CHAPTER 6

FRACTAL DIMENSIONS

In the first section of this chapter we we provide a more complete treatment of outer measure indirectly begun in Chapter 1. The rest of this chapter is devoted to present basic definitions of pressure related to Hausdorff and packing measures, Hausdorff and packing dimensions of sets and measures and ball-counting dimensions.

§6.1 OUTER MEASURES

In Section 1.1 we have introduced the abstract notion of measure. In the beginning of this section we want to show how to construct measures starting with functions of sets called *outer measures* which are required to satisfy much weaker conditions. Our exposition of this material is brief and the reader should find its complete treatment in all handbooks of geometric measure theory (see for ex. [Falconer, 1985], [Ma] or [Pe]).

Definition 6.1.1 An *outer measure* on a set X is a function μ defined on all subsets of X taking values in $[0,\infty]$ such that

$$\mu(\emptyset)=0, \tag{6.1.1}$$

$$\mu(A)\le\mu(B) \text{ if } A\subset B \tag{6.1.2}$$

and

$$\mu\Big(\bigcup_{n=1}^\infty A_n\Big)\le\sum_{n=1}^\infty\mu(A_n) \tag{6.1.3}$$

for any countable family $\{A_n : n=1,2,\ldots\}$ of subsets of X.

A subset A of X is called *μ-measurable* or simply *measurable with respect to the outer measure* μ if and only if

$$\mu(B)\ge\mu(B\cap A)+\mu(B\setminus A) \tag{6.1.4}$$

for all sets $B\subset X$. Check that the opposite inequality follows immediately from (6.1.3). Check also that if $\mu(A)=0$ then A is μ-measurable.

Theorem 6.1.2. If μ is an outer measure on X, then the family $\mathcal{F}$ of all μ-measurable sets is a σ-algebra and the restriction of μ to $\mathcal{F}$ is a measure.

Proof. Obviously $X \in \mathcal{F}$. By symmetry of (6.1.4), $A \in \mathcal{F}$ if and only if $A^c \in \mathcal{F}$. So, the conditions (1.1.1) and (1.1.2) of the definition of σ-algebra are satisfied. To check condition (1.2.3) that $\mathcal{F}$ is closed under countable union, suppose that $A_1, A_2, \ldots \in \mathcal{F}$ and let $B \subset X$ be any set. Applying (6.1.4) in turn to $A_1, A_2, \ldots$ we get for all $k \geq 1$

$$\begin{aligned}
\mu(A) &\geq \mu(B \cap A_1) + \mu(B \setminus A_1) \\
&\geq \mu(B \cap A_1) + \mu((B \setminus A_1) \cap A_2) + \mu(B \setminus A_1 \setminus A_2) \\
&\geq \ldots \\
&\geq \sum_{j=1}^{k} \mu\left((B \setminus \bigcup_{i=1}^{j-1} A_i) \cap \mathcal{A}_j \right) + \mu(B \setminus \bigcup_{j=1}^{k} A_j) \\
&\geq \sum_{j=1}^{k} \mu\left((B \setminus \bigcup_{i=1}^{j-1} A_i) \cap \mathcal{A}_j \right) + \mu(B \setminus \bigcup_{j=1}^{\infty} A_j)
\end{aligned}$$

and therefore

$$\mu(A) \geq \sum_{j=1}^{\infty} \mu\big((B \setminus \bigcup_{i=1}^{j-1} A_i) \cap \mathcal{A}_j\big) + \mu\big(B \setminus \bigcup_{j=1}^{\infty} A_j\big) \tag{6.1.5}$$

Since

$$B \cap \bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} (B \setminus \bigcup_{i=1}^{j-1} A_i) \cap A_j$$

using (6.1.3) we thus get

$$\mu(A) \geq \mu\big(\bigcup_{j=1}^{\infty} (B \setminus \bigcup_{i=1}^{j-1} A_i) \cap A_j\big) + \mu\big(B \setminus \bigcup_{j=1}^{\infty} A_j\big)$$

Hence condition (1.1.3) is also satisfied and $\mathcal{F}$ is a σ-algebra. To see that μ is a measure on $\mathcal{F}$ i.e. that condition (1.1.4) is satisfied, consider mutually disjoint sets $A_1, A_2, \ldots \in \mathcal{F}$ and apply (6.1.5) with $B = \bigcup_{j=1}^{\infty} A_j$ to get

$$\mu\big(\bigcup_{j=1}^{\infty} A_j\big) \geq \sum_{j=1}^{\infty} \mu(A_j)$$

Combining this with (6.1.3) we conclude that μ is a measure on $\mathcal{F}$. ♣

Now, let (X, ρ) be a metric space. An outer measure μ on X is said to be a *metric outer measure* if and only if

$$\mu(A \cup B) = \mu(A) + \mu(B) \tag{6.1.6}$$

for all *positively separated* sets $A, B \subset X$ that is satisfying the following condition

$$\rho(A,B) = \inf\{\rho(x,y) : x \in A, y \in B\} > 0$$

Recall that the Borel σ-algebra on X is that generated by open, or equivalently closed, sets. We want to show that if μ is a metric outer measure then the family of all μ-measurable sets contains this σ-algebra. The proof is based on the following version of Carathéodory's lemma.

Lemma 6.1.3. Let μ be a metric outer measure on (X, ρ). Let $\{A_n : n = 1, 2, \ldots\}$ be an increasing sequence of subsets of X and denote $A = \bigcup_{n=1}^{\infty} A_n$. If $\rho(A_n, A \setminus A_{n+1}) > 0$ for all $n \geq 1$, then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

Proof. By (6.1.3) it is enough to show that

$$\mu(A) \leq \lim_{n\to\infty} \mu(A_n) \tag{6.1.7}$$

If $\lim_{n\to\infty} \mu(A_n) = \infty$, there is nothing to prove. So, suppose that

$$\lim_{n\to\infty} \mu(A_n) = \sup_n \mu(A_n) < \infty \tag{6.1.8}$$

Let $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. If $n \geq m+2$, then $B_m \subset A_m$ and $B_n \subset A \setminus A_{n-1} \subset A \setminus A_{m+1}$. Thus B_m and B_n are positively separated and applying (6.1.6) we get for every $j \geq 1$

$$\mu\Big(\bigcup_{i=1}^{j} B_{2i-1}\Big) = \sum_{i=1}^{j} \mu(B_{2i-1}) \quad \text{and} \quad \mu\Big(\bigcup_{i=1}^{j} B_{2i}\Big) = \sum_{i=1}^{j} \mu(B_{2i}) \tag{6.1.9}$$

We have also for every $n \geq 1$

$$\begin{aligned} \mu(A) &= \mu\Big(\bigcup_{k=n}^{\infty} A_k\Big) = \mu\Big(A_n \cup \bigcup_{k=n+1}^{\infty} B_k\Big) \\ &\leq \mu(A_n) + \sum_{k=n+1}^{\infty} \mu(B_k) \leq \lim_{l\to\infty} \mu(A_l) + \sum_{k=n+1}^{\infty} \mu(B_k) \end{aligned} \tag{6.1.10}$$

Since the sets $\bigcup_{i=1}^{j} B_{2i-1}$ and $\bigcup_{i=1}^{j} B_{2i}$ appearing in (6.1.9) are both contained in A_{2j}, it follows from (6.1.8) and (6.1.9) that the series $\sum_{k=1}^{\infty} \mu(B_k)$ converges. Therefore (6.1.7) follows immediately from (6.1.10). The proof is finished. ♣

Theorem 6.1.4. If μ is a metric outer measure on (X, ρ) then all Borel subsets of X are μ-measurable.

Proof. Since the Borel sets form the least σ-algebra containing all closed subsets of X, it follows from Theorem 6.1.2 that it is enough to check (6.1.4) for every closed set $A \subset X$ and every $B \subset X$. For all $n \geq 1$ let $B_n = \{x \in B \setminus A : \rho(x, A) \geq 1/n\}$. Then $\rho(B \cap A, B_n) \geq 1/n$ and by (6.1.6)

$$\mu(B \cap A) + \mu(B_n) = \mu((B \cap A) \cup B_n) \leq \mu(B) \tag{6.1.10}$$

The sequence $\{B_n\}_{n=1}^\infty$ is increasing and, since A is closed, $B \setminus A = \bigcup_{n=1}^\infty B_n$. In order to apply Lemma 6.1.3 we shall show that

$$\rho(B_n, (B \setminus A) \setminus B_{n+1}) > 0$$

for all $n \geq 1$. And indeed, if $x \in (B \setminus A) \setminus B_{n+1}$, then there exists $z \in A$ with $\rho(x, z) < 1/(n+1)$. Thus, if $y \in B_n$, then

$$\rho(x, y) \geq \rho(y, z) - \rho(x, z) > 1/n - 1/(n+1) = \frac{1}{n(n+1)}$$

and consequently $\rho(B_n, (B \setminus A) \setminus B_{n+1}) > 1/n(n+1) > 0$. Applying now Lemma 6.1.3 with $A_n = B_n$ shows that $\mu(B \setminus A) = \lim_{n\to\infty} \mu(B_n)$. Thus (6.1.4) follows from (6.1.10). The proof is finished. ♣

§6.2 HAUSDORFF MEASURES

Let $\phi : [0, \infty) \to [0, \infty)$ be a non-decreasing function continuous at 0, positive on $(0, \infty)$ and such that $\phi(0) = 0$. Let (X, ρ) be a metric space. For every $\delta > 0$ define

$$\Lambda_\phi^\delta(A) = \inf\{\sum_{i=1}^\infty \phi(\mathrm{diam}(U_i))\} \tag{6.2.1}$$

where the infimum is taken over all countable covers $\{U_i : i = 1, 2, \ldots\}$ of A of diameter not exceeding δ. Conditions (6.1.1) and (6.1.2) are obviously satisfied with $\mu = \Lambda_\phi^\delta$. To check (6.1.3) let $\{A_n : n = 1, 2, \ldots\}$ be a countable family of subsets of X. Given $\varepsilon > 0$ for every $n \geq 1$ we can find a countable cover $\{U_i^n : i = 1, 2, \ldots\}$ of A_n of diameter not exceeding δ such that $\sum_{i=1}^\infty \phi(\mathrm{diam}(U_i^n)) \leq \Lambda_\phi^\delta(A_n) + \varepsilon/2^n$. Then the family $\{U_i^n : n \geq 1, i \geq 1\}$ covers $\bigcup_{n=1}^\infty A_n$ and

$$\Lambda_\phi^\delta(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \sum_{i=1}^\infty \phi(\mathrm{diam}(U_i^n)) \leq \sum_{n=1}^\infty \Lambda_\phi^\delta(A_n) + \varepsilon$$

Thus, letting $\varepsilon \to 0$, (6.1.3) follows proving that Λ_ϕ^δ is an outer measure. Define

$$\Lambda_\phi(A) = \lim_{\delta\to 0} \Lambda_\phi^\delta(A) = \sup_{\delta>0} \Lambda_\phi^\delta(A) \tag{6.2.2}$$

The limit exists, but may be infinite, since $\Lambda_\phi^\delta(A)$ increases as δ decreases. Since all Λ_ϕ^δ are outer measures, the same argument also shows that Λ_ϕ is an outer measure. Moreover Λ_ϕ turns out to be a metric outer measure, since if A and B are two positively separated sets in X, then no set of diameter less than $\rho(A,B)$ can intersect both A and B. Consequently

$$\Lambda_\phi^\delta(A \cup B) = \Lambda_\phi^\delta(A) + \Lambda_\phi^\delta(B)$$

for all $\delta < \rho(A,B)$ and letting $\delta \to 0$ we get the same formula for Λ_ϕ which is just (6.1.6) with $\mu = \Lambda_\phi$. The metric outer measure Λ_ϕ is called the *Hausdorff outer measure* associated to the function ϕ. Its restriction to the σ-algebra of Λ_ϕ-measurable sets, which by Theorem 6.1.4 includes all the Borel sets, is called the *Hausdorff measure* associated to the function ϕ.

As an immediate consequence of the definition of Hausdorff measure and the properties of the function ϕ we get the following.

Proposition 6.2.1 The Hausdorff measure Λ_ϕ is non-atomic.

Remark 6.2.2. A particular role is played by functions ϕ of the form $t \to t^\alpha$, $t, \alpha > 0$ and in this case the corresponding outer measures are denoted by Λ_α^δ and Λ_α.

Remark 6.2.3. Note that if ϕ_1 is another function but such that ϕ_1 and ϕ restrected to an interval $[0,\varepsilon)$, $\varepsilon > 0$, are equal, then the outer measures Λ_{ϕ_1} and Λ_ϕ are also equal. So, in fact, it is enough to define the function ϕ only on an arbitrarily small interval $[0,\varepsilon)$.

Remark 6.2.4. Notice that we get the same values for $\Lambda_\phi^\delta(A)$, and consequently also for $\Lambda_\phi(A)$, if the infimum in (6.2.1) is taken only over covers consisting of sets contained in A. This means that the Hausdorff outer measure $\Lambda_\phi(A)$ of A is its intrinsic property, i.e. does not depend on in which space the set A is contained. If we treated A as the metric space $(A, \rho|_A)$ with the metric $\rho|_A$ induced from ρ, we would get the same value for the Hausdorff outer measure.

If we however took the infimum in (6.2.1) only over covers consisting of balls, we could get different "Hausdorff measure" which (dependently on ϕ) would need not be even equivalent with the Hausdorff measure just defined. To assure this last property ϕ is from now on assumed to satisy the following condition.

There exists a function $C : (0,\infty) \to [1,\infty)$ such that for every $a \in (0,\infty)$ and every $t > 0$ sufficiently small (dependently on a)

$$C(a)^{-1}\phi(t) \le \phi(at) \le C(a)\phi(t) \tag{6.2.3}$$

Since $(ar)^t = a^t r^t$, all functions ϕ of the form $r \to r^t$, considered in Remark 6.2.2, satisfy (6.2.3) with $C(a) = a^t$. Check that all functions $r \to r^t \exp(c\sqrt{\log\log 1/t \log\log\log 1/r}$, $c \ge 0$ also satisfy (6.2.3) with a suitable function C.

Definition 6.2.5. A countable collection $\{(x_i, r_i) : i = 1, 2, \ldots\}$ of pairs $(x_i, r_i) \in X \times (0, \infty)$ is said to *cover* a subset A of X if $A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i)$, and is said to be *centered* at the set A if $x_i \in A$ for all $i = 1, 2, \ldots$. The *radius* of this collection is defined as $\sup_i r_i$ and its diameter as the diameter of the family $\{B(x_i, r_i) : i = 1, 2, \ldots\}$.

For every $A \subset X$ and every $r > 0$ let

$$\Lambda_\phi^{Br}(A) = \inf\{\sum_{i=1}^{\infty} \phi(r_i)\} \tag{6.2.4}$$

where the infimum is taken over all collections $\{(x_i, r_i) : i = 1, 2, \ldots\}$ centered at the set A, covering A and of radii not exceeding r. Let

$$\Lambda_\phi^{B}(A) = \lim_{r \to 0} \Lambda_\phi^{Br}(A) = \sup_{r > 0} \Lambda_\phi^{Br}(A) \tag{6.2.5}$$

The limit exists by the same argument as used for the limit in (6.2.2). We shall prove the following.

Lemma 6.2.6. For every set $A \subset X$

$$1 \le \frac{\Lambda_\phi(A)}{\Lambda_\phi^B(A)} \le C(2)$$

Proof. Since the diameter of any ball does not exceed its double radius, since the diameter of any collection $\{(x_i, r_i) : i = 1, 2, \ldots\}$ also does not exceed its double radius and since the function ϕ is non- decreasing and satisfies (6.2.3), we see that for every $r > 0$ small enough

$$\sum_{i=1}^{\infty} \phi(\mathrm{diam}(B(x_i, r_i))) \le \sum_{i=1}^{\infty} \phi(2r_i) \le C(2) \sum_{i=1}^{\infty} \phi(r_i)$$

and therefore $\Lambda_\phi^{2r}(A) \le C(2)\Lambda_\phi^{Br}(A)$. Thus, letting $r \to 0$,

$$\Lambda_\phi(A) \le C(2)\Lambda_\phi^B(A) \tag{6.2.6}$$

On the other hand, let $\{U_i : i = 1, 2, \ldots\}$ be a countable cover of A consisting of subsets of A. For every $i \ge 1$ choose $x_i \in U_i$ and put $r_i = \mathrm{diam}(U_i)$. Then the collection $\{(x_i, r_i) : i = 1, 2, \ldots\}$ covers A, is centered at A and

$$\sum_{i=1}^{\infty} \phi(r_i) = \sum_{i=1}^{\infty} \phi(\mathrm{diam}(U_i))$$

which implies that $\Lambda_\phi^{B\delta}(A) \le \Lambda_\phi^{\delta}(A)$ for every $\delta > 0$. Thus $\Lambda_\phi^B(A) \le \Lambda_\phi(A)$ which combined with (6.2.6) finishes the proof. ♣

Remark 6.2.7. The function of sets Λ_ϕ^B need not to be an outer measure since condition (6.1.2) need not to be satisfied. Since we will be never interested in exact computation of Hausdorff measure, only in establishing its positiveness or finiteness or in comparing the ratio of its value with some other quantities up to bounded constants, we will be mostly dealing with $\Lambda_\phi^{B\delta}$ and Λ_ϕ^B using nevertheless always the symbols $\Lambda_\phi^\delta(A)$ and $\Lambda_\phi(A)$.

§6.3 PACKING MEASURES

Let, as in the previous section, $\phi : [0, \infty) \to [0, \infty)$ be a non-decreasing function such that $\phi(0) = 0$ and let (X, ρ) be a metric space. A collection $\{(x_i, r_i) : i = 1, 2, \ldots\}$ centered at a set $A \subset X$ is said to be a *packing* of A if and only if for any pair $i \neq j$

$$\rho(x_i, x_j) \geq r_i + r_j$$

This property is not generally equivalent to requirement that all the balls $B(x_i, r_i)$ are mutually disjoint. It is obviously so if X is a Euclidean space. For every $A \subset X$ and every $r > 0$ let

$$\Pi_\phi^{*r}(A) = \sup\{\sum_{i=1}^\infty \phi(r_i)\} \tag{6.3.1}$$

where the supremum is taken over all packings $\{(x_i, r_i) : i = 1, 2, \ldots\}$ of A of radius not exceeding r. Let

$$\Pi_\phi^*(A) = \lim_{r \to 0} \Pi_\phi^{*r}(A) = \inf_{r > 0} \Pi_\phi^{*r}(A) \tag{6.3.2}$$

The limit exists since $\Pi_\phi^{*r}(A)$ decreases as r decreases. In opposite to Λ_ϕ^B the function Π_ϕ^* satisfies condition (6.1.2), however it also need not to be an outer measure since this time condition (6.1.3) need not to be satisfied. To obtain an outer measure we put

$$\Pi_\phi(A) = \inf\{\sum \Pi_\phi^*(A_i)\}, \tag{6.3.3}$$

where the supremum is taken over all covers $\{A_i\}$ of A. The reader will check easily, with similar arguments as in the case of Hausdorff measures, that Π_ϕ is already an outer measure and even more, a metric outer measure on X. It will be called the *outer packing measure* associated to the function ϕ. Its restriction to the σ-algebra of Π_ϕ-measurable sets, which by Theorem 6.1.4 includes all the Borel sets, will be called *packing measure* associated to the function ϕ.

Proposition 6.3.1. For every set $A \subset X$ it holds $\Lambda_\phi(A) \leq C(2)\Pi_\phi(A)$.

Proof. First we shall show that for every set $A \subset X$ and every $r > 0$

$$\Lambda_\phi^{2r}(A) \leq C(2)\Pi_\phi^{*r}(A) \tag{6.3.4}$$

Indeed, if there is no finite maximal (in the sense of inclusion) packing of the set A of the form $\{(x_i, r)\}$, then for every $k \geq 1$ there exists a packing $\{(x_i, r) : i = 1, \dots, k\}$ of A and therefore $\Pi_\phi^{*r}(A) \geq \sum_{i=1}^k \phi(r) = k\phi(r)$. Since $\phi(r) > 0$, this implies that $\Pi_\phi^{*r}(A) = \infty$ and (6.3.4) holds. Otherwise, let $\{(x_i, r) : i = 1, \dots, l\}$ be a maximal packing of A. Then the collection $\{(x_i, 2r) : i = 1, \dots, l\}$ covers A and therefore

$$\Lambda_\phi^{2r}(A) \leq \sum_{i=1}^{l} \phi(2r) \leq C(2) l \phi(r) \leq C(2)\Pi_\phi^{*r}(A)$$

that is (6.3.4) is satisfied. Thus letting $r \to 0$ we get

$$\Lambda_\phi(A) \leq C(2)\Pi_\phi^*(A) \tag{6.3.5}$$

So, if $\{A_n\}_{n\geq 1}$ is a countable cover of A then,

$$\Lambda_\phi(A) \leq \sum_{n=1}^{\infty} \Lambda_\phi(A_i) \leq C(2) \sum_{n=1}^{\infty} \Pi_\phi^*(A_i)$$

Hence, applying (6.3.3), the lemma follows. ♣

§6.4 DIMENSIONS

Let, similarly as in the two previous sections, (X, ρ) be a metric space. Recall (comp. Remark 6.2.2) that Λ_t, $t > 0$, is the Hausdorff outer measures on X associated to the function $r \to r^t$ and all Λ_t^δ are of corresponding meaning. Fix $A \subset X$. Since for every $0 < \delta \leq 1$ the function $t \to \Lambda_t^\delta(A)$ is non-increasing, so is the function $t \to \Lambda_t(A)$. Furthermore, if $s < t$, then for every $0 < \delta$

$$\Lambda_s^\delta(A) \geq \delta^{s-t}\Lambda_t^\delta(A)$$

which implies that if $\Lambda_t(A)$ is positive, then $\Lambda_s(A)$ is infinite. Thus there is a unique value, $\mathrm{HD}(A)$, called the *Hausdorff dimension* of A such that

$$\Lambda_t(A) = \begin{cases} \infty & \text{if } 0 \leq t < \mathrm{HD}(A) \\ 0 & \text{if } \mathrm{HD}(A) < t < \infty \end{cases} \tag{6.4.1}$$

Note that similarly as Hausdorff measures (comp. Remark 6.2.4), Hausdorff dimension is consequently also an intrinsic property of sets and does not depend on their complements. The following is an immediate consequence of the definitions of Hausdorff dimension and outer Hausdorff measures.

Theorem 6.4.1. The Hausdorff dimension is a monotonic function of sets, that is if $A \subset B$ then $\mathrm{HD}(A) \leq \mathrm{HD}(B)$.

We shall prove the following.

Theorem 6.4.2. If $\{A_n\}_{n\ge 1}$ is a countable family of subsets of X then

$$\mathrm{HD}(\cup_n A_n) = \sup_n\{\mathrm{HD}(A_n)\}.$$

Proof. Inequality $\mathrm{HD}(\cup_n A_n) \ge \sup_n\{\mathrm{HD}(A_n)\}$ is an immediate consequence of Theorem 6.4.1. Thus, if $\sup_n\{\mathrm{HD}(A_n)\} = \infty$ there is nothing to prove. So, suppose that $s = \sup_n\{\mathrm{HD}(A_n)\}$ is finite and consider an arbitrary $t > s$. In view of (6.4.1), $\Lambda_t(A_n) = 0$ for every $n \ge 1$ and therefore, since Λ_t is an outer measure, $\Lambda_t(\cup_n A_n) = 0$. Hence, by (6.4.1) again, $\mathrm{HD}(\cup_n A_n) \le t$. The proof is finished. ♣

As an immediate consequence of this theorem, Proposition 6.2.1 and formula (6.4.1) we get the following.

Proposition 6.4.3. The Hausdorff dimension of any countable set is equal to 0.

In exactly the same way as Hausdorff dimension HD one can define *packing* dimension* PD* and *packing dimension* PD using respectively $\Pi_t^*(A)$ and $\Pi_t(A)$ instead of $\Lambda_t(A)$. The reader can check easily that results analogous to Theorem 6.4.1, Theorem 6.4.2 and Proposition 6.4.3 are also true in these cases. As an immediate consequence of these definitions and Proposition 6.3.1 we get the following.

Lemma 6.4.4. $\mathrm{HD}(A) \le \mathrm{PD}(A) \le \mathrm{PD}^*(A)$ for every set $A \subset X$.

Now we shall define the third basic dimension – *ball-counting dimension* frequently also called box-counting dimension, Minkowski dimension or capacity. Let A be an arbitrary subset of the metric space (X,ρ). We first need the following.

Definition 6.4.5. For every $r > 0$ consider the family of all collections $\{(x_i, r_i)\}$ (see Definition 6.2.5) of radius not exceeding r which cover A and are centered at A. Put $N(A,r) = \infty$ if this family is empty. Otherwise define $N(A,r)$ to be the minimum of all cardinalities of elements of this family. Note that one gets the same number if one considers the subfamily of collections of radius exactly r and even only its subfamily of collections of the form $\{(x_i, r)\}$.

Now the *lower ball-counting dimensions* and *upper ball-counting dimension* of A are defined respectively by

$$\text{(6.4.2)} \qquad \underline{\mathrm{BD}}(A) = \liminf_{r\to 0}\frac{\log N(A,r)}{-\log r} \text{ and } \overline{\mathrm{BD}}(A) = \limsup_{r\to 0}\frac{\log N(A,r)}{-\log r}.$$

If $\underline{\mathrm{BD}}(A) = \overline{\mathrm{BD}}(A)$, the common value is called simply ball-counting dimension and is denoted by $\mathrm{BD}(A)$. The reader will easily prove the next theorem which explains the reason of the name box-counting dimension. The other names will not be discussed here.

Proposition 6.4.6. Fix $n \geq 1$. For every $r > 0$ let $\mathcal{L}(r)$ be any lattice in $I\!R^n$ consisting of cubes of sides of length r. For any set $A \subset I\!R^n$ let $L(A, r)$ denotes the number of cubes in $\mathcal{L}(r)$ which intersect A. Then

$$\underline{\mathrm{BD}}(A) = \liminf_{r \to 0} \frac{\log L(A, r)}{-\log r} \text{ and } \overline{\mathrm{BD}}(A) = \limsup_{r \to 0} \frac{\log L(A, r)}{-\log r}$$

Remark 6.4.7. Ball-counting dimension has properties which distinguish it qualitatively from Hausdorff and packing dimensions. For instance $\underline{\mathrm{BD}}(\overline{A}) = \underline{\mathrm{BD}}(A)$ and $\overline{\mathrm{BD}}(\overline{A}) = \overline{\mathrm{BD}}(A) =$. So, in particular there exist countable sets of positive ball-counting dimension, for example the set of rational numbers in the interval $[0, 1]$. Even more, there exist compact countable sets with this property like the set $\{1, 1/2, 1/3, \ldots, 0\} \subset I\!R$. On the other hand in many cases (see Theorem 6.6.6) all these dimensions coincide.

Now we shall provide other characterizations of ball-counting dimension, which in particular will be used to prove Lemma 6.4.8 and consequently Theorem 6.4.9 which establishes most general relations between the dimensions considered in this section.

Let $A \subset X$. For every $r > 0$ define $P(A, r)$ to be the supremum of cardinalities of all packings of the set A of the form $\{(x_i, r)\}$. First we shall prove the following.

Lemma 6.4.7. For every set $A \subset I\!R^n$ and every $r > 0$

$$N(A, 2r) \leq P(A, r) \text{ and } P(A, r) \leq N(A, r).$$

Proof. Let us start with the proof of the first inequality. If $P(A, r) = \infty$, there is nothing to prove. Otherwise, let $\{(x_i, r) : i = 1, \ldots, k\}$ be a packing of A with $k = P(A, r)$. Then this packing is maximal in the sense of inclusion and therefore the collection $\{(x_i, 2r) : i = 1, \ldots, l\}$ covers A. Thus $N(A, 2r) \leq l = P(A, r)$. The first part of Lemma 6.4.7 is proved.

If $N(A, r) = \infty$, the second part is obvious. Otherwise consider a finite packing $\{(x_i, r) : i = 1, \ldots, k\}$ of A and a finite cover $\{(y_j, r) : j = 1, \ldots, l\}$ of A centered at A. Then for every $1 \leq i \leq k$ there exists $1 \leq j = j(i) \leq l$ such that $x_i \in B(y_j(i), r)$ and every ball $B(y_j, r)$ can contain at most one element of the set $\{x_i : i = 1, \ldots, k\}$. So, the function $i \to j(i)$ is injective and therefore $k \leq l$. The proof is finished. ♣

As an immediate consequence of Lemma 6.4.7 we get the following.

$$\underline{\mathrm{BD}}(A) = \liminf_{r \to 0} \frac{\log P(A, r)}{-\log r} \quad \text{and} \quad \overline{\mathrm{BD}}(A) = \limsup_{r \to 0} \frac{\log P(A, r)}{-\log r}. \tag{6.4.3}$$

Now we are in a position to prove the following.

Lemma 6.4.8 For every set $A \subset X$ we have $\mathrm{PD}^*(A) = \overline{\mathrm{BD}}(A)$.

Proof. Take $t < \overline{\mathrm{BD}}(A)$. In view of (6.4.3) there exists a sequence $\{r_n : n = 1, 2, \ldots\}$ of positive reals converging to zero and such that $P(A, r_n) \geq r_n^{-t}$ for every $n \geq 1$. Then

$\Pi_t^{*r_n}(A) \geq r^t P(A, r_n) \geq 1$ and consequently $\Pi_t^*(A) \geq 1$. Hence $t \leq \mathrm{PD}^*(A)$ and therefore $\overline{\mathrm{BD}}(A) \leq \mathrm{PD}^*(A)$.

In order to prove the converse inequality consider $s < t < \mathrm{PD}^*(A)$. Then $\Pi_t^*(A) = \infty$ and therefore for every $n \geq 1$ there exists a finite packing $\{(x_{n,i}, r_{n,i}) : i = 1, \ldots, k(n)\}$ of A of radius not exceeding 2^{-n} and such that

$$\sum_{i=1}^{k(n)} r_{n,i}^t > 1 \tag{6.4.4}$$

Now for every $m \geq n$ let

$$l_m = \#\{i \in \{1, \ldots, k(n)\} : 2^{-(m+1)} < r_{n,i} \leq 2^{-m}\}$$

Then by (6.4.4)

$$\sum_{m=n}^{\infty} l_m 2^{-nt} > 1 \tag{6.4.5}$$

Suppose that $l_m < 2^{ns}(1 - 2^{(s-t)})$ for every $m \geq n$. Then

$$\sum_{m=n}^{\infty} l_m 2^{-nt} < (1 - 2^{(s-t)}) \sum_{m=1}^{\infty} 2^{n(s-t)} = 1$$

This contradicts (6.4.5) and shows that for every $n \geq 1$ there exists $m = m(n) \geq n$ such that

$$l_m \geq 2^{ns}(1 - 2^{(s-t)})$$

Hence $P(A, 2^{-(m+1)}) \geq 2^{ns}(1 - 2^{(s-t)})$, whence

$$\frac{\log P(A, 2^{-(m+1)})}{(m+1)\log 2} \geq \frac{sk \log 2}{(m+1)\log 2}$$

Thus, letting $n \to \infty$ (then also $m = m(n) \to \infty$) we obtain $\overline{\mathrm{BD}}(A) \geq s$. So, we are done. ♣

Combining now Lemma 6.4.4 and Lemma 6.4.8 and checking easily that $\mathrm{HD}(A) \leq \underline{\mathrm{BD}}(A)$ we obtain the following main general relation connecting all the dimensions under consideration.

Theorem 6.4.9. For every set $A \subset X$

$$\mathrm{HD}(A) \leq \min\{\mathrm{PD}(A), \underline{\mathrm{BD}}(A)\} \leq \max\{\mathrm{PD}(A), \underline{\mathrm{BD}}(A)\} \leq \overline{\mathrm{BD}}(A) = \mathrm{PD}^*(A)$$

We finish this section with the following definition.

Definition 6.4.10. Let μ be a Borel measure on (X,ρ). Then the *Hausdorff dimension* $\mathrm{HD}(\mu)$ of the measure μ is defined as

$$\mathrm{HD}(\mu)=\inf\{\mathrm{HD}(Y):\mu(X\setminus Y)=0\}$$

an analogous definition can be formulated for packing dimension.

§6.5 BESICOVITCH COVERING THEOREM

In this section we prove only one result, the Besicovitch covering theorem. Although this theorem seems to be almost always omitted in the classical geometric measure theory, we however consider it as one of most powerful geometric tools when dealing with some aspects of fractal sets. We refer the reader to Section 6.6 to verify our opinion.

Theorem 6.5.1. (Besicovitch covering theorem) Let $n\ge 1$ be an integer. Then there exists a constant $b(n)>0$ such that the following claim is true.

If A is a bounded subset of $I\!R^n$ then for any function $r:A\to(0,\infty)$ there exists $\{x_k:k=1,2,\ldots\}$ a countable subset of A such that the collection $\mathcal{B}(A,r)=\{(x_k,r(x_k)):k\ge 1\}$ covers A and can be decomposed into $b(n)$ packings of A.

In particular it follows from Theorem 6.5.1 that $\#\{B\in\mathcal{B}:x\in B\}\le b(n)$. Exactly the same proof (world by world) goes if open balls in Theorem 6.5.1 are replaced by closed ones.

For any $x\in I\!R^n$, any $0<r\le\infty$ and any $0<\alpha<\pi$ by $\mathrm{Con}(x,\alpha,r)$ we will denote any solid central cone with vertex x, radius r and angle (Lebesgue measure on the unit sphere S^{n-1}) α. The proof of Theorem 6.5.1 is based on the following obvious geometric observation.

Observation 6.5.2. Let $n\ge 1$ be an integer. Then there exists $\alpha(n)>0$ so small that the following holds.

If $x\in I\!R^n$, $0<r<\infty$, if $z\in B(x,r)\setminus B(x,r/3)$ and $x\in\mathrm{Con}(z,\alpha(n),\infty)$ then the set $\mathrm{Con}(z,\alpha(n),\infty)\setminus B(x,r/3)$ consists of two connected components (one of z and one of "∞") and that containing z is contained in $B(x,r)$.

Proof of Theorem 6.5.1. We will construct the sequence $\{x_k:k=1,2,\ldots\}$ inductively. Let

$$a_0=\sup\{r(x):x\in A\}$$

If $a_0=\infty$ then one can find $x\in A$ with $r(x)$ so large that $B(x,r(x)\supset A$ and the proof is finished.

If $a_0<\infty$ choose $x_1\in A$ so that $r(x_1)>a_0/2$. Fix $k\ge 1$ and assume that the points $x_1,x_2,\ldots,x_k$ have been already chosen. If $A\subset B(x_1,r(x_1))\cup\ldots\cup B(x_k,r(x_k))$ then the selection process is finished. Otherwise put

$$a_k=\sup\{r(x):x\in A\setminus\big(B(x_1,r(x_1))\cup\ldots\cup B(x_k,r(x_k))\big)\}$$

and take

$$x_{k+1} \in A \setminus \big(B(x_1, r(x_1)) \cup \ldots \cup B(x_k, r(x_k))\big) \tag{6.5.1}$$

such that

$$r(x_{k+1}) > a_k/2 \tag{6.5.2}$$

In order to shorten notation from now on throughout this proof we will write r_k for $r(x_k)$. By (6.5.1) we have $x_l \notin B(x_k, r_k)$ for all pairs k, l with $k < l$. Hence

$$\|x_k - x_l\| \geq r(x_k) \tag{6.5.3}$$

It follows from the construction of the sequence (x_k) that

$$r_k > a_{k-1}/2 \geq r_l/2 \tag{6.5.4}$$

and therefore $r_k/3 + r_l/3 < r_k/3 + 2r_k/3 = r_k$. Joining this and (6.5.3) we obtain

$$B(x_k, r_k/3) \cap B(x_l, r_l/3) = \emptyset \tag{6.5.5}$$

for all pairs k, l with $k \neq l$ since then either $k < l$ or $l < k$.

Now we shall show that the balls $\{B(x_k, r_k) : k \geq 1\}$ cover A. Indeed, if the selection process stops after finitely many steps this claim is obvious. Otherwise it follows from (6.5.5) that $\lim_{k\to\infty} r_k = 0$ and if $x \notin \bigcup_{k=1}^{\infty} B(x_k, r_k)$ for some $x \in A$ then by construction $r_k > a_{k-1}/2 \geq r(x)$ for every $k \geq 1$. The contradiction obtained proves that $\bigcup_{k=1}^{\infty} B(x_k, r_k) \supset A$.

The main step of the proof is given by the following.

Claim. For every $z \in I\!\!R^n$ and any cone $\mathrm{Con}(z, \alpha(n), \infty)$ ($\alpha(n)$ given by Observation 6.5.2)

$$\#\{k \geq 1 : z \in B(x_k, r_k) \setminus B(x_k, r_k/3) \text{ and } x_k \in \mathrm{Con}(z, \alpha(n), \infty)\} \leq 1 + 16^n$$

Denote by Q the set of integers whose cardinality is to be estimated. If $Q = \emptyset$, there is nothing to prove. Otherwise let $i = \min Q$. If $k \in Q$ and $k \neq i$ then $k > i$ and therefore $x_k \notin B(x_i, r_i)$. In view of this, Observation 6.5.2 applied with $x = x_i$, $r = r_i$, and the definition of Q, we get $\|z - x_k\| \geq 2r_i/3$, whence

$$r_k \geq \|z - x_k\| \geq 2r_i/3 \tag{6.5.6}$$

On the other hand by (6.5.4) we have $r_k < 2r_i$ and therefore $B(x_k, r_k/3) \subset B(z, 4r_k/3) \subset B(z, 8r_i/3)$. Thus, using (6.5.5), (6.5.6) and the fact that the n-dimensional volume of balls in $I\!\!R^n$ is proportional to the n^{th} power of radii we obtain $\#Q \leq (8r_i/3)^n/(2r_i/9)^n = 12^n$. The proof of the claim is finished.

Clearly there exists an integer $c(n) \geq 1$ such that for every $z \in I\!R^n$ the space $I\!R^n$ can be covered by at most $c(n)$ cones of the form $\mathrm{Con}(z, \alpha(n), \infty)$. Therefore it follows from the claim that for every $z \in I\!R^n$

$$\#\{k \geq 1 : z \in B(x_k, r_k) \setminus B(x_k, r_k/3)\} \leq c(n)12^n$$

Thus applying (6.5.5)

$$\#\{k \geq 1 : z \in B(x_k, r_k) \leq 1 + c(n)12^n \tag{6.5.7}$$

Since the ball $\overline{B}(0, 3/2)$ is compact, it contains a finite subset P such that $\bigcup_{x \in P} B(x, 1/2) \supset \overline{B}(0, 3/2)$. Now for every $k \geq 1$ consider the composition of the map $I\!R^n \ni x \to r_k x \in I\!R^n$ and the translation determined by the vector from 0 to x_k. Call by P_k the image of P under this translation. Then $\#P_k = \#P$, $P_k \subset \overline{B}(x_k, 3r_k/2)$ and

$$\bigcup_{x \in P_k} B(x, r_k/2) \supset \overline{B}(0, 3r_k/2) \tag{6.5.8}$$

Consider now two integers $1 \leq k < l$ such that

$$B(x_k, r_k) \cap B(x_l, r_l) \neq \emptyset \tag{6.5.9}$$

Let $y \in I\!R^n$ be the only point lying on the interval joining x_l and x_0 at the distance $r_k - r_l/2$ from x_k. As $x_l \notin B(x_k, r_k)$, by (6.5.9) we have $\|y - x_l\| \leq r_l + r_l/2 = 3r_l/2$ and therefore by (6.5.8) there exists $z \in P_l$ such that $\|z - y\| < r_l/2$. Consequently $z \in B(x_k, r_l/2 + r_k - r_l/2) = B(x_k, r_k)$. Thus applying (6.5.7) with z being the elements of P_l, we obtain the following

$$\#\{1 \leq k \leq l-1 : B(x_k, r_k) \cap B(x_l, r_l) \neq \emptyset\} \leq \#P(1 + c(n)12^n) \tag{6.5.10}$$

for every $l \geq 1$.

Putting $b(n) = \#P(1 + c(n)12^n) + 1$ this property allows us to decompose the set $I\!N$ of positive integers into $b(n)$ subsets $I\!N_1, I\!N_2, \ldots, I\!N_{b(n)}$ in the following inductive way. For every $k = 1, 2, \ldots, b(n)$ set $I\!N_k(b(n)) = \{k\}$ and suppose that for every $k = 1, 2, \ldots, b(n)$ and some $j \geq b(n)$ mutually disjoint families $I\!N_k(j)$ have been already defined so that

$$I\!N_1(j) \cup I\!N_{b(n)}(j) = \{1, 2, \ldots, j\}$$

Then by (6.5.10) there exists at least one $1 \leq k \leq b(n)$ such that $B(x_{j+1}, r_{j+1}) \cap B(x_i, r_i) = \emptyset$ for every $i \in I\!N_k(j)$. We set $I\!N_k(j+1) = I\!N_k(j) \cup \{j+1\}$ and $I\!N_l(j+1) = I\!N_l(j)$ for all $l \in \{1, 2, \ldots, b(n)\} \setminus \{k\}$. Putting now for every $k = 1, 2, \ldots, b(n)$

$$I\!N_k = I\!N_k(b(n)) \cup I\!N_k(b(n)+1) \cup \ldots$$

we see from the inductive construction that these sets are mutually disjoint, that they cover $I\!N$ and that for every $k = 1, 2, \ldots, b(n)$ the families of balls $\{B(x_l, r_l) : l \in I\!N_k\}$ are also mutually disjoint. The of proof the Besicovitch covering theorem is finished. ♣

We would like to emphasize here once more that the same statement remains true if open balls are replaced by closed ones. Also if instead of balls one considers n-dimensional cubes. Then although the proof is based on the same idea, however technically is considerably easier.

§6.6 VOLUME LEMMAS

In this section a function $\phi : [0, \infty) \to [0, \infty)$ is assumed to satisfy the same conditions as in Section 6.2 including (6.2.3) and moreover is assumed to be continuous. We start with the following.

Theorem 6.6.1. Let $n \geq 1$ be an integer and let $b(n)$ be the constant claimed in Theorem 6.5.1 (Besicovitch covering theorem). Assume that μ is a Borel probability measure on $I\!R^n$ and A is a bounded Borel subset of $I\!R^n$. If there exists $C \in (0, \infty]$, $(1/\infty = 0)$, such that

(a) for all (but countably many maybe) $x \in A$

$$\limsup_{r \to 0} \frac{\mu(B(x,r))}{\phi(r)} \geq C$$

then $\Lambda_\phi(E) \leq \frac{b(n)}{C}\mu(E)$ for every Borel set $E \subset A$. In particular $\Lambda_\phi(A) < \infty$.
or

(b) for all $x \in A$

$$\limsup_{r \to 0} \frac{\mu(B(x,r))}{\phi(r)} \leq C < \infty$$

then $\mu(E) \leq C\Lambda_\phi(E)$ for every Borel set $E \subset A$.

Proof. (a) In view of Proposition 6.2.1 we can assume that E does not intersect the exceptional countable set. Fix $\varepsilon > 0$ and $r > 0$. Since μ is a regular measure, there exists an open set $G \supset E$ such that $\mu(G) \leq \mu(E) + \varepsilon$. By openness of G and by assumption (a), for every $x \in E$ there exists $0 < r(x) < r$ such that $B(x, r(x)) \subset G$ and $(1/C + \varepsilon)\mu(B(x, r)) \geq \phi(r)$. Let $\{(x_k, r(x_k)) : k \geq 1\}$ be the cover of E obtained by applying Theorem 6.5.1. (Besicovitch covering theorem) to the set E. Then

$$\begin{aligned}\Lambda_\phi^r(E) &\leq \sum_{k=1}^{\infty} \phi(r(x_k)) \leq \sum_{k=1}^{\infty} (C^{-1} + \varepsilon)\mu(B(x_k, r(x_k))) \\ &\leq b(n)(C^{-1} + \varepsilon)\mu(\bigcup_{k=1}^{\infty} B(x_k, r(x_k))) \leq b(n)(C^{-1} + \varepsilon)(\mu(E) + \varepsilon)\end{aligned}$$

Letting $r \to 0$ we thus obtain $\Lambda_\phi(E) \leq b(n)(1/C + \varepsilon)(\mu(E) + \varepsilon)$ and therefore letting $\varepsilon \to 0$ the part (a) follows (note that the proof is correct with $C = \infty$!).

(b) Fix an arbitry $s > C$. Since for every $r > 0$ the function $x \to \mu(B(x,r))/\phi(r)$ is measurable and since the supremum of a countable sequence of measurable functions is also a measurable function, we conclude that for every $k \geq 1$ the function $\psi_k : A \to I\!R$ is measurable, where

$$\psi_k(x) = \sup\left\{\frac{\mu(B(x,r))}{\phi(r)} : r \in Q \cap (0, 1/k]\right\}$$

and Q denotes the set of rational numbers. For every $k \geq 1$ let $A_k = \psi_k^{-1}((0,s])$. In view of measurability of the functions ψ_k all the sets A_k are measurable. Take an arbitrary $r \in (0, 1/k]$. Then there exists a sequence $r_j : j = 1, 2, \ldots\}$ of rational numbers converging to r from above. Since the function ϕ is continuous and the function $t \to \mu(B(x,t))$ is non-decreasing, we have for every $x \in A_k$

$$\frac{\mu(B(x,r))}{\phi(r)} \leq \lim_{j\to\infty} \frac{\mu(B(x,r_j))}{\phi(r_j)} \leq s$$

So, if $F \subset A_k$ is a Borel set and if $\{(x_i, r_i) : i = 1, 2, \ldots\}$ is a collection centered at the set F, covering F and of radius not exceeding $0 < r \leq 1/k$, then

$$\sum_{i=1}^{\infty} \phi(r_i) \geq s^{-1} \sum_{i=1}^{\infty} \mu(B(x_i, r_i)) \geq s^{-1}\mu(F)$$

Hence, $\Lambda_\phi^r(F) \geq s^{-1}\mu(F)$ and letting $r \to 0$ we get

$$\Lambda_\phi(E) \geq \Lambda_\phi(F) \geq s^{-1}\mu(F)$$

By the assumption of (b), $\cup_k A_k = A$ and therefore, putting $B_k = A_k \backslash (A_1 \cup A_2 \cup \ldots \cup A_{k-1})$, $k \geq 1$, we see that the family $\{B_k : k \geq 1\}$ is a countable partition of A into Borel sets. Therefore, if $E \subset A$ then

$$\Lambda_\phi(E) = \sum_{k=1}^{\infty} \Lambda_\phi(E \cap A_k) \geq s^{-1} \sum_{k=1}^{\infty} \mu(E \cap A_k) = s^{-1}\mu(E)$$

So, letting $s \searrow C$ finishes the proof. ♣

In an analogous way one can prove the following.

Theorem 6.6.2. Let $n \geq 1$ be an integer and let $b(n)$ be the constant claimed in Theorem 6.5.1 (Besicovitch covering theorem). Assume that μ is a Borel probability measure on $I\!R^n$ and A is a bounded subset of $I\!R^n$. If there exists $C \in (0, \infty]$, $(1/\infty = 0)$, such that

(a) for all $x \in A$

$$\liminf_{r\to 0} \frac{\mu(B(x,r))}{\phi(r)} \leq C$$

then $\mu(E) \le b(n)C\Pi_\phi(E)$ for every Borel set $E \subset A$.
or

(b) for all $x \in A$

$$\liminf_{r\to 0} \frac{\mu(B(x,r))}{\phi(r)} \ge C < \infty$$

then $\Pi_\phi(E) \le C^{-1}\mu(E)$ for every Borel set $E \subset A$. In particular $\Pi_\phi(A) < \infty$.

Note that each Borel measure μ defined on a Borel subset B of R^n can be in a canonical way considered as a measure on $I\!R^n$ by putting $\mu(A) = \mu(A \cap B)$ for every Borel set $A \subset I\!R^n$.
As a simple consequence of Theorem 6.6.1 we shall prove the following.

Theorem 6.6.3. Suppose that μ is a Borel probability measure on $I\!R^n$, $n \ge 1$, and A is a bounded Borel subset of $I\!R^n$.

(a) If $\mu(A) > 0$ and there exists θ_1 such that for every $x \in A$

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \ge \theta_1$$

then $\mathrm{HD}(A) \ge \theta_1$.
(b) If there exists θ_2 such that for every $x \in A$

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \le \theta_2$$

then $\mathrm{HD}(A) \le \theta_2$.
Proof. (a) Take any $0 < \theta < \theta_1$. Then, by the assumption, $\limsup_{r\to 0} \mu(B(x,r))/r^\theta \le 1$. Therefore applying Theorem 6.6.1(b) with $\phi(t) = t^\theta$, we obtain $\Lambda_\theta(A) \ge \mu(A) > 0$. Hence $\mathrm{HD}(A) \ge \theta$ by definition (6.4.1) and consequently $\mathrm{HD}(A) \ge \theta_1$.

(b) Take now an arbitrary $\theta > \theta_2$. Then by the assumption $\limsup_{r\to 0} \mu(B(x,r))/r^\theta \ge 1$. Therefore applying Theorem 6.6.1(a) with $\phi(t) = t^\theta$ we obtain $\Lambda_\theta(A) < \infty$, whence $\mathrm{HD}(A) \le \theta$ and consequently $\mathrm{HD}(A) \le \theta_1$. The proof is finished. ♣

Recall that the Hausdorff dimension of a Borel measure has been defined in Definition 6.4.10. As a consequence of Theorem 6.6.3 we shall prove the following.

Corollary 6.6.4. Suppose that μ is a Borel probability measure on $I\!R^n$, $n \ge 1$.

(a) If there exists θ_1 such that for μ-a.e. $x \in I\!R^n$

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \ge \theta_1$$

then $\mathrm{HD}(\mu) \ge \theta_1$

(b) If there exists θ_2 such that for μ-a.e. $x \in I\!R^n$

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \le \theta_2$$

then $\mathrm{HD}(\mu) \le \theta_2$.

Proof. (a) Let $Y \subset I\!\!R^n$ be a Borel set such that $\mu(Y) = 1$. By the assumption there exists a bounded Borel subset $A \subset Y$ with $\mu(A) > 0$ such that for every $x \in A$

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \geq \theta_1$$

Thus, applying Theorem 6.6.3(a) we get $\mathrm{HD}(Y) \geq \mathrm{HD}(A) \geq \theta_1$ and taking infimum, $\mathrm{HD}(\mu) \geq \theta_1$.

(b) Decompose now the space $I\!\!R^n$ into a countable union $\cup_k X_k$ of Borel bounded sets X_k and let $X \subset I\!\!R^n$ be a Borel set of measure 1 whose every point satisfies the assumptions of Corollary 6.6.4. Applying for every $k \geq 1$ Theorem 6.6.3(b) with $A = X \cap X_k$ we get $\mathrm{HD}(X \cap X_k) \leq \theta_2$ and we are done applying Theorem 6.4.2 since $\mathrm{HD}(\mu) \leq \mathrm{HD}(X)$. ♣

Definition 6.6.5. Let X be a Borel bounded subset of $I\!\!R^n$, $n \geq 1$. A Borel probability measure on X is said to be a *geometric measure* with an exponent $t \geq 0$ if and only if there exists a constant $C \geq 1$ such that

$$C^{-1} \leq \frac{\mu(B(x,r))}{r^t} \leq C$$

for every $x \in X$ and every $0 < r \leq 1$.

We shall prove the following.

Theorem 6.6.6. If X is a Borel bounded subset of $I\!\!R^n$, $n \geq 1$, and μ is a geometric measure on X with an exponent t, then $\mathrm{BD}(X)$ exists and

$$\mathrm{HD}(X) = \mathrm{PD}(X) = \mathrm{BD}(X) = t$$

Moreover the three measures μ, Λ_t and Π_t on X are equivalent with bounded Radon-Nikodym derivatives.

Proof. The last part of the theorem follows immediately from Theorem 6.6.1 and Theorem 6.6.2 applied for $A = X$. Consequently also $t = \mathrm{HD}(X) = \mathrm{PD}(X)$ and therefore, in view of Theorem 6.4.9, we only need to show that $\overline{\mathrm{BD}}(X) \leq t$. And indeed, let $\{(x_i, r) : i = 1, \dots, k\}$ be a packing of X. Then

$$kr^t \leq C \sum_{i=1}^{k} \mu(B(x_i, r)) \leq C$$

and therefore $k \leq Cr^{-t}$. Thus $P(X, r) \leq Cr^{-t}$, whence $\log P(X, r) \leq \log C - t \log r$. Applying now formula (6.4.3) finishes the proof. ♣

In particular it follows from this theorem that every geometric measure admits exactly one exponent. Lots of examples of geometric measures will be provided in the next chapters.

BIBLIOGRAPHICAL NOTES

The history of notions and development of the geometric measure theory is very long, rich and complicated and its outline exceeds the scope of this book. We refer the interested reader to the books [Falconer, 1985] and [Mattila, 1995].