CHAPTER 7. CONFORMAL EXPANDING REPELLERS

Conformal expanding repellers (abbreviation: CER's) were defined already in Chapter 4 and some basic properties of expanding sets and repellers in dimension 1 were discussed in Chapter 4.5. A more advanced geometric theory in the real case was done in Chapter 4.6. But now we have a new tool: Frostman Lemma from Chapter 5. Armed with the theory of Gibbs measures and with the pressure function we are able to develop a geometric theory of CER's with Hausdorff measures and dimension playing the crucial role. We shall expose this theory for $C^{1+\varepsilon}$-expanding repellers in $\mathbb{R}^d$. We shall apply this theory to study the boundary of the immediate basin of attraction to a sink for a rational mapping of the Riemann sphere in the case the basin is simply-connected and the mapping on the boundary being expanding, for example for the mapping $z \mapsto z^2 + c$ for $|c|$ small, for a quasicircle invariant under the action of a quasi-Fuchsian group and the boundary of the "snowflake". We shall derive from this an information about the radial growth of the derivative of the Riemann mapping from the unit disc to the simply-connected domain.

Section 7.1.1. Pressure function and dimension.

Let $f : X \to X$ be a conformal expanding repeller in $\mathbb{R}^d$. We restrict our considerations about Gibbs measures and of the function $\phi \mapsto P(\phi)$ from Chapters 3 and 2b to $\phi = -t \log |f'|$ restricted to X. As $|f'| \geq \lambda > 1$ the function

$$P(t) := P(-t \log |f'|)$$

is finite and strictly decreasing from $+\infty$ to $-\infty$. In particular there exists exactly one parameter t_0 such that $P(t_0) = 0$. Our first result is the following.

Theorem 7.1.1. Let t_0 be defined by $P(t_0) = 0$. Write ϕ for $-t_0 \log |f'|$ restricted to X. Then each Gibbs state m corresponding to the function ϕ is a geometric measures with the exponent t_0.

Proof. Take an arbitrary ball $B(x, r)$ in $\mathbb{R}^d$ with $x \in X$. Let $n \geq 0$ be the largest integer so that

$$|(f^n)'(x)| r C \leq \xi, \tag{7.1.1}$$

where C is the distortion constant corresponding to the Hölder continuous function $\log |f'|$. Then by the Distortion Lemma for Iteration (Ch.4.5)

$$f_x^{-n}(B(f^n(x), \xi) \supset B(x, \xi |(f^n)'(x)|^{-1} C^{-1}) \supset B(x, r). \tag{7.1.2}$$

Now take n_0 such that $\lambda^{n_0 - 1} \geq C^2$. We then obtain

$$|(f^{n+n_0})'| r C^{-1} > \xi. \tag{7.1.3}$$

Hence, again by the Distortion Lemma for Iteration

$$(7.1.4) \qquad f_x^{-n-n_0}(B(f^{n+n_0}(x),\xi) \subset B\big(x, \xi|(f^{n+n_0})'(x)|^{-1}C\big) \subset B(x,r).$$

By the Gibbs property of the Gibbs measure m, with an apriopriate constant $D \geq 1$, we can write ???(Ch3)

$$\begin{aligned} D^{-1}|(f^n)'(x)|^{-t_0}D^{-1} &= \exp S_n\phi(x) \\ &\leq m(f_x^{-n}(B(f^n(x),xi)) \\ &\leq D\exp S_n\phi(x) = D|(f^n)'(x)|^{-t_0}. \end{aligned}$$

Using (7.1.2) and (7.1.3) and the fact that $\sup|(f^{n_0})'| \leq L^{n_0}$, where L is a Lipschitz constant for f, we obtain from this

$$m\big(B(x,r)\big) \leq D|(f^n)'(x)|^{-t_0} \leq D|(f^{n+n_0})'|^{-t_0}L^{n_0t_0} \leq D(L^{n_0}\xi^{-1}C^{-1})^{t_0}r^{t_0}.$$

A lower estimate of $m(B(x,r))$ by Const.r_0^t can be obtained similarly, with the use of (7.1.4) and (7.1.1).

As an immediate consequence of this theorem and Theorem 5.6.6 we get the following.

Corollary 7.1.2. The Hausdorff dimension of X, $\mathrm{HD}(X) = t_0$. Moreover it is equal to the packing and Minkowski dimensions, and all the Gibbs states corresponding to the potential $\phi = -t_0\log|f'|$, as well as t_0-dimensional Hausdorff and packing measures are all mutually equivalent with bounded Radon-Nikodym derivatives.

When we studied the pressure function $\phi \mapsto P(\phi)$ in Chapters 2 and 3 the linear functional $\psi \mapsto \int \psi d\mu_\phi$ appeared. This was the differential of P at ϕ (after restricting to 1 dimensional subspaces in the continuous functions space). Here the presence of an ambient smooth structure (1-dimensional) distingushes ψ's of the form $-t\log|f'|$. We obtain a link between the ergodic theory and the geometry of the embedding of X in $\mathbb{C}$ or $\mathbb{R}^d$.

Definition 7.1.3. Let μ be an ergodic f-invariant probability measure on X. Then by Birkhoff Ergodic Theorem for μ-almost every $x \in X$, $\lim_{n\to\infty}\frac{1}{n}\log|(f^n)'(x)|$ exists and is equal to $\int \log|f'|d\mu$. We call this number *Lyapunov characteristic exponent* and we denote it by $\chi_\mu(f)$. In our case of expanding maps in this Chapter obviously $\chi_\mu(f) > 0$.

This definition does not demand the expanding property. It makes sense for an arbitrary invariant subset X of $\mathbb{R}^d$ or the Riemann sphere $\bar{\mathbb{C}}$ for f conformal (or differentiable in the real case) defined on a neighbourhood of X. There is no problem with the integrability because $\log|f'|$ is upper bounded on X. (We do not exclude the possibility $\chi_\mu = -\infty$.) So the notion of Lyapunov characteristic exponent will play a crucial role also in subsequent chapters where non-expanding invariant sets will be studied.

Lemma 7.1.4. (Volume Lemma) Let μ be a Gibbs state for a conformal expanding repeller $X \in I\!R^d$ and a Hölder continuous potential ϕ on X . Then for μ-almost every point $x \in X$ there exists limit

$$\lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Moreover, this limit is almost everywhere constant and is equal to $\mathrm{h}_\mu(f)/\chi_\mu(f)$.

Proof. Since all the Gibbs states of ϕ are mutually equivalent with bounded Radon-Nikodym derivatives, we may assume that μ is the only invariant (whence ergodic) Gibbs state of ϕ. We may also assume that $\mathrm{P}(\phi) = 0$. By the Gibbs property and the Bounded Distortion Property for Iteration there exists $E \geq 1$ such that for all $x \in X$ and all $r > 0$ small enough

$$E^{-1} < \frac{\mu(B(x,r))}{\exp S_n\phi(x)} < E,$$

n is the largest integer such that $|(f^n)'(x)|rC \leq \xi$. Then

$$-\log E + S_n\phi(x) \leq \log \mu(B(x,r) \leq \log E + S_n\phi(x)$$

and

$$\log(\xi(rC||f'||^{-1}) - \log|(f^n)'(x)| \leq \log r \leq \log(\xi C^{-1}) - \log|(f^n)'(x)|.$$

Combining these four inequalities we obtain

$$\frac{-\log E + S_n\phi(x)}{\log(\xi C^{-1}) - \log|(f^n)'(x)|} \leq \frac{\log\big(\mu(B(x,r))}{\log r} \leq \frac{\log E + S_n\phi(x)}{\log(\xi C^{-1}) - \log|(f^n)'(x)|}.$$

Now, in view of the Birkhoff Ergodic Theorem, there exists a Borel set X_0 such that $\mu(X_0) = 1$ and for all $x \in X_0$

$$\lim_{n \to \infty} \frac{1}{n} S_n\phi(x) = \int \phi \, d\mu \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \log|(f^n)'(x)| = \chi_\mu(f).$$

Combining the last four inequalities along with the observation that $n \to \infty$ as $r \to 0$, we conclude that for all $x \in X_0$

$$\lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = \frac{\mathrm{h}_\mu(f)}{\chi_\mu(f)}.$$

The proof is finished. ♣

As an immediate consequence of this lemma and Lemma 5.6.4 we get the following.

Theorem 7.1.5. If μ is a Gibbs state for a conformal expanding repeller $X \in I\!R^d$ and a Hölder continuous potential ϕ on X, then

$$\mathrm{HD}(\mu) = \mathrm{h}_\mu(f)/\chi_\mu(f).$$

Section 2. Fluctuations for Gibbs measures.

Theorem 7.2.1. Let $f : X \to X$ be a holomorphic expanding repeller. Let ϕ be a Hölder continuous function and let μ_ϕ be it's Gibbs measure. Then, with $\kappa = \mathrm{HD}(\mu_\phi)$, either

(a) $\kappa = \mathrm{HD}(X)$ and $\mu_\phi \asymp \mathrm{H}_1$ on X (if $\psi = \phi + \kappa \log|f'| - \mathrm{P}(\phi)$ is a coboundary) or

(b) $\mu_\phi \perp \mathrm{H}_1$ (if $\psi = \phi + \kappa \log|f'| - \mathrm{P}(\phi)$ is not a coboundary) and moreover, there exists $c_0 > 0$, ($c_0 = \sqrt{2\sigma^2(\psi)/\chi_{\mu_\phi}(f)}$), such that with the gauge function $\phi_c(r) = r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})$, where $\log_3$ means the iteration of the log function 3 times:

(c) $\mu_\phi \perp \mathrm{H}_{\phi_c}$ for all $0 < c < c_0$, and

(d) $\mu_\phi \ll \mathrm{H}_{\phi_c}$ for all $c > c_0$.

Proof. If ψ is a coboundary, then it follows from equality $\phi - \psi = -\kappa \log|f'| + \mathrm{P}(\phi)$ that $\mathrm{P}(\phi) = \mathrm{P}(\phi - \psi) = \mathrm{P}(\phi) + \mathrm{P}(-\kappa \log|f'|)$. Thus $\mathrm{P}(-\kappa \log|f'|) = 0$ and the part (a) follows immediately from Theorem 7.1.1 and the observation saying that the potential cohomological up to an additive constant have the same Gibbs states.

Suppose now that ψ is not a coboundary. As in the previous section let $\lambda = \inf|f'| > 1$. Recall that then there exists $\tau > 0$ small enough that firstly, $f\mid_A$ is one - to - one for all sets $A \subset X$ with $\mathrm{diam}(A) \le \tau$ and secondly, $d(f(y), f(z)) \ge \lambda d(y,z)$ if $d(y,z) < \tau$. Fix $x \in X$ and $r > 0$ such that $r < \tau$ Define $n = n(x,r)$ to be the least number such that $\mathrm{diam}(f^n(B(x,r))) \ge \tau$. Since $f : X \to X$ is topologically exact, $n(x,r)$ is finite. By definition of n, $\mathrm{diam}(f^j(B(x,r))) < \tau$ for all $j = 0, 1, \ldots n-1$. Therefore $f^n\mid_{(B(x,r))}$ is one - to - one and

$$m_\phi(f^n(B(x,r))) = \int_{B(x,r)} \exp(P(\phi)n - S_n\phi(z))dm_\phi(z),$$

where m_ϕ is the fixed point of the dual operator of $\mathcal{L}_\phi$, the transfer operator associated with the function $\phi - \mathrm{P}(\phi)$. Since the function ϕ is Hölder continuous, it follows from ??? that there exists a constant $K > 0$ such that $|S_n\phi(z) - S_n\phi(x)| < K$ for all $x \in X$, $z \in B(x,r)$, and $n = n(x,r)$. Hence, we get

$$\begin{aligned} e^{-K}\exp(P(\phi)n - S_n\phi(x))m_\phi(B(x,r)) &\le m_\phi(f^n(B(x,r))) \\ &\le e^K \exp(P(\phi)n - S_n\phi(x))m_\phi(B(x,r)) \end{aligned}$$

or equivalently

$$\begin{aligned} e^{-K}\exp(S_n\phi(x) - P(\phi)n)m_\phi(f^n(B(x,r))) &\le m_\phi(B(x,r)) \\ &\le e^K \exp(S_n\phi(x) - P(\phi)n)m_\phi(f^n(B(x,r))). \end{aligned}$$

Similarly, since $\log|f'(z)|$ is a Hölder continuous function, there exists a constant $K_1 > 0$ such that

$$|S_n(\log|f'(z)|) - S_n(\log|f'(x)|)| \le K_1$$

for all $x \in X$, $z \in B(x,r)$, and $n = n(x,r)$. Then

$$\frac{|(f^n)'(z)|}{|(f^n)'(x)|} \le e^{K_1}$$

Therefore

$$\begin{aligned}\operatorname{diam}(B(x,r))e^{-K_1}|(f^n)'(x)| &\le \operatorname{diam}(f^n(B(x,r)))\\ &\le \operatorname{diam}(B(x,r))|(f^n)'(x)|e^{K_1}.\end{aligned}$$

Hence, $\operatorname{diam}(B(x,r))e^{-K_1}|(f^n)'(x)| \le \tau\|f'\|$, or equivalently $r \le \frac{1}{2}e^{K_1}|(f^n)'(x)|^{-1}\tau\|f'\|$. Also, since $\operatorname{diam}(f^n(B(x,r))) \ge \tau$ we get similarly $r \ge \frac{1}{2}e^{-K_1}|(f^n)'(x)|^{-1}\tau$. In conclusion,

$$\text{(7.2.1)} \qquad \frac{1}{2}e^{-K_1}|(f^n)'(x)|^{-1}\tau \le r \le \frac{1}{2}e^{K_1}|(f^n)'(x)|^{-1}\tau\|f'\|.$$

Hence, denoting $\sqrt{\log(|(f^n)'(x)|)\log_3(|(f^n)'(x)|)}$ by $g_n(x)$, we get

$$\begin{aligned}\frac{m_\phi(B(x,r))}{r^\kappa\exp(c\sqrt{\log 1/r\log_3 1/r})} &\le \frac{e^K\exp(S_n\phi(x)-P(\phi)n)m_\phi(f^n(B(x,r)))}{r^\kappa\exp(c\sqrt{\log 1/r\log_3 1/r})}\\ &\le \frac{e^K\exp(S_n\phi(x)-P(\phi)n)}{(\frac{1}{2}\frac{1}{|(f^n)'(x)|e^{K_1}}\tau)^\kappa\exp(c\sqrt{\log(2e^{-K_1}\frac{|(f^n)'(x)|}{\tau\|f'\|})\log_3(2e^{-K_1}\frac{|(f^n)'(x)|}{\tau\|f'\|})})}\\ &\le \frac{Q\exp(S_n\phi(x)-P(\phi)n)}{|(f^n)'(x)|^{-\kappa}\exp(cg_n(x))},\end{aligned}$$

where Q is a large enough constant. Similarly,

$$\begin{aligned}&\frac{m_\phi(B(x,r))}{r^\kappa\exp(c\sqrt{\log 1/r\log_3 1/r})}\\ &\ge \frac{e^{-K}\exp(S_n\phi(x)-P(\phi)n)m_\phi(f^n(B(x,r)))}{r^\kappa\exp(c\sqrt{\log 1/r\log_3 1/r})}\\ &\ge \frac{e^{-K}\exp(S_n\phi(x)-P(\phi)n)m_\phi(f^n(B(x,r)))}{(\frac{1}{2}|(f^n)'(x)|^{-1}e^{K_1}\tau\|f'\|)^\kappa\exp(c\sqrt{\log(2e^{K_1}|(f^n)'(x)|\tau^{-1})\log_3(2e^{K_1}|(f^n)'(x)|\tau^{-1})})}\\ &\ge \frac{Q_1\exp(S_n\phi(x)-P(\phi)n)m_\phi(f^n(B(x,r)))}{|(f^n)'(x)|^{-\kappa}\exp(cg_n(x))} \quad (Q_1 \text{ large enough})\\ &\ge \frac{Q_2\exp(S_n\phi(x)-p(\phi)n)}{|(f^n)'(x)|^{-\kappa}\exp(cg_n(x))},\end{aligned}$$

where $Q_2 = Q_1\min\{m_\phi(f^n(B(x,r))) : x \in X, r > 0\}$. Note that, since the topological support of μ_ϕ and m_ϕ is equal to X, using the the Bounded Distortion Theorem we get $Q_2 > 0$ because $f^n(B(x,r)) \supset B(f^n(x), r|(f^n)'(x)|K^{-1}) \supset B(f^n(x), R)$, where $R = K^{-1}\tau e^{-K_1}$. Finally, we get

$$0 < \frac{Q_2\exp(S_n\phi(x)-P(\phi)n)}{|(f^n)'(x)|^{-\kappa}\exp(cg_n(x))} \le \frac{m_\phi(B(x,r))}{r^\kappa\exp(c\sqrt{\log 1/r\log_3 1/r})} \le \frac{Q\exp(S_n\phi(x)-P(\phi)n)}{|(f^n)'(x)|^{-\kappa}\exp(cg_n(x))}$$

Hence,

$$\left[\frac{Q_2 \exp(S_n\phi(x) - P(\phi)n)}{|(f^n)'(x)|^{-\kappa} \exp(cg_n(x))}\right] \le \log\left[\frac{m_\phi(B(x,r))}{r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})}\right].$$
$$\le \log\left[\frac{Q \exp(S_n\phi(x) - P(\phi)n)}{|(f^n)'(x)|^{-\kappa} \exp(cg_n(x))}\right]$$

Therefore,

$$\begin{aligned}
&\log Q_2 + S_n\phi(x) - P(\phi)n + \kappa \log|(f^n)'(x)| - cg_n(x) \\
&\quad \le \log\left[\frac{m_\phi(B(x,r))}{r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})}\right] \\
&\quad \le \log Q + S_n\phi(x) - P(\phi)n + \kappa \log|(f^n)'(x)| - cg_n(x),
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\log Q_2 + g_n(x)\left[\frac{S_n\phi(x) - P(\phi)n + \kappa \log|(f^n)'(x)|}{g_n(x)} - c\right] \\
&\quad \le \log\left[\frac{m_\phi(B(x,r))}{r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})}\right] \\
&\quad \le \log Q + g_n(x)\left[\frac{S_n\phi(x) - P(\phi)n + \kappa \log|(f^n)'(x)|}{g_n(x)} - c\right]. \qquad (7.2.1)
\end{aligned}$$

So, in order to proceed further, we are tempted to evaluate the following upper limit.

$$\limsup_{n\to\infty} \frac{S_n\psi(x)}{\sqrt{\log|(f^n)'(x)| \log_3|(f^n)'(x)|}}$$

First, by the Birkhoff ergodic theorem, for every $\epsilon > 0$ there exists $X_1 \subset X$ such that $m_\phi(X_1) = 1$ and $\forall x \in X_1$ there exists $N > 0$ such that $\forall n \ge N$

$$n(\chi_\mu - \epsilon) \le \log|(f^n)'(x)| \le n(\chi_\mu + \epsilon).$$

Second, since ψ is not a coboundary, by the Law of Iterated Logarithms (see ???), there exists $X_2 \subset X$ such that $m_\phi(X_2) = 1$, and forall $x \in X_2$

$$\limsup_{n\to\infty} \frac{S_n\psi(x)}{\sqrt{n \log_2(n)}} = \sqrt{2\sigma^2}.$$

Therefore, for all $x \in X_1 \bigcap X_2$

$$\begin{aligned}
\limsup_{n\to\infty} \frac{S_n\psi(x)}{g_n(x)} &\le \limsup_{n\to\infty} \frac{S_n\psi(x)}{\sqrt{n(\chi_\mu - \epsilon) \log_2(n(\chi_\mu - \epsilon))}} \\
&\le \limsup_{n\to\infty} \frac{S_n\psi(x)}{\sqrt{(\chi_\mu - 2\epsilon)}\sqrt{n \log_2(n))}} \\
&= \frac{\sqrt{2\sigma^2}}{\sqrt{(\chi_\mu - 2\epsilon)}}.
\end{aligned}$$

and

$$\limsup_{n\to\infty} \frac{S_n\psi(x)}{g_n(x)} \geq \limsup_{n\to\infty} \frac{S_n\psi(x)}{\sqrt{n(\chi_\mu+\epsilon)\log_2(n(\chi_\mu+\epsilon))}}$$
$$\geq \limsup_{n\to\infty} \frac{S_n\psi(x)}{\sqrt{(\chi_\mu+2\epsilon)}\sqrt{n\log_2(n))}}$$
$$= \frac{\sqrt{2\sigma^2}}{\sqrt{(\chi_\mu+2\epsilon)}}.$$

In conclusion, forall $x \in X_1 \bigcap X_2$, we have

$$\frac{\sqrt{2\sigma^2}}{\sqrt{(\chi_\mu+2\epsilon)}} \leq \limsup_{n\to\infty} \frac{S_n\psi(x)}{g_n(x)} \leq \frac{\sqrt{2\sigma^2}}{\sqrt{(\chi_\mu-2\epsilon)}}$$

Hence, letting $\epsilon \searrow 0$,

$$\limsup_{n\to\infty} \frac{S_n\psi(x)}{g_n(x)} = \sqrt{\frac{2\sigma^2}{\chi_\mu}}.$$

Thus, with

$$\beta_n(c) = g_n(x)\left[\frac{S_n\phi(x) - P(\phi)n + \kappa\log|(f^n)'(x)|}{g_n(x)} - c\right],$$

we obtain

$$\limsup_{n\to\infty} \beta_n(c) = \begin{cases} \infty & \text{if } \sqrt{\frac{2\sigma^2}{\chi_\mu}} > c \\ -\infty & \text{if } \sqrt{\frac{2\sigma^2}{\chi_\mu}} < c \end{cases} \tag{7.2.3}$$

Since, $\lim_{n\to\infty} g_n(x) = \infty$, and this is because f is expanding. By (7.2.1)

$$\limsup_{n\to\infty}[\log Q_2 + \beta_n(c)] \leq \limsup_{r\to 0} \log\left[\frac{m_\phi(B(x,r))}{r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})}\right]$$
$$\leq \limsup_{n\to\infty}[\log Q + \beta_n(c)].$$

By (7.2.3), it gives

$$\limsup_{r\to 0} \log\left[\frac{m_\phi(B(x,r))}{r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})}\right] = \begin{cases} \infty & \text{if } \sqrt{\frac{2\sigma^2}{\chi_\mu}} > c \\ -\infty & \text{if } \sqrt{\frac{2\sigma^2}{\chi_\mu}} < c \end{cases}.$$

In other words,

$$\limsup_{r\to 0} \frac{m_\phi(B(x,r))}{r^\kappa \exp(c\sqrt{\log 1/r \log_3 1/r})} = \begin{cases} \infty & \text{if } \sqrt{\frac{2\sigma^2}{\chi_\mu}} > c \\ 0 & \text{if } \sqrt{\frac{2\sigma^2}{\chi_\mu}} < c \end{cases}.$$

Therefore, by Theorem 5.??, $\mu_\phi \perp \mathrm{H}_{\phi_c}$ for all $c < \sqrt{\frac{2\sigma^2}{\chi_\mu}}$ and $\mu_\phi \ll \mathrm{H}_{g_c}$ for all $c > \sqrt{\frac{2\sigma^2}{\chi_\mu}}$. The proof is finished. ♣

Note that this proof is done without the use of Markov partitions. Note also that the last display in the proof of Theorem 7.2.1 is known as a refined volume lemma.

Section 7.3. Radial behaviour of the Riemann map,I.

In this section $f : X \to X$ continues to be a conformal expanding repeller and we assume additionally that X is a Jordan curve. We then call f a boundary conformal repeller. Let $\Omega \subset \overline{\mathbb{C}}$ be one of the components of $\overline{\mathbb{C}} \setminus X$ and fix $z_0 \in \Omega$. Let $D^1 = \{z : |z| < 1\}$. In view of Caratheodory's theorem, any Riemann map $R : D^1 \to \Omega$ (conformal homeomorphism) sending zero to z_0 (which is unique up to rotation) extends homeomorphically to $\overline{D^1}$. For more information about a Riemann map we refer the reader to [?]. We also assume that there exists an open set $A \subset \overline{\mathbb{C}}$ surronding $\partial\Omega$ such that $f(A \cap \Omega) \subset \Omega$. With all these assumptions we speak about the expanding map $f : \partial\Omega \to \partial\Omega$ as a conformal boundary repeller with the Jordan domain Ω. This enables us to lift the map f to a closed topological annulus $B \subset \overline{D^1}$ one of whose boundary components is the circle S^1 by setting $g = R^{-1} \circ f \circ R : B \to D^1$, provided that $R(B) \subset A$. Of course g extends continuously to a map $S^1 \to S^1$ given by the same formula $R^{-1} \circ f \circ R$. Therefore, by the Schwarz reflection principle, g extends analytically to an open annulus F surronding S^1. We begin with the following.

Lemma 7.3.1. The map $g : S^1 \to S^1$ is expanding.

Proof. Since $f : X \to X$ is an expanding map, and g and f are topologically conjugate, taking $A = B(X, r)$ with $r > 0$ sufficiently small, we may assume that $\Omega \cap f^{-1}(A \cap \Omega)$ contains a connected component that is contained in $A \cap \Omega$ and whose closure forms an open neighborhood of $\partial\Omega$ in $\overline{\Omega}$. Since R establishes a topological conjugacy between g and f, the same remains true for for the map g, the set $F = R^{-1}(A)$ corresponding to A and the open disk D^1 corresponding to Ω. Since f has no critical points in some sufficiently small neighbourhood of X and since g and f are topologically conjugate on $S^1 \cup (F \cap S^1)$, it follows from these inclusions that if $G \subset F \cap (D^1 \cup S^1)$ is a connected simply connected set, then for every connected component G_1 of $g^{-1}(G) \cap F \cup S^1$, $g|_{G_1}$ is a 1-to-1 mapping onto G. Therefore, a straightforward inductive argument shows that given $z \in S^1 \cup F$, $0 < r < \delta/2$, and $z_n \in S^1 \cup F$, such that $g^k(z_n) \in S^1 \cup F$ for all $k = 0, 1, 2, \ldots, n$, and $g^n(z_n) = z$, there exists a continuous inverse branch $g_{z_n}^{-n} : B(z, r) \cap (F \cup S^1) \to F \cup S^1$ of f^n analytic on $B(z, r) \cap F$ and mapping $B(z, r) \cap (F \cup S^1)$ onto the component of $g^{-n}(B(z, r) \cap (F \cup S^1))$ containing z_n. Then $g_{z_n}^{-n}(z) = z_n$ and due to Schwarz's reflection principle, $g_{z_n}^{-n}$ extends analytically onto $B(z, r)$ and $g_{z_n}^{-n}(B(z, r))$ is contained in $S^1 \cup F \cup F^*$, where F^* is the inversion of F with respect to the circle S^1. Hence the maps $g_{z_n}^{-n} : B(z, r) \to S^1 \cup F \cup F^*$, $n \geq 1$, $z_n \in g^{-n}(z) \cap (F \cup F^* \cup S^1)$ form a normal family, and since g is topologically conjugate with f and $f : X \to X$ is expanding, the diameters of the sets $g_{z_n}^{-n}(B(z, r))$ converge uniformly to zero with respect to $z \in F \cup F^* \cup S^1$ and $z_n \in g^{-n}(z)$. Hence, the derivatives $(g_{z_n}^{-n})'(z)$ also converge uniformly to zero, and consequently there exists $q \geq 1$ such that $|(g_{z_q}^{-q})'(z)| < 1$ for all z, q, and z_q satisfying our requirements. Thus if $x \in S^1$, then $|(g^q)'(x)| = 1/|(g_x^{-q})'(g^q(x))| > 1$. The proof is finished. ♣

Our next result is the following the following.

Theorem 7.3.2. If μ is an ergodic probability measure of of positive entropy on S^1, then the non-tangent limit

$$\lim_{x \to z} \frac{-\log |R'(x)|}{\log(1-|x|)}$$

exists for μ-almost every point $z \in S^1$, is constant almost everywhere (denote it by $\chi_\mu(R)$), and

$$\chi_\mu(R) = 1 - \frac{\chi_{\mu \circ R^{-1}}(f)}{\chi_\mu(g)}.$$

Proof. Given $z \in S^1$, $0 < \alpha < \pi/2$, and $0 < r < 1$ let

$$S_{\alpha,r}(z) = z \cdot (1 + \{x \in \mathbb{C} \setminus \{0\} : |x| \leq r, \pi - \alpha \leq \mathrm{Arg}(x) \leq \pi + \alpha\}).$$

Given $0 < \alpha < \pi/2$ fix $\alpha < \alpha' < \pi/2$. In view of the SDT we can find $r_0 > 0$ so small that $\overline{S_{\alpha,r_0}(z)} \setminus \{z\} \subset D^1$ and

$$g^n(S_{\alpha,r_0}(z)) \setminus B(0, 1-r_0) \subset S_{\alpha',r_0}(g^n(z)) \tag{7.3.1}$$

for all $z \in S^1$ and all $n \geq 1$. Fix now $z \in S^1$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log |(g^n)'(z)| = \chi_\mu(g) \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \log |(f^n)'(R(z))| = \chi_{\mu \circ R^{-1}}(f).$$

By the Birkhoff ergodic theorem the set of those points z has μ measure 1. Fix also $\varepsilon > 0$ and consider n_0 so large that for all $n \geq n_0$

$$\exp(n(\chi_\mu(g) - \varepsilon)) \leq |(g^n)'(z)| \leq \exp(n((\chi_\mu(g) + \varepsilon)), \tag{7.3.2}$$

and

$$\exp(n(\chi_{\mu \circ R^{-1}}(f) - \varepsilon)) \leq |(f^n)'(R(z))| \leq \exp(n((\chi_{\mu \circ R^{-1}}(f) + \varepsilon)). \tag{7.3.3}$$

Let $x \in S_{\alpha,r_0}(z) \setminus B(0, 1-r_0)$ and consider the least integer n such that $g^n(x) \notin B(0, 1-r_0)$. Takeing r_0 smaller if necessary we may assume that $n \geq n_0$. Then by the chain rule

$$R'(x) = (f_{R(x)}^{-m})'(R(g^n(x))) \cdot R'(g^n(x)) \cdot (g^n)'(x).$$

In view of the Koebe Distortion Theorem $R'(g^n(x)) \asymp 1$. Invoking the Koebe Distortion Theorem again and (7.3.1) we conclude that $(1-|x|)|(g^n)'(x)| \asymp 1 - |g^n(x)| \asymp 1$. Denote the comparability constant by $C \geq 1$. Takeing the logarithms of both sides of the last formula, using (7.3.2), and the Koebe Distortion Theorem third time, we get for all n large enough.

$$-\log C \leq \log(1-|x|)+n(\chi_m u(g)+2\varepsilon)$$

and

$$\log(1-|x|)+n(\chi_m u(g)-2\varepsilon) \leq \log C.$$

Thus, for all n large enough

$$\frac{-\log(1-|x|)}{\chi_\mu(g)+3\varepsilon} \leq m \leq \frac{-\log(1-|x|)}{\chi_\mu(g)-3\varepsilon}.$$

So, using (7.3.2), (7.3.3), and the Koebe Distortion Theorem we get

$$\begin{aligned}|R'(x)| &\preceq |(f^n)'(R(x))|^{-1}|(g^n)'(x)| \\ &\leq \exp(-m(\chi_{\mu\circ R^{-1}}(f)-2\varepsilon))\exp(n((\chi_\mu(g)+2\varepsilon)) \\ &\leq \frac{-\log(1-|x|)}{\chi_\mu(g)-3\varepsilon}\big(\chi_\mu(g)-\chi_{\mu\circ R^{-1}}(f)+4\varepsilon).\end{aligned}$$

Thus

$$\frac{-\log|R'(x)|}{\log(1-|x|} \preceq \frac{\chi_\mu(g)-\chi_{\mu\circ R^{-1}}(f)+4\varepsilon)}{\chi_\mu(g)-3\varepsilon}$$

and similarly

$$\frac{-\log|R'(x)|}{\log(1-|x|} \geq \frac{\chi_\mu(g)-\chi_{\mu\circ R^{-1}}(f)-4\varepsilon)}{\chi_\mu(g)+3\varepsilon}$$

So, letting $\varepsilon \to 0$ (and therefore $x \to z$) finishes the proof. ♣

In order to make use of this result we shall provide a simple proof of the following Proposition.

Section 7.4 Radial behaviour of the Riemann map,II.

Keepin notation from the previous sections we shall prove here the following .

Theorem 7.4.1. Depending on whether $c(\omega)=0$ or $c(\omega)\neq 0$, either $\partial\Omega$ is real-analytic and the Riemann map $R: D^1 \to \Omega$ and its derivative R' extend holomorphically beyond ∂D^1 or for almost every $z \in \partial D^1$

$$\text{(7.4.1)} \qquad \limsup_{r\to 1}|R'(rz)|\exp c\sqrt{\log(1/1-r)\log_3(1/1-r)} = \begin{cases}\infty & \text{if } c \leq c(\omega)\\ 0 & \text{if } c > c(\omega)\end{cases}$$

and

$$\text{(7.4.1)} \qquad \limsup_{r\to 1}\big(|R'(rz)|\exp c\sqrt{\log(1/1-r)\log_3(1/1-r)}\big)^{-1} = \begin{cases}\infty & \text{if } c \leq c(\omega)\\ 0 & \text{if } c > c(\omega)\end{cases}$$

Moreover the radial limsup can be replaced by the nontangential one.

Proof. Let $n > 0$ be the least integer for which $g^n(rz) \in B(0, r_0)$ for some fixed $r_0 < 1$. We have $R'(rz) = ((f^n)'(R(rz)))^{-1} \cdot R'(g^n(rz)) \cdot (g^n)'(rz)$. Hence, for some constant $K > 0$ independent of r and z

$$K^{-1} \le \frac{|R'(rz)|}{|((f^n)'(R(rz)))^{-1}| \cdot |R'(g^n(rz))|} \le K.$$

By the bounded distortion theorem the rz in the denominator can be replaced z and n depends on r as described by (7.2.1) with r replaced by $1-r$.Now we proceed as in the proof of Lemma 7.2.1 replacing deviations of $S_n(\phi) - P(\phi)n + \kappa \log|(f^n)'(x)|$ by the deviations of $\log|(g^n)'(x)| - \log|(f^n)'(x)|$. The proof is finished. ♣

Section 7.5. Pressure versus integral means of the Riemann map

In this section we establish a close relation between integral means of derivatives of the Riemann map and topological pressure of the function $-t\log|f'|$. Given $t \in I\!R$ define

$$\beta(t) = \limsup_{r\to 1} \frac{\log \int_{S^1} |R'(rz)|^t dl(z)}{-\log(1-r)}$$

We shall prove the following.

Theorem 7.5.1. If the lifted map $g : S^1 \to S^1$ is of the form $z \mapsto z^d$, $d \ge 2$, then

$$\beta(t) = t - 1 + \frac{\mathrm{P}(f, -t\log|f'|)}{\log d}.$$

Proof. Fix $0 < r < 1$ and divide the circle S^1 into $[2\pi/(1-r)]$ arcs of length $1-r$ and one arc of length $\le 1-r$. Denote these arcs by $I_1, I_2, \dots, I_k$ and I_{k+1} respectively, where $k = [2\pi/(1-r)]$. Then

$$\int_{S^1} |R'(rz)|^t dl(z) = \sum_{j=1}^{k+1} \int_{I_j} |R'(rz)|^t dl(z) = \sum_{j=1}^{k+1} \int_{I_j} |R'(g^n(rz))|^t \frac{(|g^n)'(rz)|^t}{|(f^n)'(R(rz))|^t} dl(z).$$

Fix now $n = n(r)$ to be the first integer for which $|g^n(z)| < 1/2$. Note that n is independent of z and that there exists a constant $A \ge 1$ such that $A^{-1} \le |R'(w)| \le A$ for all $w \in B(0, 1/2)$. Hence

$$\int_{S^1} |R'(rz)|^t dl(z) \asymp \sum_{j=1}^{k+1} \int_{I_j} \frac{(|g^n)'(rz)|^t}{|(f^n)'(R(rz))|^t} dl(z). = \sum_{j=1}^{k+1} \int_{I_j} \frac{(d^{nt}}{|(f^n)'(R(rz))|^t} dl(z).$$

Now, by the Mean Value Theorem for every $j = 1, 2, \dots, k, k+1$ there exists $z_j \in I_j$ such that

$$\int_{I_j} \frac{(d^{nt}}{|(f^n)'(R(rz))|^t} dl(z) = l(I_j) \frac{(d^{nt}}{|(f^n)'(R(rz_j))|^t}.$$

Hence, as $l(I_j) = 1 - r$ for all $j = 1, \dots, k$

$$\log \int_{S^1} |R'(rz)|^t dl(z)$$
$$= \log(1-r) + \log\left(\sum_{j=1}^{k} \frac{(d^{nt}}{|(f^n)'(R(rz_j))|^t} + \frac{(d^{nt}}{|(f^n)'(R(rz_{k+1}))|^t} \cdot \frac{l(I_{k+1})^t}{(1-r)^t}\right) + O(1)$$
$$= \log(1-r) + nt\log d +$$
$$+ \log\left(\sum_{j=1}^{k} \exp\sum_{u=0}^{n-1} -t\log|f'(R(g^u(rz_j)))| + |(f^n)'(R(rz_j))|^t \frac{l(I_{k+1})^t}{(1-r)^t}\right) + O(1)$$

By our definition of n, $(1/2)^d \le r^{d^n} \le 1/2$; hence $d\log(1/2) \le d^n \log r \le \log(1/2)$. Since there exists a constant $B \ge 1$ such that $B^{-1}(1-r) \le -\log r \le B(1-r)$ for all r sufficiently close to 1, we get $B^{-1}\log 2 \le d^n(1-r) \le Bd\log 2$. Therefore $-\log B + \log\log 2 \le n\log d + \log(1-r) \le \log B + \log\log 2 + \log d$. Hence $n\log d - C \le -\log(1-r) \le n\log d + C$ for some universal constant C. Thus

$$\frac{\log\int_{S^1}|R'(rz)|^t dl(z)}{-\log(1-r)} = -1 + t +$$
$$\frac{1}{n\log d}\log\left(\sum_{j=1}^{k}\exp\sum_{u=0}^{n-1} -t\log|f'(R(g^u(rz_j)))| + \exp\sum_{u=0}^{n-1} -t\log|f'(R(g^u(rz_{k+1})))|\frac{l(I_{k+1})^t}{(1-r)^t}\right)$$
$$+ o(1).$$

Now, using the fact that $0 \le l(I_{k+1})/(1-r) \le 1$, it follows from the definition of pressure and the bounded distortion property that

$$\limsup_{r\to 1} \frac{\log\int_{S^1}|R'(rz)|^t dl(z)}{-\log(1-r)} = -1+t+\frac{1}{\log d}\mathrm{P}(g, -t\log|f'\circ R|) = -1+t+\frac{\mathrm{P}(f, -t\log|f'|)}{\log d}.$$

The proof is finished. ♣

Section 7.6 Harmonic measure.

In this section we keep the notation of the previous one. The measure $l \circ R^{-1}$, the image of the image of the Lebesgue measure l on S^1 under the Riemann map is said to be the harmonic measure of $\partial\Omega$ with respect to (or viewed from) the point z_0. Since all the Riemann maps differ by compositions with M̈bius transformations preserving t he unit circle, all the harmonic measures are strongly equivalent and corresponding Radon-Nikodyn derivatives are bounded away from zero and infinity. In particular all the harmonic measures induce the same strong measure class, which as long as we are only interested in metric properties of this class, enables us to speak generally about a harmonic measure without specifying the point z_0. For more information about harmonic measure we refer the reader to [?]. Our first aim is to represent harmonic measure as a Gibbs measure and then to apply the results of the previous section. Since the Hausdorff dimension of the circle S^1 is equal to 1, it follows from Theorem 7.1.1 that $P(g,\phi)=0$, where $\phi=-\log|g'|$. Of course the Lebesgue measure l on S^1 is equivalent with H_1. Since R is a topological conjugacy between g and f on S^1, $P(f,\phi\circ R^{-1})=P(g,\phi)=0$. Since moreover R^{-1} is a metric conjugacy between metric dynamical systems $(f,\mu_\phi\circ R^{-1})$ and (g,μ_ϕ), we therefore have

$$\mathrm{h}_{\mu_{\phi\circ R^{-1}}}(f)+\int\phi\circ R^{-1}\,d\mu_{\phi\circ R^{-1}}=\mathrm{h}_{\mu_\phi}(g)+\int\phi\,d\mu_\phi=0.$$

Hence, the uniqueness of an equilibrium state for $\phi\circ R^{-1}$ results in the following.

Theorem 7.6.2. The harmonic measure ω is the Gibbs state of the map $f:X\to X$ and the Hölder continuous potential $-\log|g'|\circ R^{-1}$.

We now want to argue that $\mathrm{HD}(\omega)=1$. This is a general result due to Makarov (see ???) true for any simply connected domain Ω with no dynamics involved. We shall however provide here a proof in the dynamical context only which is shorter and simpler than the general one.

Proposition 7.6.3. If ω is a harmonic measure on the boundary of a Jordan domain Ω, then $\chi_l(R)(z)=0$ for l-a.e. $z\in S^1$, where l denotes the Lebesgue measure on S^1.

Proof. Fix $z\in S^1$ and $0<r<1$. Then by the Koebe distortion theorem $2^{-1}(1-r)^{-3}\le|R'(z)|\le(1-r)^{-3}$ which implies that

$$2\le\frac{-\log|R'(rz)|}{\log(1-r)}\le 4$$

for all r sufficiently close to 1. Since the Gibbs state of the function $-\log|g'|$ is a g-invariant probability measure absolutely continuous with respect to the Lebesgue measure it follows from Theorem7.6.2 that $\chi_l(R)(z)$ is constant for l-a.e. $z\in S^1$. Hence

$$\begin{aligned}\chi_l(R)&=\int\chi_l(R)\,dl=\int\lim_{r\to1}\frac{\log|R'(rz)|}{-\log(1-r)}\,dl(z)\\&=\lim_{r\to1}\frac{1}{-\log(1-r)}\int\log|R'(rz)|\,dl(z).\end{aligned}$$

Since $\log|R'|$ is a harmonic function we can continue the above chain of equalities writing

$$\chi_l(R)=\lim_{r\to1}\frac{\log|R'(0)|}{-\log(1-r)}=0.$$

The proof is finished. ♣

Corollary 7.6.4. Suppose that an expanding map $f:\partial\Omega\to\partial\Omega$ is a conformal boundary repeller, and Ω is a Jordan domain. Then $\mathrm{HD}(\omega)=1$, for ω, the harmonic measure viewed from Ω.

proof. Let μ be the Gibbs state on S^1 corresponding to the potential $-\log|g'|$. Simultaneously μ is the unique probability g-invariant measure equivalent with the Lebesgue measure l on S^1. Then $\mu \circ R^{-1}$ is a probability f-invariant measure equivalent to ω. In view of Theorem7.6.2 and Lemma7.6.3, $\chi_{\mu\circ f^{-1}}(f) = \chi_\mu(g) > 0$. Since $R : S^1 \to \partial\Omega$is a topological conjugacy between $g : S^1 \to S^1$ and $f : \partial\Omega \to \partial\Omega$, $\mathrm{h}_{\mu\circ f^{-1}}(f) = \mathrm{h}_\mu(g)$. Thus

$$\frac{\mathrm{h}_{\mu\circ f^{-1}}(f)}{\chi_{\mu\circ f^{-1}(f)}} = \frac{\mathrm{h}_\mu(g)}{\chi_\mu(g)}.$$

Since $\mathrm{HD}(\mu) = 1$, an immediate application of Lemma 7.1.4 (Volume Lemma) finishes the proof. ♣

Theorem 7.6.5. Let $f : \partial\Omega \to \partial\Omega$ be a conformal boundary repeller, where Ω is a Jordan domain. Then either

(a) $\omega \asymp \mathrm{H}_1$ on $\partial\Omega$ (if $\log|g'|$ and $\log|f' \circ R|$ are cohomologous) or

(b) $\omega \perp \mathrm{H}_1$ (if $\log|g'|$ and $\log|f' \circ R|$ are not cohomologous), and then there exists $c_0 > 0$ such that with the gauge function $\phi_c(t) = t\exp(c\sqrt{\log(1/t)\log_3(1/t)})$,

$$\omega \perp \mathrm{H}_{\phi_c} \quad \text{for all} \quad 0 \le c < c_0$$

and

$$\omega \ll \mathrm{H}_{\phi_c} \quad \text{for all } c > c_0.$$

Proof. If $\log|g'|$ and $\log|f' \circ R|$ are cohomologous, then also the functions $-\log|g' \circ R^{-1}|$ and $-\log|f'|$ are cohomologous (with respect to the map $f : \partial\Omega \to \partial\Omega$). By Theorem 7.a.2 ω is a Gibbs state of the potential $-\log|g' \circ R^{-1}|$. Since $\mathrm{P}(f, -\log|f'|) = \mathrm{P}(f, -\log|g' \circ R^{-1}|) = \mathrm{P}(g, -\log|g'|) = 0$, hence, by Theorem 7.1.1, $\mathrm{HD}(\partial\Omega) = 1$, and the Gibbs states of the potential $-\log|f'|$ are equivalent to the 1-dimensional Hausdorff measure on $\partial\Omega$. So, the part (a) of Theorem 7.6.5 is proved.

Suppose now that $\log|g'|$ and $\log|f' \circ R|$ are not cohomologous. Then $-\log|g' \circ R^{-1}|$ and $-\log|f'|$ are not cohomologous. Let μ be the invariant Gibbs state of $-\log|g' \circ R^{-1}|$. By Corollary 7.6.4 and Theorem 7.a.2, $\mathrm{HD}(\mu) = 1$. Hence $\psi = -\log|g' \circ R^{-1}| + \mathrm{HD}(\mu)\log|f'| - \mathrm{P}(f, -\log|g' \circ R^{-1}|)$ is not a coboundary and, since by Theorem 7.6.2 and Lemma 7.6.3, $\int \log|f'|\,d\mu = \int \log|g' \circ R^{-1}|\,d\mu$, the potential ψ is not cohomologous to a constant. The second part of Theorem 7.6.6 is now an immediate consequence of Theorem 7.0.a. ♣

Theorem 7.6.6. If $f : \partial\Omega \to \partial\Omega$ is a conformal boundary repeller, Ω is a Jordan domain and the functions $\log|g'|$ and $\log|f'| \circ R$ are cohomologous, (this is exactly the case of Theorem 7.6.5(b)), then Ω is an analytic curve. If additionally f extends analytically onto $\overline{\mathbb{C}}$, then f is analytically conjugate with a Möbius transformation and $\partial\Omega$ is a geometric circle.

Proof. If the functions $\log|g'|$ and $\log|f'| \circ R$ are cohomologous, then according to Theorem 7.6.5(a) the boundary $\partial\Omega$ is a rectifiable Jordan curve. So, in view of Riesz's theorem, the map $R : S^1 \to \partial\Omega$ transports the measure class of the Lebesgue measure on S^1 onto the measure class of H_1 on $\partial\Omega$. Let now $R_* : S^1 \to \partial\Omega$ be the restriction to the unit circle of the Riemann map induced by the second component of the complement of $\partial\Omega$. For technical reason, which will be clear at the end of this proof we assume here that the Riemann map R_* is defined on $D^1_* = \{z : |z| > 1\}$, the complement of the closure of the unit disk D^1. Since $\partial\Omega$ is a Jordan curve and since f has no critical points on $\partial\Omega$, there exists an open neighborhood A_* of S^1 such that $f(A_* \cap \overline{\mathbb{C}} \setminus \partial\Omega)$ is well-defined and moreover $f(A_* \cap \overline{\mathbb{C}} \setminus \partial\Omega) \subset \overline{\mathbb{C}} \setminus \partial\Omega)$. Therefore, on D^1_* intersected with a sufficiently thin open annulus surronding S^1, we can define $g_* = R_*^{-1} \circ f \circ R_*$, the lift of f via the Riemann map R_*. Set

$$h = R_*^{-1} \circ R|_{S^1} : S^1 \to S^1.$$

Composing, if necessary, the Riemann maps R and R_* with appropriate rotations, we may assume that 1 is a fixed point of g and g_* and $h(1) = 1$. Our first objective is to demonstrate that h is real-analytic. Indeed, Let $\mu_1 = \phi_1 l$ and $\mu_2 = \phi_2 l$ be the (unique) probability m easures equivalent with the Lebesgue measure on the circle, respectively invariant under the action of g and g_*. Since, also by Riesz's theorem, the map $R_* : S^1 \to \partial\Omega$ transports the measure class of the Lebesgue measure on S^1 onto the measure class of H_1 on $\partial\Omega$, the homeomorphism $h : S^1 \to S^1$ sends the measure class of the Lebesgue measure on S^1 onto itself. Since h establishes conjugacy between g and g_*, it therefore maps the invariant measure μ_1 onto some probability g_*-ivariant measure equivalent with the Lebesgue measure. Since such a measure is unique, it must be equal to μ_2. Symbolically, $\mu_1 \circ h^{-1} = \mu_2$. Define now two functions $M, N : S^1 \to [0,1]$ by setting

$$M(z) = \mu_1([0,z]) = \int_1^z \phi_1 \, dl$$

and

$$N(z) = \mu_2([0,z]) = \int_1^z \phi_2 \, dl.$$

Since $\mu_1 \circ h^{-1} = \mu_2$ and $h(1) = 1$, the functions M and N are related by the formula $N(h(z)) = M(z)$. Since M and N are strictly increasing, we may solve the last equation for h to get $h = M \circ N^{-1}$. We are now done because the real analyticity of ϕ_1 and ϕ_2 (see ???) implies that the functions M and N are real-analytic and this in turn results in real analyticity of $h = M \circ N^{-1}$. Hence h extends to an analytic map $\tilde{h}$ defined on an open neighborhood H of S^1 in $\mathbb{C}$. Since $h : S^1 \to S^1$ preserves orientation, decreasing H if necessary, we get $\tilde{h}(H \cap \overline{D^1_*}) \subset \overline{D^1_*}$. Thus we produced two continuous maps $R : \overline{D^1} \to \mathbb{C}$ and $R_* \circ \tilde{h} : H \cap \overline{D^1_*} \to \mathbb{C}$ which are analytic on D^1 and $H \cap D^1_*$ respectively and which coincide on their common boundary, the unit circle S^1. Thus R and $R_* \circ \tilde{h}$ glue together to an analytic map $S : D^1 \cup H \to \overline{\mathbb{C}}$. So, since $S(S^1) = R(S^1) = \partial\Omega$, the proof of the first part is finished.

If now f extends analytically to $\overline{\mathbb{C}}$, that is if f is a rational function, then by the maximum principle $f(\Omega) \subset \Omega$, $f(\overline{\mathbb{C}} \setminus \overline{\Omega}) \subset \overline{\mathbb{C}} \setminus \overline{\Omega}$, and both maps g and g_* are defined on D^1 and D^1_* respectively. Since these maps are surjective, since they extend continuously to $\overline{D^1}$ and $\overline{D^1_*}$ respectively, and since they preserve the unit circle S^1, it follows from the Schwarz reflection principle that they extend analytically to $\overline{\mathbb{C}}$. Since g and g^* preserve S^1 (we use this fact second time) they must be finite Blaschke products. Since $f(\Omega) \subset \Omega$ and

$f(\overline{\mathbb{C}}\setminus\overline{\Omega}) \subset \overline{\mathbb{C}}\setminus\overline{\Omega}$, by the Montel theorem both Ω and $\overline{\mathbb{C}}\setminus\Omega$ are components of the Fatou set of f. And since $f|_{\partial\Omega}$ is expanding, both Ω and $\overline{\mathbb{C}}\setminus\overline{\Omega})$ are basins of immediate attraction to stable fixed points. Conjugating f if necessary with a Möbius transformation we may assume that this fixed point in D^1 is 0 and that one in D^1_* coincides with ∞. But every Blaschke product B preserving S^1 and having 0 as its fixed point, preserves the Lebesgue measure on S^1. In order to see it consider an arbitrary continuous function $\phi : S^1 \to \mathbb{R}$ and then its harmonic extension $\tilde{\phi} : D^1 \to \mathbb{R}$. Since $\phi \circ B$ is also harmonic, we have

$$\int_{S^1} \phi \circ B \, dl = \phi \circ B(0) = \phi(0) = \int_{S^1} \phi \, dl.$$

which means that B preserves the Lebesgue measure l. Thus, $\mu_1 = \mu_2 = l$ and consequently $M = N$ and h is the identity map on S^1. In conclusion $\tilde{h}$ is an identity map and R and $R_* = \tilde{h} \circ R_*$ coincide on a neighborhood of S^1. Thus R and R_* glue together to an analytic map $\tilde{R} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Since $\tilde{R}$ is injective it must be a Möbius transformation and $\tilde{R}^{-1} \circ f \circ R^{-1}$ is a finite Blaschke product. The proof is finished. ♣

Section 7.7. Geometric examples.

This last section of this chapter is devoted to explore applications of previous sections to geometric examples like the Koch's snowflake and Carleson's example. Following the idea of the proof of Theorem 7.2.1 and copeing with a biggere number of technicalities one can prove the following.

Theorem 7.7.1. Let Ω be a simply connected domain in $\mathbb{C}$ with $\partial\Omega$ a Jordan curve . Let ∂_j, $j = 1, 2 \ldots, k$ be a finite family of compact arcs in $\partial\Omega$ with pairwise disjoint interriors. Denote $\bigcup \partial_j$ by ∂ (we do not assume that this curve is connected. Assume that there exists a family of conformal maps f_j, $j = 1, \ldots, k$, (which may reverse the orientation on $\mathbb{C}$) on neighbourhoods U_j of ∂_j. For every j assume that $f_j(\Omega \cap U - J) \subset \Omega$, $|f'| > 1$ on U_j and

$$f_j(\partial\Omega \cap U_j) \subset \partial\Omega. \tag{7.7.1}$$

Assume also the Markov partition property: for every $j = 1, \ldots k$, $f_j(\partial_j) = \bigcup_{s \in I_j} \partial_s$ for some subset $I_j \subset \{1, 2, \ldots, k\}$. Consider the $k \times k$ matrix $A = A_{jk}$ where $A_{jk} = 1$ if $k \in I_j$ and $A_{jk} = 0$ if $k \notin I_j$. Then there exists a transition parameter $c(\omega, \partial)$ such that the claims of tTheorem 7.6.5 and 7.6.6 hold.

Example 1(*the snowflake*). To every side of an equilateral triangle, in the middle we glue from outside as small as three times. To every side of the resulting polygon we we glue again an equilateral triangle as small as three times and so on infinitely many times. The triangles do not overlap in this construction and the boundary of the resulting domain Ω is a Jordan curve.This Ω is called the Koch's snowflake. It was first describe by Helge Koch in 1904. Denote the curve in $\partial\Omega$ joining a point $x \in \partial\Omega$ to $y \in \partial\Omega$ in the clockwise direction just by xy. For every $\partial A_i A_{i+1(mod 12)} \subset \partial\Omega$, $i = 0, 1, \ldots, 11$, we consider its civering by the curves 12, 23, 45, 56 in Ω (see Fig.2). This covering together with the affine maps

$$
\begin{aligned}
&12,\ 34 \to 12 \text{ (preserving orientation on } \partial\Omega)\\
&23 \to 61 \text{ (reversing orientation)}\\
&56 \to 36 \text{ (preserving orientation)}\\
&45 \to 63 \text{ (reversing orientation)}
\end{aligned}
$$

gives a Markov partition of ∂_i satisfying the assumptions of Theorem 6.7.1. Since $\partial\Omega$ (and every its subcurve) is definitely not real-analytic ($\mathrm{HD}(\partial\Omega) = \log 4/\log 3$), the assertion of Theorem 6.7.1 is valid with $c(\omega, \partial_i) > 0$. We may denote $c(\omega, \partial_i)$ by $c(\omega)$ since it is independent of ∂_i by symmetry.

Example 2(*Carleson's domain*). We recall Carleson's construction from [Ca2]. We fix a broken line γ with the first and last segment lying in the same straight line in $I\!R^2$, with no other segments intersecting the segment $\overline{1, d-1}$ (see Fig. 3). Then we take a regular polygon Ω^1 with vertices $T_0, T_1, \ldots, T_n$ and glue to every side of it, from outside, the rescaled, not mirror reflected, curve γ so that the ends of the glued curve coincide with the ends of the side. The resulting curve bounds a second polygon Ω^2. Denote its vertices by $A_0, A_1, \ldots$ (Fig. 4). Then we glue again the rescaled γ to all sides of Ω^2 and a third order polygon Ω^3 with vertices $B_0, B_1, \ldots$. Then we bild Ω^4 with vertices $C_0, C_1, \ldots \Omega^5$ with $D_0, D_1, \ldots$ etc. Assume that there is no self-intersecting of the curves $\partial\Omega^n$ in this construction. Moreover assume that in the limit we obtain a Jordan curve $\mathcal{L} = \mathcal{L}(\Omega^1, \gamma) = \partial\Omega$. The natural Markov partition of each curve $T_i T_{i+1}$in $\mathcal{L}$ into curves $A_j A_{j+1}$ with $f(A_j A_{j+1}) = T_i T_{i+1}$, considered by Carleson does not satisfy the property (6.7.1) so we cannot succeed with it. Instead we proceed as follows: Define in an affine fashion

$$f(B_{d(j-1)+1} B_{dj-1}) = A_1 A_{d-1}$$

for every $j = 1, 2 \ldots, d$. Divide now every arc $B_{dj-1} A_j$ for $j = 1, 2 \ldots, d$ and $A_j B_{dj+1}$, $j = 1, 2 \ldots, d$ into curves with ends in the vertices of the polygon Ω^4 : $C^j \in B_{dj_1} A_j$, $\tilde{C}^j \in A_j B_{dj+1}$ respectively, the closest to $A_j (\neq A_j)$. Let for $j = 1, 2, \ldots, d-1$,

$$
\begin{aligned}
&f(C^j A_j) = B_{dj-1} A_j, f(B_{dj-1} C^j) = A_{d-1} B_{d^2-1},\\
&f(A_j \tilde{C}^j) = A_j B_{dj+1}, f(\tilde{C}^j B_{dj+1}) = B_1 A_1.
\end{aligned}
$$

This gives a transitive aperiodic Markov partition of $B_1 B_{d^2-1}$. We can consider instead of the broken line γ in the construction of Ω, the line $\gamma^{(2)}$, consisting of d^2 segments, which arises by glueing to every side of γ a rescaled γ. Consecutive gluing of the rescaled $\gamma^{(2)}$ to the polygon $\Omega^{(1)}$ gives consecutively Ω^3, Ω_5 etc. The same construction as above gives a Markov partition of $D_1 D_{d^2-1}$ in $T_i T_{i+1}$. By continuing this procedure we approximate $T_i T_{i+1}$, so from Theorem 6.7.1 and from symmetry we deduce that there exists a transition parameter $c(\omega)$ such that the assertion of Theorem 6.6.5(b) is satisfied. Observe that

Carleson's assumption that the broken line $1, 2, \ldots, d-1$ has no self-intersections has not been needed in these considerations. Also the assumption that $\Omega^{(1)}$ is a regular polygon can be omitted; one can prove that $c(\omega)$ doesnot depend on T_iT_{i+1} by considering a transitive, aperiodic Markov partition which involves all the sides of Ω^1 simultaneously.

Section 7.8. Multifractal analysis of Gibbs state. So far an aboundance of papers and some books on multifractal analysis have appeared written by physists as well as by mathematicians. In this section we deal with multifractal analysis of Gibbs states of Hölder continuous potentials following perhaps most closely the approach presented in [PW] although there are substantial differences as for example we do not use Markov partitions and symbolic representation of our system. We begin with a few general definitions. Let us suppose that X is a compact subset of some Euclidean space $I\!\!R^d$ and ν be a Borel probability measure on X. Given $x \in X$ we define the upper and lower pointwise dimension of ν at x by putting respectively

$$\overline{d}_\nu(x) = \limsup_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} \text{ and } \underline{d}_\nu(x) = \liminf_{r \to 0} \frac{\log \nu(B(x,r))}{\log r}.$$

If $\underline{d}_\nu(x) = \overline{d}_\nu(x)$, we call the common value the pointwise dimension of ν at x and we denote it by $d_\nu(x)$. The multifractal analysis is a description of the fine-scale geometry of the set X whose constituent components are the sets

$$K_\nu(\alpha) = \{x \in X : d_\nu(x) = \alpha\}.$$

The $f_\nu(a)$-spectrum for dimensions, the function which will mostly occupy our attention in this section is definedby

$$f_\nu(a) = \mathrm{HD}(K_\nu(\alpha)).$$

We obtain a natural decomposition of the set X as

$$X = \bigcup_{\alpha \ge 0} K_n u(\alpha) \cup \{x \in X : d_\nu(x) \text{ does not exist}\}.$$

It is important to emphasize that the union of the sets $K_n u(\alpha)$ may not be equal all of X. Shereshevsky in [Sh] showed that for the class C^2 Axiom A surface diffeomorphisms the set $X \setminus \bigcup_{\alpha \ge 0} X_\alpha$ is dense and has positive Hausdorff dimension for any Gibbs state ν. His proof can be modified for conformal expanding repellors. We begin our more technical considerations by fixing an invariant Gibbs state μ_ϕ corresponding to a Hölder continuous potential ϕ. Since we will be interesting in the measure μ_ϕ only, subtracting from $\mathrm{P}(\phi)$ its topological pressure we may assume that $\mathrm{P}(\phi) = 0$. We now define a two-parameter family of functions $\phi_{q,t} : X \to I\!\!R$, $q, t \in I\!\!R$ by setting

$$\phi_{q,t} = -t \log |f'| + q\phi.$$

We shall prove the following.

Lemma 7.8.1. For every $q \in I\!\!R$ there exists a unique $t = T(q)$ such that $\mathrm{P}(\phi_{q,t}) = 0$.

Proof. Since by (.), $\frac{\partial}{\partial t}\mathrm{P}(\phi_{q,t}) = -\int \log|f'| d\mu_{q,t}$, where $\mu_{q,t}$ is the equilibrium (Gibbs state) of the the function $\phi_{q,t}$, it follows that

$$-\log\lambda \le \frac{\partial}{\partial t}\mathrm{P}(\phi_{q,t} \le -\log\lambda_1,$$

where $1 < \lambda_1 = \inf_X |f'| \le \sup_X |f'| = \lambda$. Hence the continuous function $t \mapsto \mathrm{P}(\phi_{q,t})$ is strictly decreasing and moreover $\lim_{t\to-\infty}\mathrm{P}(\phi_{q,t}) = +\infty$ and $\lim_{t\to+\infty}\mathrm{P}(\phi_{q,t}) = -\infty$. Hence, using the Darboaux theorem we conclude that there exists a unique $t \in I\!R$ such that $\mathrm{P}(\phi_{q,t}) = 0$.The proof is complete. ♣

Let k be a strictly convex map on an interval I, hence $k'' > 0$ for all $x \in I$. The Legendre transform of k is the function g of a new variable p defined by

$$g(p) = \max_I\{px - k(x)\}$$

everywhere where the maximum exists. it can be proved that the domain of g is either a point, an interval or a semiline. It is also easy to show that g is strictly convex and that the Legendre transform is involutive. We then say that the functions f and g form a Legendre transform pair. The following theorem (see [Ro]) gives a useful characterization of a Legendre transform pair.

Theorem 7.8.2. Two strictly convex functions k and g form a Legendre transform pair if and only if $g(-k'(q)) = k(q) - qk'(q)$

We are now in position to formulate our main theorem in this section.

Theorem 7.8.3.
(a) The pointwise dimension $d_{\mu_\phi}(x)$ exists for μ_ϕ-almost every $x \in X$ and

$$d_{\mu_\phi}(x) = \frac{-\int \phi d\mu_\phi}{\int \log|f'|d\mu_\phi}.$$

(b) The function $q \mapsto T(q)$, $q \in I\!R$ is real analytic, $T(0) = \mathrm{HD}(X)$, $T'(q) < 0$ and $T''(q) \ge 0$.
(c) The function $-T'(q)$ attains values in an interval (α_1, α_2) where $0 \le \alpha_1 \le \alpha_2 < \infty$.
(d) For every $q \in I\!R$, $f_{\mu_\phi}(-T'(q)) = T(q) - qT'(q)$.
(e) If the measures μ_ϕ and the invariant Gibbs state equivalent to the HD-dimensional Hausdorff measure (see Theorem 7.1.1) do not coincide, then the functions $f_{\mu_\phi}(\alpha)$ and $T(q)$ form a Legendre transform pair.

The proof of this theorem will consist of several lemmas and observations. First notic that since $\mathrm{P}(\phi) = 0$, the part (a) is an immediate consequence of Lemma 7.1.4. We shall now prove the part (b). To this end we consider the function $c : I\!R^2 \to C^\theta(X)$, defined by the formula $c(t,q) = \phi_{q,t} = -t\log|f'| + q\phi$, where θ is a Hölder exponent of the function ϕ. Since by a result of Ruelle (see [Ru]) the topological pressure $\mathrm{P} : C^\theta \to I\!R$ is a real analytic function, the real analyticity of $T(q)$ follows immediately from the Implicite Function Theorem once we verify the non-degeneracy assumption. Due to (.) the latter is

$$\frac{\partial \mathrm{P}(c(t,q))}{\partial t}|_{(t_0,q_0)} = -\int_X \log|f'| d\mu_{q_0,t_0},$$

where μ_{q_0,t_0} is the invariant Gibbs state of the function ϕ_{q_0,t_0}. Differentiating with respect to q the equatlity $\mathrm{P}(c(T(q),q)) = 0$ we obtain

$$0 = \frac{\partial \mathrm{P}(c(t,q))}{\partial t}|_{(t(q),q)} \cdot T'(q) + \frac{\partial \mathrm{P}(c(t,q))}{\partial q}|_{(t(q),q)} = -\int \log|f'| d\mu_q \cdot T'(q) + \int \phi d\mu_q,$$

where μ_q is the equilibrium state of the potential $\phi_q = -T(q)\log|f'| + q\phi$. Since $\mathrm{P}(\phi) = 0$, using the variational principle we therefore obtain

$$T'(q) = \frac{\int \phi d\mu_q}{\int \log|f'| d\mu_q} \le \frac{-\mathrm{h}_{\mu_q(f)}}{\int \log|f'| d\mu_q} < 0. \tag{7.8.1}$$

The equality $T(0) = \mathrm{HD}(X)$ is just Corollary 7.1.2. The inequality $T''(q) \ge 0$ is contained in the following.

Lemma 7.8.4. The function $q \mapsto T(q)$ is convex. It is not strictly convex if and only if μ_ϕ is equal to the invariant Gibbs state equivalent to the HD-dimensional Hausdorff measure (see Theorem 7.1.1).

Proof. Differentiating the formula

$$0 = \frac{\partial \mathrm{P}(c(t,q))}{\partial t}|_{(t(q),q)} \cdot T'(q) + \frac{\partial \mathrm{P}(c(t,q))}{\partial q}|_{(t(q),q)}$$

we obtain

$$\begin{aligned} T''(q) &= -\frac{T'(q)^2 \frac{\partial^2 \mathrm{P}(\phi_{q,t})}{\partial t^2} + 2T'(q)\frac{\partial^2 \mathrm{P}(\phi_{q,t})}{\partial q \partial t} + \frac{\partial^2 \mathrm{P}(\phi_{q,t}}{\partial q^2}}{\frac{\partial \mathrm{P}(\phi_{q,t})}{\partial t}} \\ &= \frac{T'(q)^2 \frac{\partial^2 \mathrm{P}(\phi_{q,t})}{\partial t^2} + 2T'(q)\frac{\partial^2 \mathrm{P}(\phi_{q,t})}{\partial q \partial t} + \frac{\partial^2 \mathrm{P}(\phi_{q,t}}{\partial q^2}}{\chi_{\mu_q}}, \end{aligned}$$

where $\chi_{\mu_q} = \int \log|f'| d\mu_q$ is the Lyapunov charatceristic exponent of the measure μ_q. Invoking () we see that

$$\frac{\partial^2 \mathrm{P}}{\partial t^2} = \sigma^2_{\mu_q}(-\log|f'|), \ \frac{\partial^2 \mathrm{P}}{\partial q \partial t} = \sigma^2_{\mu_q}(\log|f'|, \phi) \ \frac{\partial^2 \mathrm{P}}{\partial q^2} = \sigma^2_{\mu_q}(\phi),$$

where, using notation, $\int g d\mu_q = \mu_q(g)$

$$\sigma^2_{\mu_q}(g,h) = \sum_{k=0}^\infty \left(\mu_q(g \cdot h \circ \sigma^k) - \mu_q(g)\mu_q(h)\right) = \sum_{k=0}^\infty \left(\mu_q(h \cdot g \circ \sigma^k) - \mu_q(g)\mu_q(h)\right).$$

Hence, we can write

$$T'(q)^2\frac{\partial^2 \mathrm{P}}{\partial t^2}+2T'(q)\frac{\partial^2 \mathrm{P}}{\partial q\partial t}+\frac{\partial^2 \mathrm{P}}{\partial q^2}=$$

$$=T'(q)^2\sum_{k=0}^{\infty}\bigl(\mu_q(\log|f'|\cdot\log|f'|\circ\sigma^k)-\chi_{\mu_q}^2\bigr)+T'(q)\sum_{k=0}^{\infty}\bigl(\mu_q(-\log|f'|\cdot\phi\circ\sigma^k)$$

$$+\chi_{\mu_q}\mu_q(\phi)\bigr)+T'(q)\sum_{k=0}^{\infty}\bigl(\mu_q(\phi(-\log|f'|\circ\sigma^k))+\chi_{\mu_q}\mu_q(\phi)\bigr)+\sum_{k=0}^{\infty}\bigl(\mu_q(\phi\cdot\phi\circ\sigma^k)-\mu_q(\phi)^2\bigr)$$

$$=\sum_{k=0}^{\infty}\mu_q\bigl(-T'(q)\log|f'|(-T'(q)\log|f'|\circ\sigma^k+\phi\circ\sigma^k)\bigr)$$

$$+\sum_{k=0}^{\infty}\mu_q\bigl(\phi(-T'(q)\log|f'|(-T'(q)\log|f'|\circ\sigma^k+\phi\circ\sigma^k)\bigr)-\sum_{k=0}^{\infty}\bigl(-T'(q)\chi_{\mu_q}+\mu_q(\phi)\bigr)^2$$

$$=\sum_{k=0}^{\infty}\mu_q\bigl((-T'(q)\log|f'|+\phi)(-T'(q)\log|f'|+\phi)\circ\sigma^k-(-T'(q)\chi_{\mu_q}+\mu_q(\phi))^2$$

$$=\sigma_{\mu_q}^2(-T'(q)\log|f'|+\phi).$$

In view of () $sg_{\mu_q}^2(-T'(q)\log|f'|+\phi)\ge 0$ and $sg_{\mu_q}^2(-T'(q)\log|f'|+\phi)=0$ if and only if the function $-T'(q)\log|f'|+\phi$ is cohomologous to a constant, say to a. It follows then from (7.8.1) that $a=\int a d\mu_q=\int(-T'(q)\log|f'|+\phi)d\mu_q=0$. Therefore $T'(q)\log|f'|$ is cohomologous to ϕ and, as $\mathrm{P}(\phi)=0$, also $\mathrm{P}(T'(q)\log|f'|)=0$. Thus, by Corollary 7.1.2 and Theorem 7.1.1, $T'(q)=-\mathrm{HD}(X)$ and consequently ϕ is cohomologous to the function $-\mathrm{HD}(X)\log|f'|$. This implies that $\mu_\phi=\mu_{-\mathrm{HD}(X)\log|f'|}$, the latter being the equilibrium (invarinat Gibbs) state of the potential $-\mathrm{HD}(X)\log|f'|$. So, applying Corollary 7.1.2 and Theorem 7.1.1 again finishes the proof. ♣

So, the part (b) is proved. Now, (b) implies that $-T'(q)>0$ and it follows from (7.8.1) that

$$-T'(q)=\frac{-\int\phi d\mu_q}{\int\log|f'|d\mu_q}\le\frac{\sup(-\phi)}{\inf|f'|}\le\sup(-\phi).$$

So, the part (c) is also proved. In order to part (d), given $\alpha\ge 0$ define

$$\tilde K(\alpha)=\left\{x\in X:\lim_{n\to\infty}\frac{-\sum_{j=0}^{n-1}\phi\circ f^j(x)}{\sum_{j=0}^{n-1}\log|f'(f^j(x))}=\alpha\right\}.$$

We shall prove the following.

Lemma 7.8.5. For every $\alpha\ge 0$, $\tilde K(\alpha)=K_{\mu_\phi}(\alpha)$.

Proof. Since the measures μ_ϕ and m_ϕ are equivalent with bounded Radon-Nikodym derivatives $K_{\mu_\phi}(\alpha)=K_{m_\phi}(\alpha)$. Thus, it suffices to show that $\tilde K(\alpha)=K_{m_\phi}(\alpha)$. As in the previous section let $\lambda=\inf|f'|>1$. Recall that then there exists $\tau>0$ small enough that firstly, $f\mid_A$ is one - to - one for all sets $A \subset X$ with $\mathrm{diam}(A) \leq \tau$ and secondly, $d(f(y), f(z)) \geq \lambda d(y, z)$ if $d(y, z) < \tau$. Fix $x \in X$ and $r > 0$ such that $r < \tau$ Define $n = n(x, r)$ to be the least number such that $\mathrm{diam}(f^n(B(x, r))) \geq \tau$. Since $f : X \to X$ is topologically exact, $n(x, r)$ is finite. By definition of n, $\mathrm{diam}(f^j(B(x, r))) < \tau$ for all $j = 0, 1, ...n - 1$. Therefore $f^n\mid_{(B(x,r))}$ is one - to - one and

$$m_\phi(f^n(B(x, r))) = \int_{B(x,r)} \exp(-S_n\phi(z))dm_\phi(z),$$

Since the function ϕ is Hölder continuous, it follows from ??? that there exists a constant $K > 0$ such that $|S_n\phi(z) - S_n\phi(x)| < K$ for all $x \in X$, $z \in B(x, r)$, and $n = n(x, r)$. Hence, we get

$$e^{-K} \exp(P - S_n\phi(x))m_\phi(B(x, r)) \leq m_\phi(f^n(B(x, r))) \leq e^K \exp(-S_n\phi(x))m_\phi(B(x, r))$$

or equivalently
(7.8.2)
$$e^{-K} \exp(S_n\phi(x))m_\phi(f^n(B(x, r))) \leq m_\phi(B(x, r)) \leq e^K \exp(S_n\phi(x))m_\phi(f^n(B(x, r))).$$

Similarly, since $\log |f'(z)|$ is a Hölder continuous function, there exists a constant $K_1 > 0$ such that

$$|S_n(\log |f'(z)|) - S_n(\log |f'(x)|)| \leq K_1$$

for all $x \in X$, $z \in B(x, r)$, and $n = n(x, r)$. Then

$$\frac{|(f^n)'(z)|}{|(f^n)'(x)|} \leq e^{K_1} \tag{7.8.3}$$

Therefore

$$\begin{aligned} \mathrm{diam}(B(x, r))e^{-K_1}|(f^n)'(x)| &\leq \mathrm{diam}(f^n(B(x, r))) \\ &\leq \mathrm{diam}(B(x, r))|(f^n)'(x)|e^{K_1}. \end{aligned}$$

Hence, $\mathrm{diam}(B(x, r))e^{-K_1}|(f^n)'(x)| \leq \tau\|f'\|$, or equivalently $r \leq \frac{1}{2}e^{K_1}|(f^n)'(x)|^{-1}\tau\|f'\|$. Also, since $\mathrm{diam}(f^n(B(x, r))) \geq \tau$ we get similarly $r \geq \frac{1}{2}e^{-K_1}|(f^n)'(x)|^{-1}\tau$. In conclusion,

$$\frac{1}{2}e^{-K_1}|(f^n)'(x)|^{-1}\tau \leq r \leq \frac{1}{2}e^{K_1}|(f^n)'(x)|^{-1}\tau\|f'\|. \tag{7.8.4}$$

Using the Bounded Distortion Theorem we also conclude the existence of some constant $K_2 > 0$ such that

$$f^n(B(x, r)) \supset B\big(f^n(x), K_2^{-1}r|(f^n)'(x)|\big) \supset B\big(f^n(x), K_2^{-1}\tau\mathrm{e}^{-K_1}\big).$$

Since the measure is positive on non-empty open sets $Q_\phi = \inf\{m_\phi(B(z, K_2^{-1}\tau\mathrm{e}^{-K_1} : z \in X\} > 0$. Hence, it follows from (7.8.2) that

$$Q_\phi\mathrm{e}^{-K} \exp\big(S_n\phi(x)\big) \leq m_\phi(B(x, r)) \leq i\mathrm{e}^K \exp\big(S_n\phi(x)\big)$$

Using this and (7.8.4) we conclude that

$$\text{(7.8.5)} \quad \frac{K + S_n\phi(x)}{-K_1 - \log 2 - S_n(\log|f'|)(x)} \le \frac{\log\big(m_\phi(B(x,r))\big)}{\log r} \le \frac{-K + \log Q_\phi + S_n\phi(x)}{K_1 - \log 2 - S_n(\log|f'|)(x)},$$

where $n = n(x,r)$. Thus, if $x \in |^K(\alpha)$, then $x \in K_{m_\phi}(\alpha)$ since $\lim_{r\to 0} n(x,r) = \infty$. Since the range of the function $r \mapsto n(x,r)$ contains all the positive integers, (7.8.5) also implies that if $x \in K_{m_\phi}(\alpha)$ then $x \in |^K(\alpha)$. The proof is complete. ♣

Now, we shall prove the following.

Lemma 7.8.6. For every $q \in I\!R$, $\mu_q(K_{\mu_\phi}(-T'(q)) = 1$ and $f_{\mu_\phi}(-T'(q)) = T(q) - qT'(q)$.
Proof. The proof of the first part of this lemma follows immediately from Lemma 7.7.4, (7.8.1) and the Birkhoff ergodic theorem. Let $m_q = m_{\phi_q}$. Since (7.8.5) has been obtained for an arbitrary Hölder continuous potential ϕ writing $\phi_q = -T(q)\log|f'| + q\phi$, we get

$$\begin{aligned} \frac{A_1 - T(q)S_n(\log|f'|)(x) + qS_n\phi(x)}{A_2 - S_n(\log|f'|)(x)} &\le \frac{\log\big(m_q(B(x,r))\big)}{\log r} \\ \text{(7.8.5)} \qquad\qquad &\le \frac{A_3 - T(q)S_n(\log|f'|)(x) + qS_n\phi(x) + S_n\phi(x)}{A_4 - S_n(\log|f'|)(x)}, \end{aligned}$$

for all$x \in X$, all $r > 0$ and some constants $A_1 - A_4 \in I\!R$, where we remaind that $n = n(x,r)$ was defined in the proof of Lemma 7.8.4. This formula immediately implies that for all $x \in K_{m_\phi}(-T'(q)) = \tilde{K}(-T'(q))$, the equality sign written due to Lemma 7.8.4, we have $d_{m_q}(x) = T(q) - qT'(q)$. Looking and this and the equality $m_q(K_{\mu_\phi}(-T'(q)) = 1$ (as m_q and μ_q are equivalent with bounded Radon-Nikodym derivatives), we conclude from Theorem 5.?.? that

$$f_{\mu_\phi}(-T'(q)) = \mathrm{HD}(K_{\mu_\phi}(-T'(q))) = T(q) - qT'(q).$$

So, the proof of part (d) is also finished.

If now the three measures μ_ϕ, and $\mu_{-\mathrm{HD}\log|f'|}$ do not coincide, then $T(q)$ is strictly convex and due to (d) and Theorem 7.8.2 the part (f) is proved. This also shows that the function $-T'(q)$ is strictly decreasing, let (α_1, α_2) denote its range (comp. (c)) and let $\beta : (\alpha_1, \alpha_2) \to I\!R$ be the function inverse to $-T'(q)$ which is also real analytic. Then it follows from lemma 7.8.6 that the function $f_{\mu_\phi}(\alpha) = T(\beta(a)) - \alpha\beta(\alpha)$, $\alpha \in (\alpha_1, \alpha_2)$ is also real analytic and the part (e) is proved. Thus the proof of Theorem 7.8.3 is complete. ♣

Section 7.9. Real analyticity of the density functions. In this section we consider potentials of the form $-t\log|f'|$, fixed points of the corresponding conjugate transfer operators m_t and invariant Gibbs states μ_t. Our aim is to show that the Radon-Nikodym derivative $\frac{d\mu_t}{dm_t}$ has a real-analytic extension. We begin with the following.

Definition 7.9.1. A conformal expanding repeller $f : X \to X$ is said to be real-analytic if it is contained in a finite union of pairwise disjoint real-analytic curves which will be denoted by $\Gamma = \Gamma_f$. Frequently in such a context we will alternatively speak about real analyticity of the set X.

The main (and only) result of this section is the following.

Theorem 7.9.2. If $f : X \to X$ is an orientation preserving conformal expanding repeller, $X \subset C$, then the Radon-Nikodym derivative $\rho = d\mu_t/dm_t$ has a real-analytic real-valued extension on a neighbourhood of X in $\mathbb{C}$. If f is real-analytic, then ρ has a real-analytic extension on a neighbourhood of X in Γ.

Proof. Observe that since f is conformal and orientation preserving, f is holomorphic on a neighbourhood of X in $\mathbb{C}$. Take $r > 0$ so small that for every $x \in X$, every $n \geq 1$ and every $y \in f^{-n}(x)$ the holomorphic inverse branch $f_y^{-n} : B(x, 2r) \to \mathbb{C}$ sending x to y is well-defined. Suppose first that f is real-analytic. We need to show that there exists a holomorphic complex-valued extension of ρ on a neighbourhood of X in $\mathbb{C}$. Taking an appropriate atlas we may assume that X is contained in a real axis (if a closed curve is a component of Γ we can use Arg). For all $k \geq 1$ and all $y \in f^{-k}(x)$ let $\nu(k, y) = 1$ or -1 depending as f_y^{-k} preserves or reverses the orientation on Γ. So

$$|(f_y^{-k})'(z)| = \nu(k, y)((f_y^{-k})'(z))$$

for all $z \in J(f) \cap B(x, r)$. Consider the following sequence of complex analytic functions on $z \in B(x, r)$

$$g_n(z) = \sum_{y \in f^{-n}(x)} \Big(\nu(n, y)((f_y^{-n})'(z))\Big)^t \exp(-nP(t)).$$

There is no problem here with raising to the t-th power since $B(x, r)$, the domain of all $\nu(n, y)(f_y^{-n})'$ is simply connected. Since the latter functions are positive in $\mathbb{R}$, we can choose the branches of the t-th powers to be also positive in $\mathbb{R}$. By Koebe's Distortion Theorem for every $z \in B(x, r/2)$, every $n \geq 1$ and every $y \in f^{-n}(x)$ we have $|(f_y^{-n})'(z)| \leq K|(f_y^{-n})'(x)|$. Hence $|g_n(z)| \leq K g_n(x)$. Since, by (3.4.2) with $u = 1$ and $c = p(t)$, the sequence $g_n(x)$ converges, we see that the functions $\{g_n|_{B(x,r/2)}\}_{n \geq 1}$ are uniformly bounded. So they form a normal family in the sense of Montel. Since $g_n(z)$ converges for all $z \in X \cap B(x, r/2)$, it follows that g_n converges to an analytic function g on $B(x, r/2)$ whose restriction to Γ is by our construction an extension of ρ.

Let us pass now to the proof of the first part of this proposition. That is, we relax the Julia real analyticity assumption and we want to construct a real-analytic real-valued extension of ρ to a neighbourhood of X in $\mathbb{C}$. Our strategy is to work in $\mathbb{C}^2$, to use an appropriate version of Montel's theorem and, in general, to proceed similarly as in the first part of the proof. So, fix $v \in X$. Identify now $\mathbb{C}$, where our f acts, to $\mathbb{R}^2$ with coordinates x, y, the real and complex part of z. Embed this into $\mathbb{C}^2$ with x, y complex. Denote the above

$\mathbb{C} = \mathbb{R}^2$ by $\mathbb{C}_0$. We may assume that $v = 0$ in $\mathbb{C}_0$. Given $k \geq 0$ and $v_k \in f^{-k}(v)$ define the function $\rho_{v_k} : B_{\mathbb{C}_0}(0, 2r) \to \mathbb{C}$ (the ball in $\mathbb{C}_0$) by setting

$$\rho_{v_k}(z) = \frac{(f_{v_k}^{-k})'(z)}{(f_{v_k}^{-k})'(0)},$$

Since $B_{\mathbb{C}_0}(0, 2r) \subset \mathbb{C}_0$ is simply connected and ρ_{v_n} nowhere vanishes, all the branches of logarithm $\log \rho_{v_n}$ are well defined on $B_{\mathbb{C}_0}(0, 2r)$. Choose this branch that maps 0 to 0 and denote it also by $\log \rho_{v_n}$. By Koebe's Distortion Theorem $|\rho_{v_k}|$ and $|\mathrm{Arg}\rho_{v_k}|$ are bouned on $B(0, r)$ by universal constants K_1, K_2 respectively. Hence $|\log \rho_{v_k}| \leq K = (\log K_1) + K_2$. We write

$$\log \rho_{v_k} = \sum_{m=0}^{\infty} a_m z^m$$

and note that by Cauchy's inequalities

$$|a_m| \leq K/r^m. \tag{7.9.1}$$

We can write for $z = x + iy$ in $\mathbb{C}_0$

$$\mathrm{Re} \log \rho_{v_k} = \mathrm{Re} \sum_{m=0}^{\infty} a_m (x + iy)^m = \sum_{p,q=0}^{\infty} \mathrm{Re}\Big(a_{p+q}\binom{p+q}{q} i^q\Big) x^p y^q := \sum c_{p,q} x^p y^q.$$

In view of (2.1), we can estimate $|c_{p,q}| \leq |a_{p+q}| 2^{p+q} \leq K r^{-(p+q)} 2^{p+q}$. Hence $\mathrm{Re} \log \rho_{v_k}$ extends, by the same power series expansion $\sum c_{p,q} x^p y^q$, to the polydisc $D_{\mathbb{C}^2}(0, r/2)$ and its absolute value is bounded there from above by K. Now for every $k \geq 0$ consider a real-analytic function b_k on $B_{\mathbb{C}_0}(0, 2r)$ by setting

$$b_k(z) = \sum_{v_k \in f^{-k}(0)} |(f_{v_k}^{-k})'(z)|^t \exp(-kP(t)).$$

By (3.4.2) the sequence $b_k(0)$ is bounded from above by a constant L. Each function b_k extends to the function

$$B_k(z) = \sum_{v_k \in f^{-k}(0)} |(f_{v_k}^{-k})'(0)|^t \mathrm{e}^{t \mathrm{Re} \log \rho_{v_k}(z)} \exp(-kP(t)).$$

whose domain, similarly as the domains of the functions $\mathrm{Re} \log \rho_{v_k}$, contains the polydisc $D_{\mathbb{C}^2}(0, r/2)$. Finally we get for all $k \geq 0$ and all $z \in D_{\mathbb{C}^2}(0, r/4)$

$$\begin{aligned}
|B_k(z)| &= \sum_{v_k \in f^{-k}(0)} |(f_{v_k}^{-k})'(0)|^t \mathrm{e}^{\mathrm{Re}(t \mathrm{Re} \log \rho_{v_k}(z))} \exp(-kP(t)) \\
&\leq \sum_{v_k \in f^{-k}(0)} |(f_{v_k}^{-k})'(0)|^t \mathrm{e}^{t |\mathrm{Re} \log \rho_{v_k}(z)|} \exp(-kP(t)) \\
&\leq \mathrm{e}^{Kt} \sum_{v_k \in f^{-k}(0)} |(f_{v_k}^{-k})'(0)|^t \exp(-kP(t)) \leq \mathrm{e}^{Kt} L.
\end{aligned}$$

Now by Cauchy's integral formula (in $D_{\mathbb{C}^2}(0, r/4)$) for the second derivatives we prove that the family B_n is equicontinuous on, say, $D_{\mathbb{C}^2}(0, r/5)$. Hence we can choose a uniformly convergent subsequence and the limit function G is complex analytic and extends ρ on $X \cap B(0, r/5)$, by (3.4.2). Thus we have proved that ρ extends to a complex analytic function in a neighbourhood of every $v \in X$ in $\mathbb{C}^2$, i.e. real analytic in $\mathbb{C}_0$. These extensions coincide on the intersections of the neighbourhoods, otherwise X is real analytic and we are in the case considered at the beginning of the proof. ♣