qquad m(T(A)) = \int_A g\, dm$$

for all special sets $A \subset X$ such that $A \cap \Gamma = \emptyset$ and

$$\text{(c)} \qquad m(T(B)) \geq \int_B g\, dm$$

for all special sets $B \subset X$ such that $B \cap \text{Crit}(T) = \emptyset$.

Moreover, if (a) is replaced by

$$\text{(a')} \qquad \overline{A} \cap (\Gamma \setminus (\text{Crit}(T) \setminus X_0(T))) = \emptyset,$$

then for any $x \in \text{Crit}(T) \setminus X_0(T)$

$$\text{(d)} \qquad m(\{T(x)\}) \leq g(x) m(\{x\}) \leq q(x) m(\{T(x)\})$$

where $q(x)$ denotes the maximal number of preimages of single points under the transformation T restricted to a sufficiently small neighbourhood of x.

The proof of property (b) is a simplification of the proof of Lemma 9.1.4 and the proof of property (c) is a simplification of the proof of Lemma 9.1.5. The proof of (d) uses the same technics and is left for the reader.

§9.2. SULLIVAN'S CONFORMAL MEASURES AND DYNAMICAL DIMENSION, I.

Let, as in Chapter 8, X denote a compact subset of the extended complex plane $\overline{\mathbb{C}}$ and let $f \in \mathcal{A}(X)$ which means that $f : X \to X$ is a continuous map that can be analytically extended to an open neighbourhood $U(f)$ of X.

Let $t \geq 0$. Any $|f'|^t$–conformal measure for $f : X \to X$ is called a t–conformal Sullivan's measure or even shorter a t–conformal measure. Rewritting the defintion (9.1.1) it means that

$$\text{(9.2.1)} \qquad m(f(A)) = \int_A |f'|^t\, dm$$

for every special set $A \subset X$. An obvious but important property of conformal measures is formulated in the following

Lemma 9.2.1. If $f : X \to X$ is locally eventually onto, then every Sullivan's conformal measure is positive on nonempty open sets of X.

In particular it follows from this lemma that if f is locally eventually onto, then for every $r > 0$

$$M(r) = \inf\{m(B(x,r)) : x \in X\} > 0 \tag{9.2.2}$$

Denote by $\delta(f)$ the infinium over all exponents $t \geq 0$ for which a t–conformal measure for $f : X \to X$ exists.

Our aim in the two subsequent sections is to show the existence of conformal measures and even more to establish more explicite dynamical characterization of the number $\delta(f)$. As a matter of fact we are going to prove that under some additional assumptions $\delta(f)$ consides with the dynamical dimension $\mathrm{DD}(X)$ of X, the hyperbolic dimension $\mathrm{HyD}(X)$ of X, and the conformal dimension $\mathrm{CD}(X)$ of X which are defined as follows.

$$\mathrm{DD}(X) = \sup\{\mathrm{HD}(\mu) : \mu \in M_e^+(f)\}$$
$$\mathrm{HyD}(X) = \sup\{\mathrm{HD}(Y) : f(Y) \subset Y,\ \ \overline{Y} = Y, \text{ and } f|_Y \text{ is an open expanding map}\}$$
$$\mathrm{CD}(X) = \sup\{\mathrm{HD}(Y) : f|_Y \text{ is a conformal expanding repeller}\}$$

In this section we shall prove the following two results.

Lemma 9.2.2. If $f : X \to X$ is locally eventually onto, then $\mathrm{DD}(X) \leq \delta(f)$.

Proof. Our main idea "to get to a large scale" is the same as in [SulDU]. However to carry it out we use Pesin theory described in Ch.8.2 instead of Mane's partition introduced in [Mane] and applied in [SulDU]. So, let $\mu \in M_e^+(f)$ and let m be a t–conformal measure. We again work in the natural extension $(\tilde{X}, \tilde{f}, \tilde{\mu})$. Fix $\varepsilon > 0$ and let $\tilde{X}(\varepsilon)$ and $r(\varepsilon)$ be given by Corollary 8.2.4. In view of the Birkhoff ergodic theorem there exist a measurable set $\tilde{F}(\varepsilon) \subset \tilde{X}(\varepsilon)$ such that $\tilde{\mu}(\tilde{F}(\varepsilon)) = \tilde{\mu}(\tilde{X}(\varepsilon))$ and an increasing sequence $\{n_k = n_k(\tilde{x}) : k \geq 1\}$ such that $\tilde{f}^{n_k}(\tilde{x}) \in \tilde{X}(\varepsilon)$ for every $k \geq 1$. Let $F(\varepsilon) = \pi(\tilde{F}(\varepsilon))$. Then $\mu(F(\varepsilon)) = \tilde{\mu}(\pi^{-1}(F(\varepsilon)) \geq \tilde{\mu}(\tilde{F}(\varepsilon)) \geq 1 - 2\varepsilon$. Consider now $x \in F(\varepsilon)$ and take $\tilde{x} \in \tilde{F}(\varepsilon)$ such that $x = \pi(\tilde{x})$. Since $\tilde{f}^{n_k}(\tilde{x}) \in \tilde{X}(\varepsilon)$ and since $\pi(\tilde{f}^{n_k}(\tilde{x}) = f^{n_k}(x)$, Corollary 8.2.4 produces a holomorphic inverse branch $f_x^{-n_k} : B(f^{n_k}(x), r(\varepsilon)) \to \overline{\mathbb{C}}$ of f^{n_k} such that $f_x^{-n_k} f^{n_k}(x) = x$ and

$$f_x^{-n_k}\big(B(f^{n_k}(x), r(\varepsilon))\big) \subset B\big(x, K|(f^{n_k})'(x)|^{-1} r(\varepsilon)\big)$$

Set $r_k(x) = K|(f^{n_k})'(x)|^{-1} r(\varepsilon)$. Then by Corollary 8.2.4 and t– conformality of m

$$m(B(x, r_k(x))) \geq K^{-t}|(f^{n_k})'(x)|^{-t} m\big(B(f^{n_k}(x), r(\varepsilon))\big) \geq M(r(\varepsilon))^{-1} K^{-2t} r(\varepsilon)^{-t} r_k(x)^t$$

Therefore, it follows from Theorem 5.5.1 (Besicovitch covering theorem) that $\mathrm{H}_t(F(\varepsilon)) \leq M(r(\varepsilon)) K^{2t} r(\varepsilon)^t b(2) < \infty$. Hence $\mathrm{HD}(F(\varepsilon)) \leq t$. Since $\mu\big(\bigcup_{n=1}^\infty F(1/n)\big) = 1$, it implies that $\mathrm{HD}(\mu) \leq t$. This finishes the proof. ♣

Theorem 9.2.3. If $f : X \to X$ is locally eventually onto and X is a repelling set for f, then $\mathrm{CD}(X) = \mathrm{HyD}(X) = \mathrm{DD}(X)$.

Proof. Obviously $\mathrm{CD}(X) \le \mathrm{HyD}(X)$. In order to see that $\mathrm{HyD}(X) \le \mathrm{DD}(X)$ notice only that in our setting every set Y appearing in the definition of hyperbolic dimension is a conformal repeller and therefore in view of ??? there exists $\mu \in M_e^+(f|_Y) \subset M_e^+(f)$ such that $\mathrm{HD}(\mu) = \mathrm{HD}(Y)$. In order to prove that $\mathrm{DD}(X) \le \mathrm{CD}(X)$ we will use Katok's theory from Section 8.6 applied to μ, an arbitrary ergodic invariant measure of positive entropy. First, for every integer $n \ge 0$ define on X a new continuous function

$$\phi_n = \max\{-n, \log|f'|\}, \quad \phi_n \ge \log|f'|$$

and note that $\phi_n \searrow \log|f'|$ pointwise on X. Since

$$\phi_n \le \max\{0, \log|f'|\} \text{ and } \int \max\{0, \log|f'|\} < \infty,$$

it follows from the Lebesgue dominated convergence theorem that $\lim_{n\to\infty} \int \phi_n \, d\mu = \chi_\mu(f) = \int \log|f'| \, d\mu > 0$. Fix $\varepsilon > 0$. Then for all n sufficiently large, say $n \ge n_0$, $\int \phi_n d\mu \le \chi_\mu/(1-\varepsilon)$ which implies that

$$\mathrm{h}_\mu(f) = \mathrm{HD}(\mu)\chi_\mu \ge (1-\varepsilon)\mathrm{HD}(\mu) \int \phi_n \, d\mu. \tag{9.2.4}$$

Fix such $n \ge n_0$. Let $X_k \subset X$, $k \ge 0$, be the sequence produced in Theorem 8.6.1 for the measure μ and the function $-\mathrm{HD}(\mu)\phi_n$ and let μ_k be an equilibrium state of the map $f|_{X_k}$ and the potential $-\mathrm{HD}(\mu)\phi_n$ restricted to X_k. It follows from the second part of Theorem 8.6.1 that $\lim_{k\to\infty} \int \phi_n \, d\mu_k = \int \phi_n \, d\mu > 0$. Thus by Theorem 9.6.1 and (9.2.4)

$$\begin{aligned}
\liminf_{k\to\infty} \big(\mathrm{h}_{\mu_k} - \mathrm{HD}(\mu) \int \phi_n \, d\mu_k\big) &= \liminf_{k\to\infty} \mathrm{P}\big(f|_{X_k}, -\mathrm{HD}(\mu)\phi_n\big) \\
&\ge \mathrm{h}_\mu(f)\mathrm{HD}(\mu) \int \phi_n \, d\mu \\
&\ge -\mathrm{HD}(\mu) \int \phi_n \, d\mu
\end{aligned}$$

Hence, for all k large enough

$$\begin{aligned}
\mathrm{h}_{\mu_k} &\ge \mathrm{HD}(\mu) \int \phi_n \, d\mu_k - 2\varepsilon \mathrm{HD}(\mu) \int \phi_n \, d\mu \ge \mathrm{HD}(\mu) \int \phi_n \, d\mu_k - 3\varepsilon \mathrm{HD}(\mu) \int \phi_n \, d\mu_k \\
&= (1-3\varepsilon)\mathrm{HD}(\mu) \int \phi_n \, d\mu_k \ge (1-3\varepsilon)\mathrm{HD}(\mu) \int \log|f'| \, d\mu_k.
\end{aligned}$$

Thus

$$\mathrm{HD}(X_k) \ge \mathrm{HD}(\mu_k) = \frac{\mathrm{h}_{\mu_k}(f)}{\chi_{\mu_k}} \ge (1-3\varepsilon)\mathrm{HD}(\mu).$$

So, letting $\varepsilon \to 0$ finishes the proof. ♣

In this section $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is assumed to be a rational map of degree ≥ 2 and X is its Julia set $J(f)$. Neverthless it is worth to mention that some results proved here continue to hold under weaker assumption that $f|_X$ is open or X is a perfect locally maximal set for f.

Lemma 9.3.1. If $z\in J(f)$ and $\overline{\{f^n(z):n\ge 0\}}\cap \mathrm{Crit}(f)=\emptyset$, then the series $\sum_{n=1}^\infty |(f^n)'(z)|^{\frac{1}{3}}$ diverges.

Proof. By the assumption there exists $\varepsilon>0$ such that for every $n\ge 0$ the map f restricted to the ball $B(f^n(z),\varepsilon)$ is injective. Since f is uniformly continuous there exists $0<\alpha<1$ such that for every $x\in\overline{\mathbb{C}}$

$$f(B(x,\alpha\varepsilon))\subset B(f(x),\varepsilon). \tag{9.3.1}$$

Suppose that the series $\sum_{n=1}^\infty |(f^n)'(z)|^{\frac{1}{3}}$ converges. Then there exists $n_0\ge 1$ such that $\sup_{n\ge n_0}(2|(f^n)'(z)|)^{\frac{1}{3}}<1$. Choose $0<\varepsilon_1=\varepsilon_2=\ldots=\varepsilon_{n_o}<\alpha\varepsilon$ so small that for every $n=1,2,\ldots,n_0$

$$f^n \text{ restricted to the ball } B(z,\varepsilon_n) \text{ is injective.} \tag{9.3.2}$$

and

$$f^n(B(z,\varepsilon_n))\subset B(f^n(z),\varepsilon) \tag{9.3.3}$$

For every $n\ge n_0$ define ε_{n+1} inductively by

$$\varepsilon_{n+1}=(1-(2|(f^n)'(z)|)^{\frac{1}{3}})\varepsilon_n. \tag{9.3.4}$$

Then $0<\varepsilon_n<\alpha\varepsilon$ for every $n\ge 1$.

Assume that (9.3.2) and (9.3.3) are satisfied for some $n\ge n_0$. Then by the Köbe Distortion Theorem ??? and (9.3.4) the set $f^n(B(z,\varepsilon_{n+1}))$ is contained in the ball centered at $f^n(z)$ and of radius

$$\varepsilon_{n+1}|(f^n)'(z)|\frac{2}{(1-\varepsilon_{n+1}/\varepsilon_n)^3}=\frac{2\varepsilon_{n+1}|(f^n)'(z)|}{2|(f^n)'(z)|}=\varepsilon_{n+1}<\alpha\varepsilon.$$

Therefore, since f is injective on $B(f^n(z),\varepsilon)$, formula (9.3.2) is satisfied for $n+1$ and using also (9.3.1) we get

$$f^{n+1}(B(z,\varepsilon_{n+1}))=f\big(f^n(B(z,\varepsilon_{n+1}))\big)\subset f(B(f^n(z),\alpha\varepsilon))\subset B(f^{n+1}(z),\varepsilon).$$

Thus (9.3.3) is satisfied for $n+1$.

Let $\varepsilon_n \downarrow \varepsilon_0$. Since the series $\sum_{n=1}^{\infty} |(f^n)'(z)|^{\frac{1}{3}}$ converges, it follows from (9.3.4) that $\varepsilon_0 > 0$. Clearly (9.3.2) and (9.3.3) remain true with ε_n replaced by ε_0. It follows that the family $\{f^n|_{B(z,\frac{1}{2}\varepsilon_0)}\}_{n=1}^{\infty}$ is normal and consequently $z \notin J(f)$. This contradiction finishes the proof. ♣

As an immediate consequence of this lemma and of Birkhoff's Ergodic Theorem we get the following.

Corollary 9.3.2. If μ be an ergodic f–invariant measure for which there exists a compact set $Y \subset J(f)$ such that $\mu(Y) = 1$ and $Y \cap Crit(f) = \emptyset$, then $\chi_\mu \geq 0$.

Let now Ω be a finite subset of $\bigcup_{n=1}^{\infty} f^n(\mathrm{Crit}(f))$ such that

$$\Omega \cap \{f^n(c) : n = 1, 2 \ldots\} \neq \emptyset \qquad \text{for every } c \in \mathrm{Crit}(f) \tag{9.3.5}$$

and

$$\Omega \cap \mathrm{Crit}(f) = \emptyset. \tag{9.3.6}$$

Sets satisfying these conditions exist since no critical point of f lying in $J(f)$ can be periodic. Now let $V \subset J(f)$ be an open neighbourhood of Ω and define $K(V)$ to be the set of those points of $J(f)$ whose forward trajectory avoids V. Equivalently this means that

$$K(V) = \{z \in J(f) : f^n(z) \notin V \text{ for every } n \geq 0\} = \bigcap_{n=0}^{\infty} f^{-n}(J(f) \setminus V)$$

Hence $K(V)$ is a compact subset of $J(f)$ and $f(K(V)) \subset K(V)$. Consequently we can consider dynamical system $f|_{K(V)} : K(V) \to K(V)$. Note that $f(K(V)) = K(V)$ does not hold for all sets V and that usually $f^{-1}(K(V)) \not\subset K(V)$. Simple considerations based on (9.3.5) and the definition of sets $K(V)$ give the following.

Lemma 9.3.3. $\mathrm{Crit}(f|_{K(V)}) \subset \mathrm{Crit}(f) \cap K(V) = \emptyset$, $K(V)_0(f) = \mathrm{Sing}(f) \subset \partial V$, and $-t \log |f'|$ is a well–defined continuous function on $K(V)$.

Fix now $z \in K(V)$ and set $E_n = f|_{K(V)}^{-n}(z)$, $n \geq 0$. Then $E_{n+1} = f|_{K(V)}^{-1}(E_n)$ and therefore the sequence $\{E_n\}$ satisfies (9.1.9) with $D = K(V)$. Take $t \geq 0$ and let $c(t, V)$ be the transition parameter associated to this sequence and the function $-t \log |f'|$. Put $\mathrm{P}(t, V) = \mathrm{P}(f|_{K(V)}, -t \log |f'|)$. We shall prove the following.

Lemma 9.3.4. $c(t, V) \leq \mathrm{P}(t, V)$.

Proof. Fix $\theta > 0$. Let $\varepsilon_0 > 0$ so small as required in Lemma 3 of [PFR - Przyt] ???. By the definition of topological pressure (see ???) one can find an integer $n_0 \geq 1$ such that

$$\sum_{x \in F_n(\varepsilon)} \exp(S_n(-t \log |f'|)(x)) \leq \exp((\mathrm{P}(t, V) + \theta)n)$$

for every $n \geq n_0$ and every (n, ε)–separated set $F_n(\varepsilon)$ of $K(V)$. By [PFR - Przyt] ???, the set E_n can be divided into (n, ε)–separated sets whose number does not exceed $C\exp(\theta n)$. Therefore

$$\sum_{x \in E_n} \exp(S_n(-t\log|f'|)(x)) \leq C\exp(\theta n) \cdot \exp((\mathrm{P}(t, V) + \theta)n)$$

for every $n \geq n_0$. Thus, from the definition of the transition parameter we conclude that $c(t, V) \leq \mathrm{P}(t, V) + 2\theta$. Letting $\theta \searrow 0$ we finally obtain $c(t, V) \leq \mathrm{P}(t, V)$, proving the lemma. ♣

The standard straightforward arguments showing continuity of topological pressure prove also the following.

Lemma 9.3.5. The function $t \mapsto c(t, V)$ is continuous.

Set

$$s(V) = \inf\{t \geq 0 : c(t, V) \leq 0\} < +\infty$$

We shall prove the following.

Lemma 9.3.6. $s(V) \leq \mathrm{DD}(J(f))$.

Proof. Suppose that $\mathrm{DD}(J(f)) < s(V)$ and take $0 \leq \mathrm{DD}(J(f)) < t < s(V)$. From this choice and by Lemma 9.3.4 we have $0 < c(t, V) \leq \mathrm{P}(t, V)$ and by Variational Principle ??? there exists $\mu \in M_e(f_{K(V)}) \subset M_e(f)$ such that $\mathrm{P}(t, V) \leq \mathrm{h}_\mu(f) - t\chi_\mu(f) + c(t, V)/2$. Therefore, by Corollary 9.3.2 and Lemma 9.3.3 we get $\mathrm{h}_\mu(f) \geq c(t, V)/2 > 0$ and applying additionally Theorem 8.1.1 (Ruelle's inequality), $\chi_\mu(f) > 0$. Hence, it follows from Theorem 8.4.1 that

$$t \leq \mathrm{HD}(\mu) - \frac{1}{2}\frac{c(t, V)}{\chi_\mu} < \mathrm{HD}(\mu) \leq \mathrm{DD}(J(f))$$

This contradiction finishes the proof. ♣

Let m be a limit measure on $K(V)$ associated to the sequence E_n and the function $-s(V)\log|f'|$. Since $c(0, V) \geq 0$ and $s(V) < \infty$, it follows from Lemma 9.3.5 that $c(s(V), V) = 0$. Therefore, applying Lemma 9.1.4 and Lemma 9.1.5 with $\Gamma = \partial V$ we see that $m(f(A)) \geq \int_A |f'|^{s(V)}\,dm$ for any special set $A \subset K(V)$ and $m(f(A)) = \int_A |f'|^{s(V)}\,dm$ for any special set $A \subset K(V)$ such that $A \cap \partial V = \emptyset$. Treating now m as a measure on $J(T)$ and using straightforward measure–theoretic arguments we deduce from this that

$$m(f(A)) \geq \int_A |f'|^{s(V)}\,dm \tag{9.3.7}$$

for any special set $A \subset J(f)$ and

$$m(f(A)) = \int_A |f'|^{s(V)}\,dm \tag{9.3.8}$$

for any special set $A \subset J(f)$ such that $A \cap \bar{V} = \emptyset$. Now we are in position to prove the following.

Lemma 9.3.7. For every Ω there exist $0 \le s(\Omega) \le \mathrm{DD}(J(f))$ and a Borel probability measure m on $J(f)$ such that

$$m(f(A)) \ge \int_A |f'|^{s(\Omega)}\,dm$$

for any special set $A \subset J(f)$ and

$$m(f(A)) = \int_A |f'|^{s(\Omega)}\,dm$$

for any special set $A \subset J(f)$ disjoint from Ω.

Proof. For every $n \ge 1$ let $V_n = B(\Omega, \frac{1}{n})$ and let m_n be the measure on $J(f)$ satisfying (9.3.7) and (9.3.8) for the neighbourhood V_n. Using Lemma 9.1.6 we shall show that any weak–* limit m of the sequence of measures $\{m_n\}_{n=1}^\infty$ satisfies the requirements of Lemma 9.3.7. Indeed, first observe that the sequence $\{s(V_n)\}_{n=1}^\infty$ is nondecreasing and denote its limit by $s(\Omega)$. Therefore the sequence of continuous functions $g_n = |f'|^{s(V_n)}$, $n = 1, 2, \ldots$, defined on $J(f)$ converges uniformly to the continuous function $g = |f'|^{s(\Omega)}$. Let A be a special subset of $J(f)$ such that

$$\overline{A} \cap (\mathrm{Sing}(f) \cup \Omega) = \emptyset. \tag{9.3.9}$$

Then one can find a compact set $\Gamma \subset J(f)$ disjoint from $\overline{A}$ and such that $\mathrm{Int}(\Gamma) \supset \mathrm{Sing}(f) \cup \Omega$. So, using also Lemma 9.3.3, we see that for any n sufficiently large, say $n \ge q$,

$$\overline{V_n} \subset \Gamma \qquad \text{and} \qquad \overline{V_n} \cap \mathrm{Crit}(f) = \emptyset. \tag{9.3.10}$$

Therefore, by (9.3.7) and (9.3.8), we conclude that Lemma 9.1.6 applies to the sequence of measures $\{m_n\}_{n=q}^\infty$ and the sequence of functions $\{g_n\}_{n=q}^\infty$. Hence, the first property required in our lemma is satisfied for any special subset of $J(f)$ disjoint from $\mathrm{Crit}(f)$ and since $A \cap \Gamma = \emptyset$, the second property is satisfied for the set A. So, since any special subset of $J(f)$ disjoint from $\mathrm{Sing}(f) \cup \Omega$ can be expressed as a disjoint union of special sets satisfying (9.3.9), an easy computation shows that the second property is satisfied for all special sets disjoint from $\mathrm{Sing}(f) \cup \Omega$. Therefore, in order to finish the proof, it is enough to show that the second requirement of the lemma is satisfied for every point of the set $\mathrm{Sing}(f)$. First note that by (9.3.10) and (9.3.8), formula (a') in Lemma 9.1.6 is satisfied for every $n \ge q$ and every $x \in \mathrm{Crit}(f) \setminus J(f)_0(f)$. As $f : J(f) \to J(f)$ is an open map, the set $J(f)_0(f)$ is empty and $\mathrm{Sing}(f) = \mathrm{Crit}(f)$. Consequently formula (d) of Lemma 9.1.6 is satisfied for any critical point $c \in J(f)$ of f. Since $g(c) = |f'(c)|^{s(\Omega)} = 0$, this formula implies that $m(f()) \le 0$. Thus $m(\{f(c)\}) = 0 = |f'(c)|^{s(\Omega)} m(\{c\})$. The proof is finished. ♣

In order to simplify our exposition, given $t \geq 0$ and a finite set $F \subset J(f)$ let us call any measure satisfying requirements of Lemma 9.3.7 with $s(\Omega)$ replaced by t and Ω replaced by F, (t, F)-conformal.

Lemma 9.3.8. If a finite subset G of $\overline{\bigcup_{n=1}^{\infty} f^n(\mathrm{Crit}(f)) \setminus \mathrm{Crit}(f)}$ has a nonempty intersection with the set $\overline{\{f^n(x) : n \geq 0\}}$ for every $x \in \mathrm{Crit}(f)$, then there exists an (s, G)–conformal measure m for $f : J(f) \to J(f)$ with $0 \leq s \leq \mathrm{DD}(J(f))$.

Proof. For $x \in \mathrm{Crit}(f)$ choose a point $y(x) \in G \bigcap \overline{\{f^n(x) : n \geq 0\}}$ and let $n_j = n_j(x)$, $j \geq 1$ be a sequence of positive integers (not necessarily tending to ∞) such that $y(x) = \lim_{j\to\infty} f^{n_j}(x)$ and $f^{n_j}(x) \notin \mathrm{Crit}(f)$ for every $j \geq 1$. For every $j \geq 1$ define the set $\Omega_j = \{f^{n_j}(x) : x \in \mathrm{Crit}(f)\}$ and note that it satisfies conditions (9.3.5) and (9.3.6). Let m_j be the corresponding measure produced in Lemma 9.3.7.

Now we proceed similarly as in the proof of Lemma 9.3.7. Passing to a subsequence we can suppose that the sequence $\{s(\Omega_j)\}_{j=1}^{\infty}$ converges and let $A \subset J(f)$ be a special set such that $\overline{A} \cap (G \cup \mathrm{Sing}(f)) = \emptyset$. Then one can find a compact subset Γ of $J(f)$ disjoint from $\overline{A}$ and such that $\mathrm{Int}(\Gamma) \supset G \cup \mathrm{Sing}(f)$. Therefore, it follows from this and the definition of $y(x)$ that for every j large enough $\Omega_j \subset \Gamma$. Hence Lemma 9.1.6 applies. Since also $\Omega_j \cap \mathrm{Crit}(f) = \emptyset$ for every $j \geq 1$, the proof is finished in the same way as in Lemma 9.3.7 ♣

Lemma 9.3.9. Let m be a $(t, \cdot)$–conformal measure for $f : J(f) \to J(f)$. If for some $z \in J(f)$ the series $S(t, z) = \sum_{n=1}^{\infty} |(f^n)'(z)|^t$ diverges then $m(\{z\}) = 0$ or a positive iteration of z is a parabolic point of f. Moreover, if z itself is periodic then $m(\{f(z)\}) = |f'(z)|^t m(\{z\})$.

Proof. Suppose that $m(\{z\}) > 0$. Assume first that the point z is not eventually periodic. fhen by the definition of a conformal measure on the complement of some finite set we get $1 \geq m(\{f^n(z) : n \geq 1\}) \geq m(\{z\}) \sum_{n=1}^{\infty} |(f^n)'(z)|^t = \infty$, which is a contradiction. Hence z is eventually periodic and therefore there exist positive integers k and q such that $f^k(f^q(z)) = f^q(z)$. Since $f^q(z) \in J(f)$ and since the family of of all iterates of f on a sufficiently small neighbourhood of an attractive periodic point is normal, this implies that $|(f^k)'(f^q(z))| \geq 1$. If $|(f^k)'(f^q(z))| = \lambda > 1$ then, again by the definition of a conformal measure on the complement of some finite set, $m(\{f^q(z)\}) > 0$ and $m(\{f^{kn}(f^q(z))\}) \geq \lambda^{nt} m(\{f^q(z)\})$. Thus $m(\{f^{kn}(f^q(z))\})$ converges to ∞, which is a contradiction. Therefore $|(f^k)'(f^q(z))| = 1$ which finishes the proof of the first assertion of the lemma. In order to prove the second assertion assume that $q = 1$. Then, using the definition of conformal measures on the complement of some finite set again, we get $m(\{f(z)\}) \geq m(\{z\})|f'(z)|^t$ and on the other hand

$$m(\{z\}) = m(\{f^{k-1}(f(z))\}) \geq m(\{f(z)\})|(f^{k-1})'(f(z))|^t = m(\{f(z)\})|f'(z)|^{-t}.$$

Therefore $m(\{f(z)\}) = m(\{z\})|f'(z)|^t$. The proof is finished. ♣

Corollary 9.3.10. If for every $x \in \mathrm{Crit}(f)$ one can find $y(x) \in \overline{\{f^n(x) : n \geq 0\}}$ such that the series $S(t, y(x))$ diverges for every $0 \leq t \leq \mathrm{DD}(J(f))$, then there exists an s–conformal measure for $f : J(f) \to J(f)$ with $0 \leq s \leq \mathrm{DD}(J(f))$.

Proof. If for $t > 0$ and $z \in J(f)$ the series $S(t, z)$ diverges then $z \notin \mathrm{Crit}(f)$. Therefore the set $G = \{y(x) : x \in \mathrm{Crit}(f)\}$ satisfies the assumtions of Lemma 9.3.8. Denote by m the (s, G)–conformal measure ($0 \le s \le \mathrm{DD}(J(f))$) produced in this lemma. Suppose first that G does not contain a non–periodic point eventually falling into a parabolic point. Since for every $t \ge 0$ and $z \in J(f)$ the divergence of the series $S(t, z)$ implies the divergence of the series $S(t, f(z))$, it follows immediately from Lemma 9.3.9 that the measure m is s-conformal.

In the general case replace every non–periodic eventually parabolic point in G by its iteration which is already parabolic and denote the set obtained in this way by G^*. Observe that G^* satisfies all the assumptions required in the first part of the proof. Therefore we are done. ♣

Now we are in position to prove the following main result of this section.

Theorem 9.3.11. $\mathrm{HyD}(J(f)) = \mathrm{DD}(J(f)) = \delta(f)$ and there exists a $\delta(f)$–conformal measure for $f : J(f) \to J(f)$.

Proof. For every $x \in \mathrm{Crit}(f)$ the set $\overline{\{f^n(x) : n \ge 0\}}$ is closed and forward invariant under f. Therefore, in view of Theorem 2.??? (Bogolubov-Krylov theorem) there exists $\mu \in M_e(f)$ supported on $\overline{\{f^n(x) : n \ge 0\}}$. By Corollary A of [Przyt, Lyap] there exists at least one point $y(x) \in \overline{\{f^n(x) : n \ge 0\}}$ such that $\limsup_{n\to\infty} |(f^n)'(y(x))| \ge 1$ and consequently the series $S(t, y(x))$ diverges for every $t \ge 0$. So, applying Lemma 9.3.10 and then looking at Lemma 9.2.2 and Theorem 9.2.3 complete the proof. ♣

§9.4. PESIN'S FORMULA AND LEDRAPPIER'S CONVERSE THEOREM.

In this section our aim is to prove two main theorems. The first one is as follows.

Theorem 9.4.1. (Pesin's formula) Assume that X is a compact subset of the closed complex plane $\overline{\mathbb{C}}$ and that $f \in \mathcal{A}(X)$. If m is a t- conformal measure for f and $\mu \in M_e^+(f)$ is absolutely continuous with respect to m, then $\mathrm{HD}(\mu) = t$.

Proof. In view of Lemma 9.2.2 we only need to prove that $t \le \mathrm{HD}(\mu)$ and in order to do this we essentially combine the arguments from the proof of Lemma 9.2.2 and from the proof of formula (8.4.1). So, we work in the natural extension $(\tilde X, \tilde f, \tilde\mu)$. Fix $0 < \varepsilon < \chi_\mu/3$ and let $\tilde X(\varepsilon)$ and $r(\varepsilon)$ be given by Corollary 8.2.4. In view of the Birkhoff ergodic theorem there exists a measurable set $\tilde F(\varepsilon) \subset \tilde X(\varepsilon)$ such that $\tilde\mu(\tilde F(\varepsilon)) \ge 1 - 2\varepsilon$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} \chi_{\tilde X(\varepsilon)} \circ \tilde f^n(\tilde x) = \tilde\mu(\tilde X(\varepsilon))$$

for every $\tilde x \in \tilde F(\varepsilon)$. Let $F(\varepsilon) = \pi(\tilde F(\varepsilon))$. Then $\mu(F(\varepsilon)) = \tilde\mu(\pi^{-1}(F(\varepsilon)) \ge \tilde\mu(\tilde F(\varepsilon)) \ge 1 - 2\varepsilon$. Consider now $x \in F(\varepsilon) \cap X_o$ and take $\tilde x \in \tilde F(\varepsilon)$ such that $x = \pi(\tilde x)$. Then by the above there exists an increasing sequence $\{n_k = n_k(x) : k \geq 1\}$ such that $\tilde{f}^{n_k}(\tilde{x}) \in \tilde{X}(\varepsilon)$ and

$$\frac{n_{k+1} - n_k}{n_k} \leq \varepsilon \tag{9.4.1}$$

for every $k \geq 1$. Moreover Corollary 8.2.4 produces holomorphic inverse branches $f_x^{-n_k} : B(f^{n_k}(x), r(\varepsilon)) \to \overline{\mathbb{C}}$ of f^{n_k} such that $f_x^{-n_k} f^{n_k}(x) = x$ and

$$f_x^{-n_k}\big(B(f^{n_k}(x), r(\varepsilon))\big) \subset B\big(x, K|(f^{n_k})'(x)|^{-1} r(\varepsilon)\big)$$

Set $r_k = r_k(x) = K^{-1}|(f^{n_k})'(x)|^{-1} r(\varepsilon)$. By Corollary 8.2.4 $r_k \leq K^{-2} \exp\big(-(\chi_\mu - \varepsilon)n_k\big) r(\varepsilon)$. So, using Corollary 8.2.4 again and (9.4.1) we can estimate

$$\begin{aligned}
r_k &= r_{k+1}|(f^{n_{k+1}-n_k})'(f^{n_k}(x))| \leq r_{k+1} K \exp\big(\chi_\mu + \varepsilon)(n_{k+1} - n_k)\big) \\
&\leq r_{k+1} K \exp\big(\chi_\mu + \varepsilon)n_{k+1}\varepsilon\big) \leq K r_{k+1} \exp\big(\chi_\mu - \varepsilon)2n_{k+1}\varepsilon\big) \leq r_{k+1} K (K^{-2} r(\varepsilon) r_{k+1}^{-1})^{2\varepsilon} \\
&= K^{1-4\varepsilon} r(\varepsilon)^{2\varepsilon} r_{k+1}^{1-2\varepsilon}
\end{aligned}$$

Take now any $0 < r \leq r_1$ and find $k \geq 1$ such that $r_{k+1} < r \leq r_k$. Then using this estimate, t-conformality of m, and invoking Corollary 8.2.4 once more we get

$$\begin{aligned}
m(B(x, r)) \leq m(B(x, r_k)) &\leq K^t |(f^{n_k})'(x)|^{-t} m(B(x, r(\varepsilon))) \\
&\leq K^{2t} r(\varepsilon)^{-t} r_k^t \\
&\leq K^{(3-4\varepsilon)t} r(\varepsilon)^{2\varepsilon t} r^{(1-2\varepsilon)t}
\end{aligned}$$

So, by Theorem 5.5.1 (Besicovitch covering theorem) $\mathrm{H}_{(1-2\varepsilon)t}(X) \geq \mathrm{H}_{(1-2\varepsilon)t}(F(\varepsilon)) > 0$, whence $\mathrm{HD}(X) \geq (1 - 2\varepsilon)t$. Letting $\varepsilon \to 0$ completes the proof. ♣