CHAPTER 4. THERMODYNAMICAL FORMALISM

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In Chapter 2 (Th. 2.4.6) we proved that for every positive expansive map of a compact space $T : X \to X$ and an arbitrary continuous function $\phi : X \to I\!R$ there exists an equilibrium state. In Remark 3.4.4 we provided a specific construction for T open distance expanding topologically transitive and ϕ Hölder. Here we shall construct this equilibrium measure with a greater care and study its miraculous regularity with respect to the "potential" function ϕ, its "mixing" properties and uniqueness. So, for the whole chapter $T : X \to X$ we fix an open, distance expanding, topologically transitive map of a compact metric space (X, ρ), with constants η, λ, ξ introduced in Ch.3.

SECTION 4.1. GIBBS MEASURES: INTRODUCTORY REMARKS.

A probability measure μ on X and Borel σ-algebra of sets is said to be a *Gibbs state (measure)* for the potential ϕ provided that there exist $P \in I\!R$ and $C \geq 1$ such that for all $x \in X$ and all $n \geq 1$

$$C^{-1} \leq \frac{\mu\big(T_x^{-n}(B(T^n(x), \xi))\big)}{\exp(S_n\phi(x) - Pn)} \leq C \tag{4.1.1}$$

If additionally μ is T-invariant we call μ *invariant Gibbs state* (or *measure*).

We denote the set of all Gibbs states of ϕ by G_ϕ. It is obvious that if μ is a Gibbs state of ϕ and ν is equivalent to μ with Radon-Nikodym derivatives uniformly bounded from above and below, then ν is also a Gibbs state. The following proposition shows that the converse is also true and identifies the constant P appearing in the definition of Gibbs states as the topological pressure of ϕ.

Proposition 4.1.1. If μ and ν are Gibbs states associated to the map T and a Hölder continuous function ϕ and the corresponding constants are denoted respectively by P, C and Q, D then $P = Q = \mathrm{P}(T, \phi)$ and the measures μ and ν are equivalent with mutual Radon-Nikodym derivatives uniformly bounded.

Proof. Since X is a compact space, there exist finitely many points $x_1, \ldots, x_l \in X$ such that $B(x_1, \xi) \cup \ldots \cup B(x_l, \xi) = X$. We claim that for every compact set $A \subset X$ every $\delta > 0$ and for all $n \geq 1$ large enough

$$\mu(A) \leq CDl \exp((Q - P)n)(\nu(A) + \delta) \tag{4.1.2}$$

By the compactness of A and by the regularity of the measure ν there exists $\varepsilon > 0$ such that $\nu(B(A, \varepsilon)) \leq \nu(A) + \delta$. Fix an integer $n \geq 1$ so large that $\xi\lambda^{-n} < \frac{\varepsilon}{2}$ and for every $1 \leq i \leq l$ let

$$X(i) = \{x \in T^{-n}(x_i) : A \cap T_x^{-n}(B(x_i, \xi)) \neq \emptyset\}.$$

Then

$$A \subset \bigcup_{i=1}^{l} \bigcup_{x \in X(i)} T_x^{-n}(B(x_i,\xi)) \subset B(A,\varepsilon)$$

and since for any fixed $1 \le i \le l$ the sets $T_x^{-n}(B(x_i,\xi))$ for $x \in T^{-n}(x_i)$ are mutually disjoint, it follows from (4.1.1) that

$$\begin{aligned}
\mu(A) &\le \mu\Big(\bigcup_{i=1}^{l} \bigcup_{x \in X(i)} T_x^{-n}(B(x_i,\xi))\Big) \le \sum_{i=1}^{l} \sum_{x \in X(i)} \mu\big(T_x^{-n}(B(x_i,\xi))\big) \\
&\le C\sum_{i=1}^{l} \sum_{x \in X(i)} \exp(S_n\phi(x) - Pn) = C\exp(Q-P)n)\sum_{i=1}^{l} \sum_{x \in X(i)} \exp(S_n\phi(x) - Qn) \\
&\le CD\exp(Q-P)n)\sum_{i=1}^{l} \sum_{x \in X(i)} \nu\big(T_x^{-n}(B(x_i,\xi))\big) \le CD\exp(Q-P)n)l\nu(B(A,\varepsilon)) \\
&\le CDl\exp(Q-P)n)(\nu(A)+\delta)
\end{aligned}$$

Exchanging the roles of μ and ν we also obtain

$$\nu(A) \le CDl\exp((P-Q)n)(\mu(A)+\delta) \tag{4.1.3}$$

for all $n \ge 1$ large enough. So, if $P \ne Q$, say $P < Q$, then it follows from (4.1.3) applied to the compact set X that $\nu(X) = 0$. Hence $P = Q$, and as by regularity of μ and ν (4.1.2) and (4.1.3) continue to be true for all Borel subsets of X, we conclude that μ and ν are equivalent with the Radon-Nikodym derivative $d\mu/d\nu$ bounded from above by CDl and from below by $(CDl)^{-1}$ (letting $\delta \to 0$).

It is left to show that $P = \mathrm{P}(T,\phi)$. Looking at the expression after the third inequality sign in our estimates of $\mu(A)$ with $A = X$ we get

$$0 = \log\mu(X) \le \log C + \log\Big(\sum_{i=1}^{l} \sum_{x \in X(i)} \exp(S_n\phi(x))\Big) - Pn.$$

Since for every i, $X(i)$ is an (η,n)-separated set, taking into account dividing by n in the definition of pressure, we can replace here $\sum_i$ by a largest summand for each n. We get $P \le P(T,\phi)$.

On the other hand for an arbitrary $x \in X$

$$\sum_{y \in T^{-n}(x)} \exp(S_n\phi(x) - Pn) \le C \sum_{y \in T^{-n}(x)} \mu\big(T_y^{-n}(B(x,\xi))\big) \le C\mu(X) = C$$

gives $\mathrm{P}(T,\phi) = \mathrm{P}_x(T,\phi) \le P$. The proof is finished. ♣

Remark 4.1.2 To prove Proposition 4.1.1 except the part identifying P as $\mathrm{P}(T,\phi)$ we used only the inequalities

$$C^{-1} \le \frac{\mu\big(T_x^{-n}(B(T^n(x),\xi)\big)\exp Pn}{\nu\big(T_x^{-n}(B(T^n(x),\xi)\big)\exp Qn} \le C.$$

We used the function ϕ in (4.1.1) and its Hölder continouity only to prove $P = Q = P(T,\phi)$. Hölder continuity allows us also to replace x in $S_n\phi(x)$ by an arbitrary point contained in $T_x^{-n}(B(T^n(x),\xi))$.

Remark 4.1.3. For $\mathcal{R} = \{R_1, ..., R_d\}$, a Markov partition of diameter smaller than ξ, (4.1.1) produces a constant C depending on $\mathcal{R}$ (see Exercise 1) such that

$$C^{-1} \le \frac{\mu\big(R_{j_0,\dots,j_{n-1}}\big)}{\exp(S_n\phi(x) - Pn)} \le C \tag{4.1.4}$$

for every admissible sequence $j_0, j_1, \dots j_{n-1}$ and every $x \in R_{j_0,\dots,j_{n-1}}$. In particular (4.1.4) holds for the shift map of a one-sided topological Markov chain.

The following completes Proposition 4.1.1.

Proposition 4.1.4. If ϕ and ψ are two arbitrary Hölder continuous functions on X, then the following conditions are equivalent:

(1) $\phi-\psi$ is cohomologous to a constant in the space of bounded functions (see Def.1.11.2).

(2) $G_\phi = G_\psi$.

(3) $G_\phi \cap G_\psi \ne \emptyset$.

Proof. Of course (2) implies (3). That (1) implies (2) is also obvious. If (3) is satisfied, that is if there exists $\mu \in G_\phi \cap G_\psi$, then it follows from (4.1.1) that

$$D^{-1} \le \exp(S_n(\phi)(x) - S_n(\psi)(x) - n\mathrm{P}(\phi) + n\mathrm{P}(\psi)) \le D$$

for some constant D, all $x \in X$ and $n \in I\!N$. Acting with the logarithms we see that the condition (3) in Proposition 3.4.5 is satisfied with ϕ and ψ replaced by $\phi-\mathrm{P}(\phi)$ and $\psi-\mathrm{P}(\psi)$ respectively. Hence, by this Proposition $\phi - P(phi)$ and $\psi - \mathrm{P}(\psi)$ are cohomologous which finishes the proof. ♣

We shall prove later that the class of Gibbs states associated to T and ϕ is not empty (Sec.3) and contains exactly one Gibbs state which is T-invariant (Corollary 4.2.9). Actually we shall prove a stronger uniqueness theorem. We shall prove that any invariant Gibbs state is an equilibrium state for T and ϕ and prove (Sec.6) uniqueness of the equilibrium state for open expanding T and Hölder continuous ϕ.

Proposition 4.1.5 A probability T-invariant Gibbs state μ is an equilibrium state for T and ϕ.

Proof. Consider an arbitrary finite partition $\mathcal{P}$ into Borel sets of diameter less than $\min(\eta, \xi)$. Then for every $x \in X$ we have $T_x^{-n}(B(T^n(x), \xi)) \supset \mathcal{P}^n(x)$, where $\mathcal{P}^n(x)$ is the element of the partition $\mathcal{P}^n = \bigvee_{j=0}^{n} \mathcal{P}$ that contains x. Hence $\mu\big(T_x^{-n}(B(T^n(x), \xi))\big) \geq \mu(\mathcal{P}^n(x))$. Therefore by the Shannon-McMillan-Breiman Theorem and (4.1.1) one obtains

$$\mathrm{h}_\mu(T) \geq \mathrm{h}_\mu(T, \mathcal{P}) \geq \int \Big(\limsup_{n\to\infty} \frac{1}{n}(n\, \mathrm{P}(T, \phi)) - S_n\phi(x)\Big) d\mu = \mathrm{P}(T, \phi) - \int \phi\, d\mu.$$

or in other words, $\mathrm{h}_\mu(T) + \int \phi\, d\mu \geq \mathrm{P}(T, \phi)$ which just means that μ is an equilibrium state. ♣

SECTION 4.2. TRANSFER OPERATOR AND ITS CONJUGATE. MEASURE WITH PRESCRIBED JACOBIAN.

Suppose first that we are in the situation of Chapter 1., i.e. T is a measurable map. Suppose that m is backward quasi-invariant with respect to T, i.e.

$$T_*(m) = m \circ T^{-1} \prec m. \tag{4.2.1}$$

Then by the Radon-Nikodym Theorem there exists an m-integrable function Φ such that for every measurable A we have $m(T^{-1}(A)) = \int_A \Phi dm$. One writes $d(m \circ T^{-1})/dm = \Phi$. In the situation of this Chapter, where T is a local homeomorphism (one does not need expanding yet) if T^{-1} has d branches on a ball $B(x, \xi)$ mapping the ball to $U_1, ..., U_d$ respectively, then $\Phi = \sum_{j=1}^d \Phi_j$ where $\Phi_j := d(m \circ (T|_{U_j})^{-1})/dm$. If we consider measures absolutely continuous with respect to a "reference measure" m then the transformation $\mu \mapsto T_*(\mu)$ can be rewritten in the language of densities with respect to m as

$$d\mu/dm \mapsto d(T_*\mu)/dm = \sum_{j=1}^d \big((d\mu/dm) \circ (T|_{U_j})^{-1}\big)\Phi_j. \tag{4.2.1a}$$

It is comfortable to define $\Psi(z) = \frac{d(m \circ T_x^{-1})}{dm}(T(z))$, i.e. $\Psi = \Phi_j \circ T$ if $X \in U_j$. Notice that Ψ is defined on a set whose T-image has full measure (a set maybe larger than a set of full measure), see Sec.6 for further discussion.

The transformation in (4.2.1a) can be considered as a linear operator $\mathcal{L}_m : L^1(m) \to L^1(m)$,

$$\mathcal{L}_m(u)(x) = \sum_{\overline{x} \in T^{-1}(x)} u(\overline{x})\Psi(\overline{x}).$$

This makes sense, because if we change u on a set A of measure 0, then even if $m(T(A)) > 0$, we have $\Phi_j|_{T(A)} = 0$ m-a.e., hence $\mathcal{L}_m(u)$ does not depend on u on $T(A)$.
We have fixed the convension that if u is not defined (on a set of measure 0) an $\Psi = 0$, then $u\Psi = 0$.

Thus we obtain the following characterization of probability T-invariant measures absolutely continuous with respect to m.

Proposition 4.2.0. A probability measure $\mu = hm$, $h \geq 0$, is T-invariant if and only if

$$\mathcal{L}_m(h) = h.$$

After this introduction, the appearence of the following linear operator, called the Perron-Frobenius-Ruelle or Ruelle or Araki or also *transfer operator*, is not surprising:

$$\mathcal{L}_\phi(u)(x) = \sum_{\overline{x} \in T^{-1}(x)} u(\overline{x}) \exp(\phi(\overline{x})).$$

If the function ϕ is fixed we omit sometimes the subscript ϕ at $\mathcal{L}$. The function ϕ is often called a *potential* function.

The transfer's conjugate operator will be our tool to find a quasi-invariant measure m such that Ψ will be a scalar multiple of $\exp\phi$, hence $\mathcal{L}_m$ will be a scalar multiple of $\mathcal{L}_\phi$. Then in turn we will look for fixed points of $\mathcal{L}_m$ to find invariant measures. Restricting our attention to $\exp\phi$, we restrict considerations to Ψ strictly positive defined everywhere. One sometimes allows ϕ to have the value $-\infty$, but we do not consider this case in our book.

Let us be now more precise. Consider $\mathcal{L}_\phi$ acting on the Banach space of continuous functions $\mathcal{L}_\phi : C(X) \to C(X)$. It is a continuous linear operator and its norm is equal to $\sup_x \sum_{\overline{x}\in T^{-1}(x)} \exp(\phi(\overline{x})) = \sup \mathcal{L}_\phi(1\!\!1)$ as this is a positive operator i.e. it maps real non-negative functions to real non-negative functions (see Ch.2.1). Consider the conjugate operator $\mathcal{L}^*_\phi : C^*(X) \to C^*(X)$. Note that as conjugate to a positive operator it is also positive, i.e. transforms measures into measures.

Lemma 4.2.1. For every $\mu \in C^*(X)$ and every Borel set $A \subset X$ on which T is injective

$$\mathcal{L}^*_\phi(\mu)(A) = \int_{T(A)} \exp(\phi \circ (T|_A)^{-1}) d\mu \tag{4.2.2}$$

Proof. It is sufficient to prove (4.2.2) for $A \subset B(x,r)$ with any $x \in X$ and r such that T is injective on $B(x, 2r)$ (say $r = \eta$). Approximate in pointwise convergence the indicator function χ_A by uniformly bounded continuous functions with support in $B = B(x, 2r)$. We have for any such function f

$$\mathcal{L}^*_\phi(\mu)(f) = \mu(\mathcal{L}_\phi(f)) = \int_{T(B)} (f \exp(\phi)) \circ (T|_B)^{-1} d\mu.$$

We used here the fact that the only branch of T^{-1} mapping $T(B)$ to the support of f is that one leading $T(B)$ to B. Passing with f to the limit χ on both sides (Lebesgue convergence theorem) gives (4.2.2). ♣

Observe that whereas $\mathcal{L}_\phi$ transports measure from the past, $\mathcal{L}^*_\phi$ pulls it back from the future with Jacobian $\exp\phi$. This is the right operator to use, to look for the missing "reference measure" m.

Definition 4.2.2. Recall from Chapter 1 (Def.1.9.4) that a measurable function $J : X \to [0,\infty)$ is called the *Jacobian* or *strong Jacobian* of a map $T : X \to X$ with respect to a measure μ if for every Borel set $A \subset X$ on which T is injective $\mu(T(A)) = \int_A J d\mu$. In particular μ is forward quasi-invariant

J is called the *weak Jacobian* if $J : X \to [0,\infty)$ and there exists a Borel set $E \subset X$ such that $\mu(E) = 0$ and for every Borel set $A \subset X$ on which T is injective, $\mu(T(A \setminus E)) = \int_A J d\mu$.

Notice that if μ is backward quasi-invariant then the condition that J is the weak Jacobian translates to $\mu(A) = \int_{T(A)} \frac{1}{J\circ(T|_A)^{-1}} d\mu$.

Corollary 4.2.3. If a probability measure μ satisfies $\mathcal{L}_\phi^*(\mu) = c\mu$ (i.e. μ is an eigenmeasure of $\mathcal{L}_\phi^*$ corresponding to a positive eigenvalue c), then $c\exp(-\phi)$ is the Jacobian of T with respect to μ.

Proof. Substitute $c\mu$ in place of $\mathcal{L}^*(\mu)$ in (4.2.2). It then follows that μ is backward quasi-invariant and $c\exp(-\phi)$ is the weak Jacobian of T with respect to μ. Since $\frac{1}{\exp(-\phi)} = \exp\phi$, it is positive everywhere, hence $c\exp(-\phi)$ is strong Jacobian of T. ♣

Theorem 4.2.4. Let $T : X \to X$ be a local homeomorphism of a compact metric space X and let $\phi : X \to I\!R$ be continuous. Then there exists a probability measure $m = m_\phi$ and a constant $c > 0$, such that $\mathcal{L}_\phi^*(m) = cm$. The function $c\exp(-\phi)$ is strong Jacobian for T with respect to the measure m.

Proof. Consider the map $l(\mu) := \frac{\mathcal{L}^*(\mu)}{\mathcal{L}^*(\mu)(1\!\!1)}$ on the convex set of probability measures on X, i.e. on $M(X)$, endowed with the weak* topology (Ch.2.1). l is continuous in this topology since $\mu_n \to \mu$ weak* implies for every $u \in C(X)$ that $\mathcal{L}^*(\mu_n)(u) = \mu_n(\mathcal{L}(u)) \to \mu(\mathcal{L}(u)) = \mathcal{L}^*(\mu)(u)$. As $M(X)$ is weak* compact (see Th.2.1.6) we can use Theorem 2.1.7 (Schauder-Tychonoff fixed point theorem) to find $m \in M(X)$ such that $l(m) = m$. Hence $\mathcal{L}^*(m) = cm$ for $c = \mathcal{L}^*(m)(1\!\!1)$. T has Jacobian equal to $c\exp(-\phi)$, by Corollary 4.2.3. ♣

Note again that we write $\exp\phi$ in order to guarantee it never vanishes, so that there exists the Jacobian for T with respect to m. To find an eigen-measure m for $\mathcal{L}^*$ (i.e. with a weak Jacobian being a multiple of $\exp(-\phi)$) we could perfectly allow $\exp\phi = 0$.

We have the following complementary fact in case Jacobian J exists.

Proposition 4.2.4a. If $T : X \to X$ is a local homeomorphism of a compact metric space X and a Jacobian J with respect to a probability measure m exists, then for every Borel set A

$$\frac{1}{d}\int_A J\,dm \le m(T(A)) \le \int_A J\,dm.$$

where d is the degree of T ($d := \sup_{x\in X} \sharp T^{-1}(\{x\})$). In particular if $m(A) = 0$, then $m(T(A)) = 0$.

Proof. Let us partition A into finitely many Borel sets, say $A_1, A_2, \dots, A_n$, of diameters so small that T restricted to each of them is injective. Then, on one hand,

$$m(T(A)) = m\Big(\bigcup_{i=1}^n T(A_i)\Big) \le \sum_{i=1}^n m(T(A_i)) = \sum_{i=1}^n \int_{A_i} J\,dm = \int_A J\,dm.$$

and on the other hand, since the multiplicity of the family $\{T(A_i) : 1 \le i \le n\}$ does not exceed d,

$$m(T(A)) = m\Big(\bigcup_{i=1}^n T(A_i)\Big) \ge \frac{1}{d}\sum_{i=1}^n m(T(A_i)) = \frac{1}{d}\sum_{i=1}^n \int_{A_i} J\,dm = \frac{1}{d}\int_A J\,dm.$$

The proof is finished. ♣

Let us go back to T open distance expanding topologically transitive map.

Proposition 4.2.5. The measure m is positive on non-empty open sets. Moreover for every $r > 0$ there exists $\alpha = \alpha(r) > 0$ such that for every $x \in X$, $m(B(x,r)) \geq a$.

Proof. For every open $U \subset X$ there exists $n \geq 0$ such that $\bigcup_{j=0}^{n} T^j(U) = X$ (Theorem 3.3.9). So, by Proposition 4.2.4a, $m(U) = 0$ would imply that
$1 = m(X) \leq \sum_{j=0}^{n} m(T^j(U)) = 0$, a contradiction.

Finally let $x_1, ..., x_m$ be an $r/2$-net in X and $\alpha := \min_{1 \leq j \leq m}\{m(B(x_j, r/2))\}$. Since for every $x \in X$ there exists j such that $\mathrm{dist}(x, x_j) \leq r/2$, hence $m(B(x,r)) \supset B(x_j, r/2$, it is enough to set $\alpha(r) := \alpha$.

Proposition 4.2.6. The measure m is a Gibbs state of ϕ and $\log c = \mathrm{P}(T, \phi)$.

Proof. We have for every $x \in X$ and every integer $n \geq 0$,

$$m(B(x,\xi)) = \int_{T_x^{-n}(B(T^n(x),\xi))} c^n \exp(-S_n\phi)\, dm.$$

Since, by Lemma 3.4.2, the ratio of the supremum and infimum of the integrand of the above integral is bounded from above by a constant $C > 0$ and from below by C^{-1}, we obtain

$$1 \geq m(B(x,\xi)) \geq C^{-1} c^n \exp(-S_n\phi(x)) m(T_x^{-n}(B(T^n(x),\xi)))$$

and

$$\alpha(\xi) \leq m(B(x,\xi)) \leq C c^n \exp(-S_n\phi(x)) m(T_x^{-n}(B(T^n(x),\xi))).$$

Hence

$$\alpha(\xi) C^{-1} \leq \frac{m(T_x^{-n}(B(T^n(x),\xi)))}{\exp(S_n\phi(x) - n\log c)} \leq C$$

and therefore m is a Gibbs state. That $\log c = \mathrm{P}(T, \phi)$ follows now from Proposition 4.1.1. ♣

We now also give a simple direct proof of equality $\log c = \mathrm{P}(T, \phi)$. First note that by the definition of $\mathcal{L}_\phi$ and a simple inductive argument, for every integer $n \geq 0$

$$\mathcal{L}_\phi^n(u)(x) = \sum_{\overline{x} \in T^{-n}(x)} u(\overline{x}) \exp(S_n\phi(\overline{x})). \tag{4.2.2a}$$

The estimate (3.4.3) translates to

$$C^{-1} \leq \mathcal{L}^n(1\!\!1)(x)/\mathcal{L}^n(1\!\!1)(y) \leq C \quad \text{for every} \quad x, y \in X. \tag{4.2.3}$$

Now $c^n = c^n m(1\!\!1) = (\mathcal{L}^*)^n(m)(1\!\!1) = m(\mathcal{L}^n(1\!\!1))$ and hence

$$\log c = \lim_{n\to\infty} \frac{1}{n} \log m(\mathcal{L}^n(1\!\!1)) = P(T, \phi).$$

The latter equality follows from (4.2.3) and Proposition 3.4.3.

Note that in the latter equality we used the property that m is a measure (positive). For m a signed eigen-measure and c a complex eigenvalue for $\mathcal{L}^*$ we would obtain only $\log|c| \le P(T,\phi)$ (one should consider a function u such that $\sup|u| = 1$ and $m(u) = 1$ rather than the function $1\!\!1$) and indeed usually the point spectrum of $\mathcal{L}^*$ is big(ref ??????).

We are in the position to prove already some ergodic properties of Gibbs states:

Theorem 4.2.7. If T is topologically mixing, then the system (T,m) is exact in the measure theoretic sense, namely for every A of positive measure $m(f^n(A)) \to 1$ as $n \to \infty$.

The topological background of this Theorem is the topological exactness: Th.3.3.10, which plays a crucial role in the proof.

Proof. Let E be an arbitrary Borel set with $m(E) > 0$. By the regularity of m we can find a compact set $A \subset E$ such that $m(A) > 0$. Fix an arbitrary $\varepsilon > 0$. As in the proof of Proposition 4.1.1, we find n and a covering of A by sets D_ν of the form $T_x^{-n}(B(x_i,\xi)), x \in X(i), i = 1,...,l$ such that $m(\bigcup_\nu D_\nu) \le m(A) + \varepsilon$. Hence $m(\bigcup_\nu (D_\nu \setminus A)) \le \varepsilon$. Since the multiplicity of this covering is at most l, we have

$$\sum_\nu m(D_\nu \setminus A) \le l\varepsilon.$$

Hence

$$\frac{\sum_\nu m(D_\nu \setminus A)}{\sum_\nu m(D_\nu)} \le \frac{l\varepsilon}{m(A)}.$$

Therefore there exists $D = D_{\nu_\phi} = T_x^{-n}(B)$, for $B = B(x_i,\xi))$ and arbitrary n large enough, such that

$$\frac{m(D \setminus A)}{m(D)} \le \frac{l\varepsilon}{m(A)}.$$

Hence

$$\frac{m(B \setminus T^n(A))}{m(B)} \le \frac{1}{m(B)} \int_{D\setminus A} c^n \exp(-S_n\phi)dm \le C\frac{m(D \setminus A)}{m(D)} \le C\frac{l\varepsilon}{m(A)}.$$

with C as in Proof of Proposition 4.2.6. By the topological exactness of T, there exists $N \ge 0$ such that for every i we have $T^N(B(x_i,\xi)) = X$. In particular $T^N(B) = X$. So, using Proposition 4.2.4a, we get

$$m(X \setminus T^N(T^n(A))) \le m(T^N(B \setminus T^n(A))) \le c^N(\inf \exp\phi)^{-N}\frac{Cl\varepsilon}{m(A)}.$$

Letting $\varepsilon \to 0$ we obtain $m(X \setminus T^N(T^n(A))) \to 0$ as $n \to \infty$, hence $m(T^{N+n}(A)) \to 1$. ♣

We have considered here a special Gibbs measure $m = m_\phi$. Notice however that by Proposition 4.1.1 the assertion of Theorem 4.2.7 holds for every Gibbs measure associated to T and ϕ.

Corollary 4.2.8. If T is a topologically transitive, open, distance expanding map, then for every Hölder potential ϕ, each corresponding Gibbs measure is ergodic.

Proof. By Ch.3.3 ???? there exists a positive integer N such that T^N is topologically mixing on a T^N-invariant open-closed set $Y \subset X$, where $\bigcup_{j=0,\dots,N-1} T^j(Y) = X$. So our $T^N|_Y$, being also an open expanding map, is exact in the measure-theoretic sense by Theorem 3.7. So if $m(E) > 0$ then for every $j = 0, ..., N-1$ we have $m(T^{Nn}T^j(E)) \to m(T^j(Y))$, hence $m(\bigcup_{n\geq 0} T^n(E)) \to 1$. For E being T-invariant this yields $m(E) = 1$. This implies ergodicity. ♣

With the use of Proposition 1.2.4 we get promised

Corollary 4.2.9. If T is a topologically transitive, open, distance expanding map, then for every Hölder potential ϕ, there is at most one corresponding invariant Gibbs measure.

SECTION 4.3. ITERATION OF TRANSFER OPERATOR. EXISTENCE OF GIBBS STATES.

It is comfortable to consider the operator $\mathcal{L}_{\overline{\phi}}$ for $\overline{\phi} = \phi - \mathrm{P}(T,\phi)$. That is $\mathcal{L}_{\overline{\phi}} = \mathrm{e}^{-\mathrm{P}(T,\phi)}\mathcal{L}_\phi$. (Recall that $\mathrm{P}(T,\phi) = \log c$.) Then for the reference measure $m = m_\phi$ satisfying $\mathcal{L}^*_\phi(m) = \mathrm{e}^{\mathrm{P}(\phi)}m$ we have $\mathcal{L}^*_{\overline{\phi}}(m) = m$ i.e.

$$\int u dm = \int \mathcal{L}_{\overline{\phi}}(u) dm \quad \text{for every} \quad u \in C(X). \tag{4.3.1}$$

For fixed ϕ we often denote $\mathcal{L}_{\overline{\phi}}$ by $\mathcal{L}_0$. By (4.2.3) for every $x,y \in X$, and non-negative integer n

$$\mathcal{L}_0^n(1\!\!1)(x)/\mathcal{L}_0^n(1\!\!1)(y) \le C. \tag{4.3.2}$$

Multiplying this inequality by $\mathcal{L}_0^n(1\!\!1)(y)$ and then integrating with respect to the variable x and y we get respectively the first and the third of the following inequalities below

$$C^{-1} \le \inf \mathcal{L}_0^n(1\!\!1) \le \sup \mathcal{L}_0^n(1\!\!1) \le C. \tag{4.3.3}$$

By (3.4.2) for every $x,y \in X$ such that $x \in B(y,\xi)$ we have an inequality more refined than (3.4.3). Namely
(4.3.4)

$$\frac{\mathcal{L}_\phi^n(1\!\!1)(x)}{\mathcal{L}_\phi^n(1\!\!1)(y)} = \frac{\sum_{\overline{x}\in T^{-n}(x)} \exp S_n\phi(\overline{x})}{\sum_{\overline{y}\in T^{-n}(y)} \exp S_n\phi(\overline{y})} \le \sup_{\overline{x}\in T^{-n}(x)} \frac{\exp S_n\phi(\overline{x})}{\exp S_n\phi(y_n(\overline{x}))} \le \exp(C_1\rho(x,y)^\alpha),$$

where $C_1 = \frac{\vartheta_\alpha(\phi)}{1-\lambda^{-\alpha}}$ and $y_n(\overline{x}) := T_{\overline{x}}^{-n}(y)$. By this estimate and by (4.3.3) we get for all $n \ge 1$ and all $x,y \in X$ such that $x \in B(y,\xi)$

$$\begin{aligned} \Big|\mathcal{L}_0^n(1\!\!1)(x) - \mathcal{L}_0^n(1\!\!1)(y)\Big| &= \Big|\frac{\mathcal{L}_0^n)(1\!\!1)(x)}{\mathcal{L}_0^n(1\!\!1)(y)} - 1\Big|\mathcal{L}_0^n(1\!\!1)(y) \le \\ C|\exp(C_1\rho(x,y)^\alpha) - 1| &\le C_2\rho(x,y)^\alpha \end{aligned} \tag{4.3.6}$$

with C_2 depending on C, C_1 and ξ.

Proposition 4.3.1. There exists a positive function $u_\phi \in \mathcal{H}_\alpha(X)$ such that $\mathcal{L}_0(u_\phi) = u_\phi$ and $\int u_\phi \, dm = 1$.

Proof. By (4.3.6) and (4.3.3) the functions $\mathcal{L}_0^n(1\!\!1)$ have uniformly bounded norms in Hölder space $\mathcal{H}_\alpha(X)$, see Ch.3.4. Hence by Arzela-Ascoli theorem there exists a limit u_ϕ for a subsequence of $u_n = \frac{1}{n}\sum_{j=0}^{n-1} \mathcal{L}_0^j(1\!\!1)$, $n = 1, \ldots$. Of course $u_\phi \in \mathcal{H}_\alpha(X)$, $C^{-1} \le u_\phi \le C$, and using (4.3.3), a straightforward computation shows that $\mathcal{L}_0(u_\phi) = u_\phi$ (compare 2.1.14). Also $\int u_\phi \, dm = \lim_{n\to\infty} \int u_n \, dm = \int 1\!\!1 \, dm = 1$. The proof is finished. ♣

Combining this proposition, Proposition 4.2.0, and Proposition 4.2.6 we get the following.

Theorem 4.3.2. For every Hölder continuous function $\phi : X \to I\!\!R$ there exists an invariant Gibbs state associated to T and ϕ, namely $\mu_\phi = u_\phi m_\phi$.

In the rest of this Section we provide a closer study of iteration of $\mathcal{L}_0$ on the real or complex Banach spaces $C(X)$ and $\mathcal{H}_\alpha$.

Definition 4.3.3. We call a continuous linear operator $Q : B \to B$ on a Banach space B *almost periodic* if for every $b \in B$ the sequence $Q^n(b), n = 0, 1, ...$ is relatively compact, i.e. its closure in B is compact (in the norm topology).

Proposition 4.3.4. The operators $\mathcal{L}_0^n$ on $C(X)$ have uniformly bounded norm for all $n = 1, 2, ...$

Proof. By the definition of $\mathcal{L}$ and by (4.3.3) for every $u \in C(X)$:

$$\sup |\mathcal{L}_0^n(u)| \le \sup |u| \sup |\mathcal{L}_0^n(1\!\!1)| \le C \sup |u| \tag{4.3.8}$$

♣

Remark that instead of referring to the form of $\mathcal{L}$ one can only refer to the fact that $\mathcal{L}$ is a positive operator, hence its norm is attained on $1\!\!1$.

Theorem 4.3.5. The operator $\mathcal{L}_0$ is almost periodic on $C(X)$. Moreover all the functions $\mathcal{L}^n(u)$ are equicontinuous and have uniformly bounded absolute values, provided $\mathcal{L}_0$ is associated to ϕ belonging to a bounded set in $\mathcal{H}_\alpha$ and u taken from a family of equicontinuous functions, of uniformly bounded absolute values.

Proof. For every $x \in X$ and $n > 0$ denote $\exp(S_n\phi(x))$ by $E_n(x)$. Consider arbitrary $x \in X, y \in B(x, \xi)$. Use the notation $y_n(\overline{x}) := T_{\overline{x}}^{-n}(y)$, the same as in (4.3.4). We have for every $u \in C(X)$

$$|\mathcal{L}_0^n(u)(x) - \mathcal{L}_0^n(u)(y)| = |\sum_{\overline{x} \in T^{-n}(x)} u(\overline{x})E_n(\overline{x}) - u(y_n(\overline{x}))E_n(y_n(\overline{x})|$$

$$\le |\sum_{\overline{x} \in T^{-n}(x)} u(\overline{x})(E_n(\overline{x}) - E_n(y_n(\overline{x})))| + |\sum_{\overline{x} \in T^{-n}(x)} E_n(y_n(\overline{x}))(u(\overline{x}) - u(y_n(\overline{x}))|$$

$$\le (\sup |u|)C_2\rho(x, y)^\alpha + C \sup_{\overline{x}} |u(\overline{x}) - u(y_n(\overline{x}))| \tag{4.3.9}$$

by (4.3.6) and (4.3.3). Denote a modulus of uniform continuity of u by h, i.e. consider an increasing function $h : I\!\!R^+ \to I\!\!R^+$ such that $\lim_{\varepsilon \searrow 0} h(\varepsilon) = 0$ and for every $z_1, z_2 \in X$ $|u(z_1) - u(z_2)| \le h(\rho(z_1, z_2))$. (4.3.9) gives:

$$|\mathcal{L}_0^n(u)(x) - \mathcal{L}_0^n(u)(y)| \le (\sup|u|)C_2\rho(x,y)^\alpha + Ch(\rho(x,y)) := g(\rho(x,y)) \tag{4.3.10}$$

We conclude that all functions $\mathcal{L}_0^n(u)$ have the same modulus of continuity g, depending on h, $\sup|\phi|$ and $||\phi||_{\mathcal{H}_\alpha}$. They are also uniformly bounded by Proposition 4.3.4. Hence by Arzela-Ascoli theorem the sequence $\mathcal{L}_0^n(u)$ is relatively compact.

If we consider a family of functions u rather than one function, we set h a modulus of continuity of the family.

♣

For $u \in \mathcal{H}_\alpha$ we obtain the fundamental estimate (4.3.11):

Theorem 4.3.6. There exist constants $C_3, C_4 > 0$ such that for every $u \in \mathcal{H}_\alpha$ and $n = 1, 2, ...$

$$\vartheta_\alpha(\mathcal{L}_0^n(u)) \le C_3\lambda^{-n\alpha}\vartheta_\alpha(u) + C_4\|u\|_\infty, \quad \lambda > 1 \tag{4.3.11}$$

Proof. Continuing the third line of (4.3.9) we obtain

$$|\mathcal{L}_0^n(u)(x) - \mathcal{L}_0^n(u)(y)| \le \|u\|_\infty C_2\rho(x,y)^\alpha + C\vartheta_{\alpha,\xi}(u)\lambda^{-n\alpha}\rho(x,y)^\alpha$$

We have applied here $\rho(\overline{x}, y_n(\overline{x})) \le \lambda^{-n}\rho(\overline{x}, \overline{y})$

This proves (4.3.11), provisionally with $\vartheta_{\alpha,\xi}$ rather than ϑ_α, with $C_3 = C$ from (3.4.3) and (4.3.3) and with $C_4 = C_2$ (recall that the latter constant is of order CC_1 where C_1 appeared in (4.3.4)). Passing to ϑ_α changes C_4 to $\max\{C_4, 2C/\xi^\alpha\}$, see (3.3.8) and Ch.3.4. ♣

Corollary 4.3.7. There exist an integer $N > 0, \tau < 1, C_5 > 0$ such that for every $u \in \mathcal{H}_\alpha$

$$\|\mathcal{L}_0^N(u)\|_{\mathcal{H}_\alpha} \le \tau\|u\|_{\mathcal{H}_\alpha} + C_5\|u\|_\infty \tag{4.3.12}$$

Proof. This Corollary immediately follows from (4.3.11) and Proposition 4.3.4. ♣

In fact (4.3.12) together with (4.3.8) imply a similar fact for itarates of $\mathcal{L}^N$, which resembles back (3.3.11). Namely the following holds

Proposition 4.3.8.

$$\exists C_6 > 0\ \forall n = 1, 2, ... \quad \|\mathcal{L}_0^{nN}(u)\|_{\mathcal{H}_\alpha} \le \tau^n\|(u)\|_{\mathcal{H}_\alpha} + C_6\|u\|_\infty \tag{4.3.13}$$

Proof. Substitute in (4.3.12) $\mathcal{L}_0^N(u)$ in place of u etc. n times using $\|\mathcal{L}_0^j(u)\|_\infty \le C\|u\|_\infty$. You obtain (4.1.13) with $C_6 = CC_5/(1-\tau)$. ♣

In Appendix we prove a general theorem by Ionescu-Tulcea and Marinescu (abbr.: ITM), which under assumptions (4.3.8), (4.3.12) gives an information about the spectrum of $\mathcal{L}_0$. Ch.3 Sec.5 is devoted to this. This occurs useful in other than expanding and Hölder cases. Here, in the next Section, assuming topological mixing of T, we shall proceed directly, not referring to ITM Theorem.

Analogously to $\mathcal{L}_0^n$ considered on $C(X)$ the convergence theorem below is a special case of a general theory of almost periodic operators, see Sec.5.

SECTION 4. CONVERGENCE OF $\mathcal{L}^n$. MIXING PROPERTIES OF GIBBS MEASURES

Recall that the considerations preceding Theorem 4.3.2 proved the following

Proposition 4.4.1 There exists a positive function $u_\phi \in \mathcal{H}_\alpha(X)$ such that $\mathcal{L}_0(u_\phi) = u_\phi$.

It is convienient to replace the operator $\mathcal{L}_0$ by $\hat{\mathcal{L}}(u) = \frac{1}{u_\phi}\mathcal{L}_0(uu_\phi)$.

If we denote the operator of multiplication by a function w by the same symbol w then we can write $\hat{\mathcal{L}}(u) = u_\phi^{-1} \circ \mathcal{L}_0 \circ u_\phi$. Since $\hat{\mathcal{L}}$ and $\mathcal{L}_0 = \mathcal{L}_{\overline{\phi}}$ are conjugate by the operator u_ϕ, their properties (e.g. spectrum) are the same. In addition, as this operator is positive, non-negative functions go to non-negative functions. Hence measures are mapped to measures for the conjugate operator.

Proposition 4.4.2. $\hat{\mathcal{L}} = \mathcal{L}_\psi$ where $\psi = \overline{\phi} + \log u_\phi - \log u_\phi \circ T = \phi - \mathrm{P}(T,\phi) + \log u_\phi - \log u_\phi \circ T$.

Proof. $\hat{\mathcal{L}}(u)(x) = \frac{1}{u_\phi(x)} \sum_{T(\overline{x})=x} u(\overline{x})u_\phi(\overline{x}) \exp\phi(\overline{x}) = \sum_{T(\overline{x})=x} u(\overline{x}) \exp(\phi(\overline{x})$
$+\log u_\phi(\overline{x}) - \log u_\phi(x))$ ♣

Note that the eigenfunction u_ϕ for $\mathcal{L}_0$ has changed to the eigenfunction $1\!\!1$ for $\hat{\mathcal{L}}$. In other words we have

Proposition 4.4.3. $\hat{\mathcal{L}}(1\!\!1) = 1\!\!1$, i.e. for every $x \in X$

$$\sum_{\overline{x}\in T^{-1}(x)} \exp\psi(\overline{x}) = 1 \tag{4.4.1}$$

♣

Note that Jacobian of T with respect to the Gibbs measure $\mu = u_\phi m$ (see Th. 4.3.2) is $(u_\phi \circ T)(\exp(-\overline{\phi}))u_\phi^{-1} = \exp(-\psi)$. So for ψ the reference measure (with Jacobian $\exp(-\psi)$) and the invariant Gibbs measure coincide.

Note that passing from $\mathcal{L}_\phi$, through $\mathcal{L}_{\overline{\phi}}$, to $\mathcal{L}_\psi$ we just replaced ϕ by cohomological (up to a constant) functions. By Proposition 4.1.3. this does not change the set of Gibbs states.

One can think of the transformation $u \mapsto u/u_\phi$ as new coordinates on $C(X)$ or $\mathcal{H}_\alpha(X)$ (real or complex-valued functions). $\mathcal{L}_0$ changes in these coordinates to $\mathcal{L}_\psi$ and the functional $m(u)$ to $m(u_\phi u)$. The latter, denote it by m_ψ, is the eigenmeasure for $\mathcal{L}_\psi^*$ with the eigenvalue 1. It is positive because the operator u_ϕ is positive (see the comment above). So $\exp(-\psi)$ is the Jacobian for m_ψ by Corollary 4.2.3. Hence by (4.4.1) m_ψ is T-invariant. This is our invariant Gibbs measure μ.

For $\hat{\mathcal{L}}$ Proposition 4.3.4 changes to

Proposition 4.4.4. $\|\hat{\mathcal{L}}\|_\infty = 1$.

Proof. $\sup|\hat{\mathcal{L}}(u)| \le \sup|u|$ because $\hat{\mathcal{L}}$ is an operator of "taking an average" of u from the past (by Proposition 4.4.3). The equality follows from $\hat{\mathcal{L}}(1\!\!1) = 1\!\!1$. ♣

Topological mixing of T gives a stronger result:

Lemma 4.4.5. If T is topologically mixing, then given any increasing function $g : I\!\!R^+ \to I\!\!R^+$ such that $\lim_{\varepsilon\searrow 0} g(\varepsilon) = 0$, $(\forall \delta_1 > 0$ and $K > 0)$ $(\exists \delta_2 = \delta_2(g, \delta_1, K) > 0$ and a positive integer $n = n(g, \delta_1, K) > 0)$ such that for all $\phi \in \mathcal{H}_\alpha$ with $||\phi||_{\mathcal{H}_\alpha} \le K$ and $u \in C(X, I\!\!R)$ with modulus of continuity g) (i.e. for every $z_1, z_2 \in X$ $|u(z_1) - u(z_2)| \le g(\rho(z_1, z_2))$, and such that $\int u d\mu = 0$ and $\sup|u| \ge \delta_1$

$$\sup|\hat{\mathcal{L}}^n(u)| \le \sup|u| - \delta_2.$$

Proof. Fix $\varepsilon > 0$ so that $h(\varepsilon) < \delta_1/2$. Let n be chosen for ε according to Proposition 3.3.11 $((\exists n)(\forall x) T^n(B(x,\varepsilon)) = X)$. By $\int u d\mu = 0$ there exist $y_1, y_2 \in X$ such that $u(y_1) \le 0, u(y_2) \ge 0$. For an arbitrary $x \in X$ choose $x' \in B(y_1, \varepsilon) \cap T^{-n}(x)$ (it exists by the definition of n). We have $u(x') \le \delta_1/2$. So

$$\hat{\mathcal{L}}^n(u)(x) = u(x') \exp S_n\psi(x') + \sum_{\overline{x}\in T^{-n}(x)\setminus\{x'\}} u(\overline{x}) \exp S_n\psi(\overline{x})$$

$$\le (\sup|u| - \delta_1/2) \exp S_n\psi(x') + \sup|u| \sum_{\overline{x}\in T^{-n}(x)\setminus\{x'\}} \exp S_n\psi(\overline{x})$$

$$\le \sup|u| \Big(\sum_{\overline{x}\in T^{-n}(x)} \exp S_n\psi(\overline{x})\Big) - (\delta_1/2) \exp S_n\psi(x') = \sup|u| - (\delta_1/2) \exp S_n\psi(x').$$

Similarly for $x' \in B(y_2, \varepsilon) \cap T^{-n}(x)$

$$\hat{\mathcal{L}}^n(u)(x) \ge -\sup|u| + (\delta_1/2) \exp S_n\psi(x').$$

Thus we proved our Lemma, with $\delta_2 := (\delta_1/2) \inf_{x\in X} \exp S_n\psi(x)$.

Note that we used here the existence of a uniform bound, $\sup|\psi| \le \sup|\phi| + 2\sup|\text{-}\log(u_\phi)|$ and $\sup|\log(u_\phi)| \le \log C$, where C depends on K, see (4.2.3), (3.4.2), (3.4.3). ♣

We obtain now Theorem which completes Prop.4.3.4 and Th.4.3.5.

Theorem 4.4.6. For every $u \in C(X, C\!\!\!\!I)$ and T topologically mixing open expanding map

(4.4.2) $$c^{-n}\mathcal{L}_\phi^n(u) - m_\phi(u)u_\phi \to 0 \quad \text{(coverges uniformly)} \quad \text{as} \quad n \to \infty$$

In particular if $\int u d\mu = 0$ then

$$\hat{\mathcal{L}}^n(u) \to 0 \tag{4.4.3}$$

Moreover the convergences in (4.4.2) and (4.4.3) are uniform in every set of equicontinuous functions u of uniformly bounded absolute values, and ϕ in a bounded set in $\mathcal{H}_\alpha(X)$.

Proof. For real-valued u, with $\int u d\mu = 0$, the sequence $a_n(u) := \sup |\hat{\mathcal{L}}^n(u)|$ is decreasing by Proposition 4.4.4. Suppose that $\lim_{n\to\infty} a_n = a > 0$. By Theorem 4.3.5 all the iterates $\hat{\mathcal{L}}^n(u)$ have a common modulus of continuity. Denote it, as in Th.4.3.5, by g. So applying Lemma 4.4.5 with this g and $\delta_1 = a$ we find n_0, δ_2 such that $\sup |\hat{\mathcal{L}}^{n_0}\big(\hat{\mathcal{L}}^n(u)\big)| \le \sup |\hat{\mathcal{L}}^n(u)| - \delta_2$ for every n. So for n such that $\sup |\hat{\mathcal{L}}^n(u)| < a + \delta_2$ we obtain $\sup |\hat{\mathcal{L}}^{n+n_0}(u)| < a$, a contradiction with the definition of a.

This proves (4.4.3). For an arbitrary $u \in C(X, I\!\!R)$ we obtain from (4.4.3) due to $\hat{\mathcal{L}}(1\!\!1) = 1\!\!1$

$$\hat{\mathcal{L}}^n(u) - \mu(u)1\!\!1 = \hat{\mathcal{L}}^n(u - \mu(u)1\!\!1) \to 0.$$

Change now coordinates on $C(X)$ to go back to $\mathcal{L}_0$ and next replace it by $c^{-1}\mathcal{L}_\phi$. One obtains (4.4.2). Given a complex-valued u decompose it into sum of real and imaginary part.

If we allow u and ϕ to vary we modify the proof. The point is that by Lemma 4.4.5, for every $\delta_1 > 0$, for every $m \ge Kn(g, \delta_1, K)/\delta_2(g, \delta_1, K)$, we get $\sup |\hat{\mathcal{L}}^m(u)| \le \delta_1$, where g is the modulus of continuity for the family $\{\hat{\mathcal{L}}^n(u)\}$ provided by Theorem 4.3.5, and K bounds the norm in $\mathcal{H}_\alpha$ of the functions ϕ. Letting $\delta_1 \to$ proves the Theorem. ♣

Note that (4.4.2) means weak*-convergence of measures

$$\lim_{n\to\infty} \sum_{\overline{x}\in T^{-n}(x)} \delta_{\overline{x}} \times \exp(S_n\phi(\overline{x}))/c^n \to u_\phi(x) m_\phi$$

for every $x \in X$.

In the sequel one can consider either $C(X, I\!\!R)$ or $C(X, \mathbb{C})$. Let us decide for $C(X, \mathbb{C})$.

Note that by $\mathcal{L}^*_\phi(m_\phi) = cm_\phi$ we have the $\mathcal{L}$-invariant decomposition $C(X) = \operatorname{span}(u_\phi) \oplus \ker(m_\phi)$. For $u \in \operatorname{span}(u_\phi)$ we have $\mathcal{L}_\phi(u) = cu$. On $\ker(m_\phi)$, by Th.4.4.6., $c^{-n}\mathcal{L}^n_\phi \to 0$ in strong topology. Denote $(\mathcal{L}_\phi)|_{\ker(m)}$ by $\mathcal{L}_{\ker,\phi}$. For $\mathcal{L}_{\ker,\phi}$ restricted to $\mathcal{H}_\alpha$ we can say more on the above convergence:

Theorem 4.4.7. There exists an integer $n > 0$ such that

$$\|c^{-n}\mathcal{L}^n_{\ker,\phi}\|_{\mathcal{H}_\alpha} < 1.$$

Proof. Again it is sufficient to consider real u with $\mu(u)=0$ and the operator $\hat{\mathcal{L}}$. Set $\delta=\min\{1/8C_4, 1/4\}$, with C_4 from (4.3.11). By Th.4.3.6. for u such that $\|u\|_{\mathcal{H}_\alpha}\le 1$ all functions $\hat{\mathcal{L}}^n(u)$ have the same modulus of continuity $g(\varepsilon)=C_7\varepsilon^\alpha$ for a constant $C_7>0$. Hence from Theorem 4.4.6. we conclude that $(\exists n_1)(\forall u : \|u\|_{\mathcal{H}_\alpha}\le 1)$

$$\|\hat{\mathcal{L}}^{n_1}(u)\|_\infty \le \delta. \tag{4.4.4}$$

Next, for n_2 satisfying $C_3\lambda^{-n_2\alpha}C_7 + C_4\delta \le 1/4$, again by Th.4.3.6. we obtain

$$\vartheta_\alpha(\hat{\mathcal{L}}^{n_2}(\hat{\mathcal{L}}^{n_1}(u)) \le 1/4.$$

hence $||\hat{\mathcal{L}}^{n_1+n_2}(u)||_{\mathcal{H}_\alpha} \le 1/2$. Theorem has been proved with $n=n_1+n_2$. ♣

Note that Th.4.4.6. could be deduced from Th.4.4.7 by uniform approximating continuous functions by Hölder ones, and using Prop. 4.3.4.

Corollary 4.4.8. The convergences in Theorem 4.4.6 for $u\in\mathcal{H}_\alpha$ are exponential. Namely there exist $0<\tau<1$ and $C\ge 0$ such that for every function $u\in\mathcal{H}_\alpha$

$$\|c^{-n}\mathcal{L}_\phi^n(u) - m_\phi(u)u_\phi\|_\infty \le \|c^{-n}\mathcal{L}_\phi^n(u) - m_\phi(u)u_\phi\|_{\mathcal{H}_\alpha} \le C\|u-m_\phi(u)u_\phi\|_{\mathcal{H}_\alpha}\tau^n. \tag{4.4.4'}$$

In particular if $\int u\,d\mu = 0$ then

$$\|\hat{\mathcal{L}}^n(u)\|_\infty \le \|\hat{\mathcal{L}}^n(u)\|_{\mathcal{H}_\alpha} \le C\|u\|_{\mathcal{H}_\alpha}\tau^n. \tag{4.4.4''}$$

Now we can study "mixing" properties of the system (T,μ) for our invariant Gibbs measure μ. Roughly speaking the speed of mixing is related to the speed of convergence of $\mathcal{L}^n_{\mathrm{ker},\phi}$ to 0.

The first dynamical (mixing) consequence of Theorem 4.4.8 is the following result known as the exponential decay of correlations, see the definition in Ch.1.11.

Theorem 4.4.10. There exists $C\ge 1$ and $\rho<1$ such that for all $f\in L^1(\mu)$, $g\in\mathcal{H}_\alpha$

$$C_n(f,g) \le C\rho^n\|f-Ef\|_1\|g-Eg\|_{\mathcal{H}_\alpha}.$$

Proof. Write $F=f-Ef, G=g-Eg$. We obtain

$$C_n(f,g) = |\int F\cdot(G\circ T^n)\,d\mu| = |\int \hat{\mathcal{L}}^n\big(F\cdot(G\circ T^n)\big)\,d\mu|$$

$$= |\int G\cdot\hat{\mathcal{L}}^n(F)\,d\mu| \le \|G\|_1 C\tau^n\|F\|_{\mathcal{H}_\alpha}.$$

We have used here a very important identity true for arbitrary F, G ($EF, EG = 0$ does not matter).

$$\mathcal{L}^n\big(F \cdot (G \circ T^n)\big) = G \cdot \mathcal{L}^n(F), \tag{4.4.5}$$

which follows immediately from the definition of $\mathcal{L}$ with an arbitrary potential ϕ. Namely

$$\mathcal{L}_\phi{}^n\big(F \cdot (G \circ T^n)\big)(x) = \sum_{\overline{x} \in T^{-n}(x)} G(x) F(\overline{x}) \exp S_n\psi(\overline{x}) = G(x)\mathcal{L}_\phi^n(F)(x)$$

♣

Exercise. Prove that for every μ square integrable functions f, g one has $\int f \cdot (g \circ T^n)\, d\mu \to Ef \cdot Eg$. (Hint: approximate f and g by Hölder functions. Of course the information on the speed of convergence would become lost.)

The convergence in the exercise is one of equivalent definitions of the *mixing* property, see Ch.1.10. We proved however earlier the stronger property: measure-theoretical exactness, Th. 4.2.7.

We can however make a better use of the exponential convergence in Theorem 4.4.10.

Theorem 4.4.11. Let (X, T) be a topologically mixing topological one-sided Markov chain with T the shift to the left and $d \geq 2$ symbols, see Ch.0. Let $\mathcal{F}$ be the σ-algebra generated by the partition $\mathcal{A}$ into sets with fixed 0-th coordinate, namely by $\mathcal{A} = \{X_1, ..., X_d\}$ where $X_j = \{(a_0, a_1, ...) \in X : a_0 = j\}$. For every $0 \leq k \leq l$ write $\mathcal{F}_k^l$ for the σ-algebra generated by $\mathcal{A}_k^l = \{\bigvee_{j=k}^l T^{-j}(\mathcal{A})$ i.e. by the sets with fixed $k, k+1, ..., l$'th coordinates. Let $\phi : X \to I\!\!R$ be Hölder continuous.

Then there exist $0 < \rho < 1, C > 0$ such that for every $k \geq 0, f : X \to I\!\!R$ measurable in $\mathcal{F}_0^k$ and g being μ_ϕ-integrable

$$\Big|\int f \cdot (g \circ T^n)\, d\mu_\phi - Ef \cdot Eg\Big| \leq C\rho^{n-k}\|f - Ef\|_1 \|g - Eg\|_1. \tag{3.4.6}$$

Proof. Assume $Ef = Eg = 0$. By Theorem 4.4.10

$$\Big|\int f \cdot (g \circ T^n)\, d\mu\Big| = \Big|\int g \cdot \hat{\mathcal{L}}^{n-k}(\hat{\mathcal{L}}^k(f))\, d\mu\Big| \leq \|g\|_1 C\rho^{n-k}\|\hat{\mathcal{L}}^k(f)\|_{\mathcal{H}_\alpha}. \tag{4.4.7}$$

Decompose f into real and imaginary parts and represent each one by the difference of nowhere negative functions. This allows in the estimates which follow to assume that $f \geq 0$.

Notice that for every cylinder $A \in \mathcal{A}$ and $x \in A$, in the expression

$$\hat{\mathcal{L}}^k(f)(x) = \sum_{T^k(y)=x} f(y) \exp S_n\psi(y)$$

there is no dependence of f on $x \in A$ because f is constant on cylinders of $\mathcal{A}_0^k$. So

$$|\frac{\sup_A \hat{\mathcal{L}}^k(f)}{\inf_A \hat{\mathcal{L}}^k(f} \leq \sup_{B \in \mathcal{A}_0^k} \sup_{y,y' \in B} \exp\big(S_n\psi(y) - S_n\psi(y')\big) \leq C$$

a constant C resulting from Ch.3.4. So

$$\sup_A |\hat{\mathcal{L}}^k(f)| \leq \frac{C}{\mu(A)} \int \hat{\mathcal{L}}^k(f)\, d\mu = \frac{C}{\mu(A)} \|f\|_1 \leq \big(\inf_{A \in \mathcal{A}} \frac{C}{\mu(A)}\big)\|f\|_1 = C'\|f\|_1,$$

where the latter equality defines C'.

It is left yet to estimate the $\vartheta_{\alpha,\xi}$ and ϑ_α pseudonorms of $\hat{\mathcal{L}}^k(f)$, cf.Ch.3.4. We assume that ξ is less than the minimal distance between the cylinders in $\mathcal{A}$. We have similarly to (4.3.6), for x, y belonging to one cylinder $A \in \mathcal{A}$,

$$\hat{\mathcal{L}}^k(f)(x) - \hat{\mathcal{L}}^k(f)(y) = \big(\frac{\hat{\mathcal{L}}^k(f)(x)}{\hat{\mathcal{L}}^k(f)(y)} - 1\big)\hat{\mathcal{L}}^k(f)(y)$$

$$\leq (\exp C_1\rho(x,y)^\alpha - 1)\|C'\|f\|_1 \leq C''\rho(x,y)^\alpha\|f\|_1.$$

for a constant C''.

Hence, $\vartheta_{\alpha,\xi}(\hat{\mathcal{L}}^k(f)) \leq |f\|_1 C''$ and, passing to ϑ_α as in Ch.3.4,

$$\vartheta_\alpha(\hat{\mathcal{L}}^k(f)) \leq |f\|_1 \max\{C'', 2C'\xi^{-\alpha}\}).$$

Thus, continuing (4.4.7), we obtain for a constant C

$$C_n(f,g) \leq \|f\|_1\|g\|_1 C\rho^{n-k}.$$

♣

An immediate corollary from Theorem 4.4.11 is, that for every $B_1 \in \mathcal{F}_0^k$ and Borel B_2 (i.e. $B_2 \in \mathcal{F}_0^\infty$)

$$|\mu(B_1 \cap T^{-n}(B_2)) - \mu(B_1)\mu(B_2)| \leq C\rho^{n-k}\mu(B_1)\mu(B_2), \tag{4.4.8}$$

compare (1.11.9). Therefore, for every non-negative integer t and every $A \in \mathcal{F}_0^k$ for the conditional measures with respect to A

$$\sum_{B \in \mathcal{A}_0^t} |\mu(T^{-n}(B)\big|A) - \mu(B)| \leq C\rho^{n-k}.$$

This means that $\mathcal{A}$ satisfies the weak Bernoulli property, hence the natural extension $(\tilde{X}, \tilde{T}, \tilde{\mu})$ is measure-theoretically isomorphic to a two-sided Bernoulli shift, see Ch.1.11.

Corollary 4.4.12. Every topologically mixing, open, distance expanding map T, with invariant Gibbs measure $\mu = \mu_\phi$ for a Hölder continuous function ϕ, has the natural extension $(\tilde{X}, \tilde{T}, \tilde{\mu})$ measure-theoretically isomorphic to a two-sided Bernoulli shift.

Proof. Let $\pi : \Sigma_A \to X$ be the coding map from a one-sided topological Markov chain, due to a Markov partition, see Ch.3.5. Since π is Hölder, the function $\phi \circ \pi$ is also Hölder continuous, hence we can discuss the invariant Gibbs measure $\mu_{\phi\circ\pi}$. For this measure we can apply Theorem 4.4.11 and its consequences. Therefore to end the proof it is enough to prove the following

Lemma 4.4.13. The measures μ_ϕ and $\mu_{\phi\circ\pi} \circ \pi^{-1}$ coincide.

Proof. $\exp(-\phi \circ \pi + P)$ is the strong Jacobian for the shift map σ and the measure $\mu_{\phi\circ\pi}$, where P is the pressure for both $(\sigma, \phi \circ \pi)$ and (T, ϕ), see Ch.4.5. Since π yields a measure-theoretical isomorphism between $\mu_{\phi\circ\pi}$ and $mu_{\phi\circ\pi} \circ \pi^{-1}$, Ch.4.5, the measure $\mu_{\phi\circ\pi} \circ \pi^{-1}$ is forward quasi-invariant under T and has the strong Jacobian $\exp(-\phi + P)$. The same is the Jacobian of μ_ϕ. Therefore both measures coincide. ♣

Chapter 3, Section 5: More on almost periodic operators

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In this Section we show how to deduce Th 4.4.6 (on convergence) and Th. 4.4.7 and Cor. 4.4.8 (exponential convergence) from general functional analysis theorems. We do not need this later on in this book, but the theorems are useful in other important situations .....

Recall (Def.4.3.3) that $Q : F \to F$ a continuous linear operator of a Banach space is called almost periodic if for every $b \in F$ the sequence $Q^n(b)$ is relatively compact. By Banach-Steinhaus theorem there is a constant $C \geq 0$ such that $\|Q^n\| \leq C$ for every $n \geq 0$.

Theorem 4.5.1. If $Q : F \to F$ is an almost periodic operator on a complex Banach space F, then

$$F = F_0 \oplus F_{\mathrm{u}}, \tag{4.5.1}$$

where $F_0 = \{x \in F : \lim_{n\to\infty} A^n(x) = 0\}$ and F_{u} is the closure of the subspace of F generated by eigenfunctions of eigenvalues of modulus 1.

Adding additional assumptions one gains additional information on this decomposition.

Definition 4.5.2. Let $F = C(X)$ and suppose $Q : F \to F$ is *positive*, namely $f \geq 0$ implies $Q(f) \geq 0$. Then Q is called *primitive* if for every $f \in C(X), f \geq 0, f \neq 0$ there exists $n \geq 0$ such that for every $x \in X$ it holds $Q^n(f)(x) > 0$. If we change the order of the quantificators to: ... for every x there exists n ..., then we call Q *nondecomposable.*

Theorem 4.5.3 For $Q : \mathbb{C}(X) \to C(X)$ (real or complex) linear positive primitive operator of spectral radius equal to 1 we have dim span$(C(X)_u = 1$ in the decomposition (4.5.1), the eigenvalue corresponding to $C(X)_u$ is equal to 1 and the eigenfunction is positive (everywhere > 0). More precisely there exists a probability measure m_Q on X and a positive function u_Q such that for every $u \in C(X)$ we have strong convergence

$$Q^n(u) \to u_Q \int u\,dm.$$

Proof. This is just a repetition of considerations of Sections 2-4. First find a probability measure m such that $Q^*(m) = m$ as in Th.4.2.4. (we leave a proof that the eigenvalue is equal to 1, to the reader). Next find for Q an eigenfunction $u_Q \geq 0$ as $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} Q^j(1\!\!1)$. We have $u_Q = f^n(u_Q) > 0$ because Q is nondecomposable. Finally for $\hat{Q}(u) := Q(uu_Q)u_Q^{-1}$ we have $\hat{Q}(1\!\!1) = 1\!\!1$ (a positive operator with this property is called *stochastic*) and we repeat Proof of Th. 4.4.6, replacing the property of topological mixing by primitivity. ♣

Notice that this yields Th.4.4.6 because of

Proposition 4.5.4. If an open expanding map T is topologically mixing then for every Hölder function ϕ the transfer operator $Q = \mathcal{L}_{\overline{\phi}}$ is primitive.

The proof is easy, it is in fact contained in Proof of Lemma 4.4.5.

Assume now only that T is topologically transitive. Let Ω^k denote the sets from spectral decomposition $X = \Omega = \bigcup_{k=1}^n \Omega^k$ as in Th.3.3.4. Write $u_Q \in C(X)$ for an eigenfunction of the operator Q as before. Notice now (exercise!) that the space $F_{\rm u}$ for the operator $Q = \mathcal{L}_{\overline{\phi}}$ is spanned by n eigenfunctions $v_t = \sum_{k=1}^n \chi_{\Omega^k} \lambda^{-tk} u_Q, \quad t = 1, ..., n,$ where χ means indicator functions, with $\lambda = \varepsilon^{2\pi i/n}$. Each v_t corresponds to the eigenvalue λ^t. Thus the set of these eigenvalues is a cyclic group.

It is also an easy exercise to describe $F_{\rm u}$ if $X = \Omega = \bigcup_{j=1}^J \bigcup_{k=1}^{k(j)} \Omega_j^k$. The set of eigenvalues is the union of J cyclic groups. It is harder to understand $F_{\rm u}$ and the corresponding set of eigenvalues for T open expanding, without assuming $\Omega = X$.

References to the above theory are:

[LL]

[LL] M. Yu. Lyubich: Perron-Frobenius theory for almost periodic operators and semigroups representations. Teoria Funkcii 46 (1986), 54-72.

[L] M. Yu. Lyubich: Entropy properties of rational endomorphisms of the Riemann sphere. ETDS (1983), 351-385.

[S]

A general theorem related to Th. 4.4.7 and Cor.4.4.8 is the following

Theorem 4.5.5 (Ionescu–Tulcea and Marinescu) Let $(F, |\cdot|)$ be a Banach space equipped with a norm $|\cdot|$ and let $E \subset F$ be its linear subspace. Moreover the linear space E is assumed to be equipped with a norm $\|\cdot\|$ which satisfies the following two conditions.

(1) Any bounded subset of the Banach space E with the norm $\|\cdot\|$ is relatively compact as a subset of the Banach space F with the norm $|\cdot|$.

(2) Moreover, if $\{x_n : n = 1, 2, \ldots\}$ is a sequence of points in E such that $\|x_n\| \le K_1$ for all $n \ge 1$ and some constant K_1, and if $\lim_{n\to\infty} |x_n - x| = 0$ for some $x \in F$, then $x \in E$ and $\|x\| \le K_1$.

Let $Q : F \to F$ be a bounded linear operator which preserves E, whose restriction to E is also bounded with respect to the norm $\|\cdot\|$ and which satisfies the following two conditions.

(3) There exists a constant K such that $|Q^n| \le K$ for all $n = 1, 2, \ldots$.

(4) $\exists N \ge 1 \quad \exists \tau < 1 \quad \exists K_2 > 0 \quad \|Q^N(x)\| \le \tau\|x\| + K_2|x|$ for all $x \in E$.

Then

(5) There exists only finitely many eigenvalues of $Q : F \to F$ of modulus 1, say $\gamma_1, \ldots, \gamma_p$.

(6) Let $F_i = \{x \in F : Q(x) = \gamma_i x\}$, $i = 1, \ldots, p$. Then $F_i \subset E$ and $\dim(E_i) < \infty$.

(7) The operator $Q : F \to F$ can be represented as

$$Q = \sum_{i=1}^{p} \gamma_i Q_i + S$$

where Q_i and S are bounded, $Q_i(F) = F_i$, $\sup_{n\geq 1} |S^n| < \infty$ and

$$Q_i^2 = Q_i, \quad Q_i Q_j = 0 \ (i \neq j), \quad Q_i S = S Q_i = 0$$

Moreover

(8) $S(E) \subset E$ and considered as a linear operator on $(E, \|\cdot\|)$, S is bounded and there exist constants $K_3 > 0$ and $0 < \tilde{\tau} < 1$ such that

$$\|S^n\| \leq K_3 \tilde{\tau}^n$$

for all $n \geq 1$.

The proof of this theorem can be found in [...] in the case $N = 1$ (see assumpion 4). Its validity for any $N \geq 1$ is mentiond in Section 9, p.145 of this paper. In Appendix ... we give a complete proof.

Now, in view of Theorem 3.4.1 and Corollary 4.3.7, Theorem 4.5.5 applies to the operator $Q = \mathcal{L}_{\overline{\phi}} : C(X) \to C(X)$ if one substitutes $F = C(X)$, $E = \mathcal{H}_\alpha(X)$. If T is topologically mixing, hence Q is primitive on $C(X)$, then $\dim(\oplus F_i) = 1$ and the corresponding eigenvalue is equal to 1, as in Th.4.5.3.

Example of application Lasota-Yorke, Rychlik: functions of bounded variation ....

Sec. 4.6. UNIQUENESS OF EQUILIBRIUM STATES

We proved already the existence (Th.4.3.2) and uniqueness (Cor.4.2.9) of invariant Gibbs states and proved that invariant Gibbs states are equilibrium states (Prop.4.1.5). Here we shall give 3 different proofs of the uniqueness of equilibrium states.

Let ν be a T-invariant measure and let a finite real function J_ν be the corresponding Jacobian in the weak sense, J_ν is defined ν-a.e. . By the invariance of ν we have $\nu(E) = 0 \Rightarrow \nu(T^{-1}(E)) = \nu(E) = 0$, i.e. ν is backward quasi-invariant. At the beginning of Sec.2 we defined in this situation $\Psi = \Phi_x \circ T$ with $\Phi_x = \frac{d\nu \circ T_x^{-1}}{d\nu}$ defined for ν-a.e. point in the domain of a branch T_x^{-1}. (In Sec.2 we used notation Φ_j for Φ_x.) Φ_x is strong Jacobian for T_x^{-1}.

Notice that for ν-a.e. z

$$(4.6.1) \qquad (J_\nu \circ T_x^{-1}) \cdot \Phi_x(z) = \begin{cases} 1, & \text{if } \Phi_x(z) \neq 0; \\ 0, & \text{if } \Phi_x(z) = 0. \end{cases}$$

Indeed, after removal of $\{z : \Phi_x(z) = 0\}$ the measures ν and $\nu \circ T^{-1}$ are equivalent, hence Jacobians of T and T_x^{-1} are mutual inverses. We can fix J_ν arbitrary, bounded, on $T^{-}(\{z : \Phi_x(z) = 0\})$.

Recall that we have defined $\mathcal{L}_\nu : L^1(\nu) \to L^1(\nu)$, the transfer operator associated with the measure ν as follows

$$\mathcal{L}_\nu(g)(x) = \sum_{y \in T^{-1}(x)} g(y)\Psi(y).$$

Remind that if T maps a set A of measure 0 to a set of positive measure, then Ψ is specified, equal to 0, on a subset of A that is mapped by T to a set of full measure ν in $T(A)$.

Then since ν is T-invariant, $\mathcal{L}_\nu(1\!\!1) = 1\!\!1$ and for every ν-integrable g we have $\int \mathcal{L}_\nu(g)\, d\nu = \int g\, d\nu$.

Lemma 4.6.1. Let $\psi : X \to I\!\!R$ be a continuous function such that $\mathcal{L}_\psi(1\!\!1) = 1\!\!1$, i.e. for every x, $\sum_{y \in T^{-1}(x)} \exp\psi(y) = 1$, and ν be an ergodic equilibrium state for ψ. Then J_ν is strong Jacobian and $J_\nu = \exp(-\psi)$ ν-almost everywhere .

Proof. The proof is based on the following computation using the inequality $1 + \log(x) \leq x$, with the equality only for $x = 1$.

$$1 = \int 1\!\!1\, d\nu \geq \int \mathcal{L}_\nu(J_\nu \exp\psi)\, d\nu = \int J_\nu \exp\psi\, d\nu$$

$$\geq \int \big(1 + \log(J_\nu \exp\psi)\big)\, d\nu = 1 + \int \psi d\nu + \int \log J_\nu\, d\nu = 1 + \int \psi d\nu + h_\nu(T) \geq 1.$$

To obtain the first inequality, write $\mathcal{L}_\nu(J_\nu \exp\psi)(x) = \sum_{y \in T^{-1}(x)} J_\nu(y)(\exp\psi(y))\Psi(y) = 1$ or 0, depending as $\Psi(y) = 1$ or 0, by (4.6.1) and by $\sum_{y \in T^{-1}(x)} \exp\psi(y) = 1$.

The last inequality follows from

$$\int \psi d\nu + h_\nu(T) = P(\psi) \geq \limsup_{n\to\infty} \frac{1}{n} \sum_{y\in T^{-n}(x)} \exp S_n\psi(y) = 0,$$

true, see Th.2.2.10, since all points in $T^{-n}(x)$ are (n,η)-separated, η defined in Ch.3.

Therefore all the inequalities in this proof must become equalities. Therefore the Jacobian $\Phi_x \neq 0$ for each branch T_x^{-1} and $J_\nu = \exp(-\psi)$, ν- a.e. ♣

Notice that we have not assumed ψ is Hölder above. Now we shall assume Hölder.

Theorem 4.6.2. There exists exactly one equilibrium state for each Hölder continuous potential ϕ.

Proof. Let ν be an equilibrium state for ϕ. As in Sec.4 set $\psi = \phi - P(T,\phi) + u_\phi \circ T - u_\phi$ and ν is also equilibrium state for ψ. Then by Lemma 4.6.1

$$\nu\big(T_z^{-n}(B(T^n(z),\xi))\big) = \int_{B(T^n(z),\xi)} \exp\,\big(S_n\psi(T_z^{-n}(x))\big)\,d\nu(x) =$$

$$\int_{B(T^n(z),\xi)} \frac{u_\phi(x)}{u_\phi(T^n(x))} \exp(S_n\phi - nP(T,\phi))(T_z^{-n}(x))\,d\nu(x).$$

So, by *pre-bounded distortion lemma* (Lemma 3.4.2),

$$\frac{\inf|u_\phi|}{\sup|u_\phi|}BC^{-1} \leq \frac{\nu\big(T_z^{-n}(B(T^n(z),\xi))\big)}{\exp(S_n\phi - nP(T,\phi))(z)} \leq \frac{\sup|u_\phi|}{\inf|u_\phi|}C,$$

where $B = \inf\{\nu(B(y,\xi)\}$. It is positive by Proposition 4.2.5.

Therefore ν is an invariant Gibbs state for ϕ; unique by Corollary 4.2.9. ♣

Remark 4.6.3. In fact already the knowledge that $\exp(-\psi)$ is weak Jacobian implies automatically that it is a strong Jacobian. Indeed by the invariance of ν we have

$$\sum_{y\in T^{-1}(x)} \Phi_y(y) = 1 = \sum_{y\in T^{-1}(x)} \exp\psi(y)$$

and each non-zero summand on the left is equal to a corresponding summand on the right. So there are no summands equal to 0.

Uniqueness: Proof II. We shall provide a new proof of Lemma 4.6.1. It is not so elementary as the previous one, but it exhibits a relation with the finite case, the *prototype lemma* in Introduction.

For every $y \in X$ denote $A(y) := T^{-1}\Big(T\big(\{y\}\big)\Big)$. Let $\{\nu_A\}$ denote the canonical system of conditional measures for the partition of X into the sets $A = A(y)$, see Ch.1.6. Since there exists a finite one-sided generator, see Lemma 2.4.5, with the use of Theorem 1.9.7 we obtain

$$0 = P(T,\,\psi) = h_\nu(T) + \int \psi\, d\nu = H_\nu\Big(\varepsilon \mid T^{-1}(\varepsilon)\Big) + \int \psi\, d\nu =$$

$$= \int \Big(\sum_{z\in A(y)} \nu_{A(y)}(\{z\})\big(-\log\big(\nu_{A(y)}(\{z\})\big) + \psi(z)\big)\Big)\, d\nu(y).$$

The latter expression is always negative except for the case $\nu_{A(y)}(z) = \exp\psi(z)$ ν-a.e. by the prototype lemma. So for a set $Y = T^{-1}\big(T(Y)\big)$ of full measure ν, for every $y\in Y$ we have

$$\nu_{A(y)}(\{y\}) = \exp\psi(y), \quad \text{in particular} \quad \nu_{A(y)}(\{y\}) \neq 0. \tag{4.6.2}$$

So for every Borel set $B\subset Y$ such that T is 1–to–1 on it, since B intersects each $A(y)\subset T^{-1}(T(B))$ at precisely one point, we obtain

$$\nu\big(T(B)\big) = \nu\Big(T^{-1}\big(T(B)\big)\Big) =$$

$$= \int_{T^{-1}\left(T(B)\right)} \left(\int_{A(y)} 1\!\!1_B(z)/\nu_{A(y)}(\{z\})\, d\nu_{A(y)}(z)\right) d\nu(y) =$$

$$= \int_{T^{-1}\left(T(B)\right)} 1\!\!1_B(y)/\nu_{A(y)}(\{y\})\, d\nu(y) = \int_B 1/\nu_{A(y)}(\{y\})\, d\nu(y)$$

Notice that we have proved in this computation a general useful fact that $1/\nu_{A(y)}(\{y\}))$ is weak Jacobian for T and ν. In absence of the property (4.6.2) that $\nu_{A(y)}(\{y\})\neq 0$ we should have subtracted the set $E = \{y : \nu_{A(y)}(y) = 0\}$ of measure 0 under the integrals.

Let us go back to our situation. By (4.6.2) this Jacobian is equal to $\exp -\psi$. Observe also that $\nu\big(T(X\setminus Y)\big) = 0$ because $X\setminus Y = T^{-1}\big(T(X\setminus Y)\big)$ and ν is T-invariant. So $\exp -\psi$ is strong Jacobian.

Uniqueness. Proof III. Due to Corollary 2.5.7 it is sufficient to prove the differentiability of the pressure $P(T,\phi)$ as a function of continuous function ϕ at Hölder ϕ in a set of directions dense in the weak topology on $C(X)$.

Lemma 4.6.4. Let $\phi : X\to I\!\!R$ be a Hölder continuous function and μ_ϕ denote the invariant Gibbs measure. Let $F : X\to I\!\!R$ be continuous. Then, for an arbitrary $x\in X$,

$$\lim_{n\to\infty} \frac{1}{n} \frac{\sum_{y\in T^{-n}(x)} S_nF \exp(S_n\phi)(y)}{\sum_{y\in T^{-n}(x)} \exp(S_n\phi)(y)} = \int F\, d\mu_\phi. \tag{4.6.3}$$

The convergence is uniform for an equicontinuous family of F's and ϕ's in a bounded set in the Banach space of Hölder functions $\mathrm{H}_\alpha(X)$.

Proof.

The above left hand side expression can be written in the form:

$$\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_\phi^n(F\circ T^j)(x)}{\mathcal{L}_\phi^n(1\!\!1)(x)}=\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}^{n-j}(F\cdot\mathcal{L}^j(1\!\!1))(x)}{\mathcal{L}^n(1\!\!1)(x)}. \tag{4.6.4}$$

where $\mathcal{L}=\mathcal{L}_0=e^{-P(T,\phi)}\mathcal{L}_\phi$, compare the beginning of Sec.3.

Since $F\cdot\mathcal{L}^j(1\!\!1)$ is an equicontinuous family of functions we obtain

$$\mathcal{L}^{n-j}(F\cdot\mathcal{L}^j(1\!\!1))(x)\to u_\phi(x)\int F\cdot\mathcal{L}^j(1\!\!1)\,dm_\phi$$

as $n-j\to\infty$, see Remark 4.4.6a.

Therefore continuing (4.6.4) we obtain

$$\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{j=0}^{n-1}u_\phi(x)\int F\cdot\mathcal{L}^j(1\!\!1)\,dm_\phi}{u_\phi(x)}=\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int F\cdot\mathcal{L}^j(1\!\!1)\,dm_\phi=\int F\,d\mu_\phi$$

because $\mathcal{L}^j(1\!\!1)$ uniformly converges to u_ϕ and $\mu_\phi=u_\phi m_\phi$. ♣

Now we shall compute the derivative $dP(T,\phi+t\gamma)/dt$ for every Hölder ϕ and γ at every t. In particular we shall prove differentiability at $t=0$. Thus our dense set of directions is spanned by Hölder functions γ.

Write

$$P_n(t)=\frac{1}{n}\log\sum_{y\in T^{-n}(x)}\exp(S_n(\phi+t\gamma))(y);$$

$$Q_n(t):=(dP_n/dt)(t)=\frac{\frac{1}{n}\sum_{y\in T^{-n}(x)}S_n\gamma(y)\exp(S_n(\phi+t\gamma))(y)}{\sum_{y\in T^{-n}(x)}\exp(S_n(\phi+t\gamma)(y)}. \tag{4.6.5}$$

By Lemma 4.6.4 $\lim_{n\to\infty}Q_n(t)=\int\gamma\,d\mu_{\phi+t\gamma}$ and the convergence is uniform with respect to t. Since, in addition, $\lim_{n\to\infty}P_n(t)=P(t)$, we conclude that $P(T,\phi+t\gamma)=\lim P_n(t)$ is differentiable with derivative being the limit of derivatives: $\lim_{n\to\infty}Q_n(t)=\int\gamma\,d\mu_{\phi+t\gamma}$, ♣

Notice that the differential (Gateaux) operator $\gamma\mapsto\int\gamma\,d\mu_\phi$ is indeed that one from Proposition 2.5.6.

Notice also that *a posteriori*, by Cor.2.5.7, we proved that for ϕ Hölder continuous, $P(T,\phi)$ is differentiable in direction of every continuous function. This is by the way obvious in general: two different supporting functionals are different restricted to any dense subspace.

§4.7. Probability laws and $\sigma^2(u,v)$.

Exponential convergences in §4.4 allow to prove the probability laws.

Theorem 4.7.1. Let (X,T) be an open distance expanding repeller and μ the invariant Gibbs measure for a Hölder function $\phi : X \to I\!R$. Then if $g : X \to I\!R$ satisfies

$$\sum_{n=0}^{\infty} \|\hat{\mathcal{L}}^n(g - \mu(g))\|_2 < \infty, \tag{4.7.1}$$

in particular if g is Hölder continuous, it satisfies CLT. If g is Hölder continuous it satisfies LIL.

Proof. First show how CLT can be deduced from Theorem 1.11.5. We can assume $\mu(g) = 0$. Let $(\tilde{X}\tilde{\mathcal{F}}, \tilde{\mu})$ be the natural extension (see Ch.1.7). Recall that $\tilde{X}$ can be viewed as the set of all T-trajectories $(x_n)_{n\in \boldsymbol{Z}}$ (or backward trajectories), $\tilde{T}((x_n)) = (x_{n+1})$ and $\pi_n((x_n)) = x_n$. It is sufficient now to check (1.11.12) for the automorphism $\tilde{T}$ the function $\tilde{g} = g \circ \pi_0$ and $\tilde{\mathcal{F}}_0 = \pi^{-1}(\mathcal{B})$ for the completed Borel σ-algebra $\mathcal{B}$. Since $\tilde{g}$ is measurable with respect to $\tilde{\mathcal{F}}_0$ it is also measurable with respect to all $\tilde{\mathcal{F}}_n = \tilde{T}^{-n}(\tilde{\mathcal{F}}_0)$ for $n \le 0$ hence $\tilde{g} = E(\tilde{g}|\tilde{\mathcal{F}}_n)$. So we need only to prove $\sum_{n\ge 0}^{\infty} \|E(\tilde{g}|\tilde{\mathcal{F}}_n)\|_2 < \infty$.

Let us start with a general fact concerning an arbitrary probability space $(X, \mathcal{F}, \mu)$ and a μ-preserving endomorphism T.

Lemma 4.7.2. Let U denote the unitary operator on $L^2(X, \mathcal{F}, \mu)$ associated to T, namely $U(f) = f \circ T$. Then for every $k \ge 0$ the operator $U^k U^{*k}$ is the orthogonal projection of $H_0 = L^2(X, \mathcal{F}, \mu)$ to $H_k = L^2(X, T^{-k}(\mathcal{F}), \mu)$.

Proof. U^* is the operator in the space conjugate to H_k which is H_k itself (a Hilbert space). $U^k(u) = u \circ T^k$ is measurable with respect to $T^{-k}(\mathcal{F})$, so the range of $U^k U^{*k}$ is indeed in $H_k = L^2(X, T^{-k}(\mathcal{F}), \mu)$.

For any $u, v \in H_0$ write $\int u \cdot v \, d\mu =< u, v >$, the scalar product. For arbitrary $f, g \in H_0$ we compute

$$< U^k U^{*k}(f), g \circ T^k >=< U^k U^{*k}(f), U^k(g) >$$
$$=< U^{*k}(f), g >=< f, U^k(g) >=< f, g \circ T^k > .$$

It is clear that all functions in $H_k = L^2(X, T^{-k}(\mathcal{F}), \mu)$ are represented by $g \circ T^k$ for $g \in L^2(X, \mathcal{F}, \mu)$. Therefore by the above equality for all $h \in H_k$ we obtain

$$< f - U^k U^{*k}(f), h >=< f, h > - < f, h >= 0. \tag{4.7.2}$$

In particular for $f \in H_k$ we conclude from (4.7.2) for $h = f - U^k U^{*k}(f)$, that $< f - U^k U^{*k}(f), f - U^k U^{*k}(f) >= 0$ hence $U^k U^{*k}(f) = f$. By (4.7.2) the projection is orthogonal. ♣

Since the conditional expectation value $f \to E(f|T^{-k}(\mathcal{F}))$ is the orthogonal projection to H_k we conclude that $E(f|T^{-k}(\mathcal{F})) = U^k U^{*k}(f)$.

Now let us pass to our special situation of Theorem 4.7.1.

Lemma 4.7.3. For every $f \in L^2(X, \mathcal{F}, \mu)$ we have $U^*(f) = \hat{\mathcal{L}}(f)$.

Proof. $< U^* f, g > = < f, Ug > = \int f \cdot (g \circ T)\, d\mu = \int \hat{\mathcal{L}}(f \cdot (g \circ T))\, d\mu =$
$\int (\hat{\mathcal{L}}(f)) \cdot g\, d\mu = < \hat{\mathcal{L}}(f), g >$. ♣

Proof of Theorem 4.7.1. Conclusion. We can assume that $\mu(g) = 0$. We have

$$\sum_{n\geq 0}^{\infty} \|E(\tilde{g}|\tilde{\mathcal{F}}_n)\|_2 = \sum_{n\geq 0}^{\infty} \|U^k U^{*k}(g)\|_2 = \sum_{n\geq 0}^{\infty} \|\hat{\mathcal{L}}^k(g)\|_2 < \infty,$$

the latter has been assumed in (4.7.1). Thus CLT has been proved by applying Theorem 1.11.5. If g is Hölder continuous it satisfies (4.7.1). Indeed $\hat{\mathcal{L}}^k(g)$ converges to 0 in the sup norm exponentially fast as $k \to \infty$ by Corollary 4.4.8 (see (4.4.4")). This implies the same convergence in L^2 hence the convergence of the above series.

Now let us prove CLT and LIL with the use of Theorem 1.11.1 for Hölder continuous g. As in Proof of Corollary 4.4.12, let $\pi : \Sigma_A \to X$ be a coding map from a 1-sided topological Markov chain of d symbols due to a Markov partition, see Ch.3.5. Since π is Hölder continuous, if g and ϕ are Hölder continuous, then the compositions $g \circ \pi, \phi \circ \pi$ are Hölder continuous. π is an isomorphism between the measures $\mu_{\phi\circ\pi}$ on Σ_A and μ_ϕ on X, see Ch.3.5 and Lemma 4.4.13. The function $g \circ \pi$ satisfies the assumptions of Theorem 1.11.1 with respect to the σ-algebra $\mathcal{F}$ associated to the partition of Σ_A into 0-th cylinders, see Theorem 4.4.11. ϕ-mixing follows from (4.4.7) and the estimate in (1.11.7) is exponential with an arbitrary δ due to the Hölder property of $g \circ \pi$. Hence, by Theorem 1.11.1, $g \circ \pi$ and therefore g satisfy CLT and LIL.

In Section 4.6 we computed the first derivative of the pressure function. Here using the same method we compute the second derivative and see that it is a respective dispersion σ^2, see Ch.1.11.

Theorem 4.7.4. For every $\phi, u, v : X \to I\!R$ Hölder continuous functions there exists the second derivative

(4.7.2) $$\frac{\partial^2}{\partial s \partial t} P(T, \phi + su + tv)|_{s=t=0} = \lim_{n\to\infty} \frac{1}{n} \int S_n(u - \mu_\phi u) S_n(v - \mu_\phi v)\, d\mu_\phi.$$

where μ_ϕ is the invariant Gibbs measure for ϕ. The latter expression for $u = v$ is the asymptotic variance $\sigma^2(u)$ discussed in CLT, Ch1.11.

Proof. By Ch.4.6, see (4.6.3), (4.6.5),

$$\text{(4.7.3)} \qquad \frac{\partial^2}{\partial s \partial t} P(T, \phi + su + tv)|_{t=0} = \frac{\partial}{\partial s} \lim_{n\to\infty} \frac{\frac{1}{n}\sum_{y\in T^{-n}(x)} S_n v(y) \exp S_n(\phi+su)(y)}{\sum_{y\in T^{-n}(x)} \exp S_n(\phi+su)(y)}.$$

Now we change the order of $\partial/\partial s$ and lim. This will be justified if we prove the uniform convergence of the resulting derivative functions.

Fixed $x \in X$ and n we abbreviate in the further notation $\sum_{y\in T^{-n}(x)}$ to $\sum_y$ and compute

$$F_n(s) := \frac{\partial}{\partial s}\Big(\frac{\sum_y S_n v(y) \exp S_n(\phi+su)(y)}{\sum_y \exp S_n(\phi+su)(y)}\Big) =$$

$$\frac{\sum_y S_n u(y) S_n v(y) \exp S_n(\phi+su)(y)}{\sum_y \exp S_n(\phi+su)(y)} -$$

$$\frac{\big(\sum_y S_n u(y) \exp S_n(\phi+su)(y)\big)\big(\sum_y S_n v(y) \exp S_n(\phi+su)(y)\big)}{\big(\sum_y \exp S_n(\phi+su)(y)\big)^2} =$$

$$\frac{\mathcal{L}^n\big((S_n u)(S_n v)\big)(x)}{\mathcal{L}^n(1\!\!1)(x)} - \frac{\mathcal{L}^n(S_n u)(x)}{\mathcal{L}^n(1\!\!1)(x)} \frac{\mathcal{L}^n(S_n v)(x)}{\mathcal{L}^n(1\!\!1)(x)}.$$

As in Section 6 we write here $\mathcal{L} = \mathcal{L}_0 = e^{-P(T,\phi)}\mathcal{L}_\phi$. It is useful to write the later expression for $F_n(s)$ in the form

$$\text{(4.7.4)} \qquad F_n(s) = \int (S_n u)(S_n v)\, d\mu_{s,n} - \int (S_n u)\, d\mu_{s,n} \int (S_n v)\, d\mu_{s,n}$$

or

$$\text{(4.7.5)} \qquad F_n(s) = \sum_{i,j=0}^{n-1} \int (u\circ T^i - \mu_{s,n}(u\circ T^i))(v\circ T^j - \mu_{s,n}(v\circ T^j))\, d\mu_{s,n},$$

where $\mu_{s,n}$ is the probability measure distributed on $T^{-n}(x)$ according to the weights $\exp(S_n(\phi+su))(y)/\sum_y \exp S_n(\phi+su)(y)$.

Note that $\frac{1}{n}F_n(s)$ with $F_n(s)$ as in the formula (4.7.5) resembles already (4.7.2) because $\mu_{s,n} \to m_{\phi+su}$ in the weak*-topology, see (4.4.3'). However we still need to work a little bit.

For each i, j denote the respective summand in (4.7.5) by $K_{i,j}$. To simplify notation denote $u\circ T^i$ by u_i and $v \circ T^j$ by v_j. We have

$$K_{i,j} = \frac{\mathcal{L}^n\big((u_i - \mu_{s,n}u_i)(v_j - \mu_{s,n}v_j)\big)(x)}{\mathcal{L}^n(1\!\!1)(x)}$$

and for $0 \le i \le j < n$, using (4.4.5) twice,

$$K_{i,j} = \frac{\mathcal{L}^{n-j}\Big(\big(\mathcal{L}^{j-i}((u-\mu_{s,n}u_i)\mathcal{L}^i(\mathbb{1}))\big)\big(v-\mu_{s,n}v_j\big)\Big)(x)}{\mathcal{L}^n(\mathbb{1})(x)}. \tag{4.7.6}$$

By Corollary 4.4.8 for $\tau < 1$ and Hölder norm $||\cdot||_{\mathcal{H}_\alpha}$ for an exponent $\alpha > 0$, transforming the integral as in Proof of Theorem 4.4.10, we get

$$||\mathcal{L}^{j-i}((u-\mu_{s,n}u_i)\mathcal{L}^i(\mathbb{1})) - u_\phi\Big(\int u_i\, dm_{\phi+su} - \mu_{s,n}u_i)\Big)||_{\mathcal{H}_\alpha} \le C\tau^{j-i}$$

where C depends only on Hölder norms of u and $\phi_s u$. The difference in the large parentheses, denote it by $D_{i,n}$, is bounded by $C\tau^{n-i}$ in the Hölder norm, again by Corollary 4.4.8.

We conclude that for all j the functions

$$L_j := \sum_{i\le j} \mathcal{L}^{j-i}((u-\mu_{s,n}u_i)\mathcal{L}^i(\mathbb{1})$$

are uniformly bounded in the Hölder norm $||\cdot||_{\mathcal{H}_\alpha}$ by a constant C depending again only on $||u||_{\mathcal{H}_\alpha}$ and $||\phi+su||_{\mathcal{H}_\alpha}$. Hence summing over $i \le j$ in (4.7.6) and applying $\mathcal{L}^{n-j}$ we obtain

$$\Big|\Big|\sum_{i=0}^{j} K_{i,j} - \sum_{i=0}^{j}\int (u_i-\mu_{s,n}u_i)(v_j-\mu_{s,n}v_j)\, dm_{\phi+su}\Big|\Big|_\infty \le C\tau^{n-j}.$$

Here C depends also on $||v||_{\mathcal{H}_\alpha}$. We can replace the first sum by the second sum because after summing over $j = 0, ..., n-1$ dividing by n and passing with n to ∞ they lead to the same result.

Next notice that $\mu_{s,n}$ can be replaced by $m_{\phi+su}$ due to dealing with n as above. Indeed,

$$\Big|\int (u_i-\mu_{\phi+su}u_i)(v_j-\mu_{\phi+su}v_j)\, dm_{\phi+su} - \int (u_i-\mu_{s,n}u_i)(v_j-\mu_{s,n}v_j)\, dm_{\phi+su}\Big| =$$

$$|(\mu_{s,n}u_i - m_{\phi+su}u_i)\cdot(\mu_{s,n}v_j - m_{\phi+su}v_j)| +$$

$$\Big|\int (u_i-\mu_{\phi+su}u_i)\cdot(\mu_{s,n}v_j - m_{\phi+su}v_j)\, dm_{\phi+su}\Big| +$$

$$|\int (\mu_{s,n}u_i - m_{\phi+su}u_i)(v_j-\mu_{\phi+su}v_j)\, dm_{\phi+su}| = I + II + III.$$

By $D_{i,n} \le C\tau^{n-i}$ and $D_{j,n} \le C\tau^{n-j}$, the summand I is bounded by $\tau^{n-i}\tau^{n-j}$. Note that the second and third summands are equal 0.

The last step is to replace $m = m_{\phi+su}$ by the invariant Gibbs measure $\mu = \mu_{\phi+su}$. Similarly as above we can replace m by μ in mu_i, mv_i. Indeed,

$$|\mu u_i - mu_i| = |\int u\cdot\mathcal{L}^i(\mathbb{1})\, d\mu - \int uu_{\phi+su}\, d\mu| = |\int u\cdot(\mathcal{L}^i(\mathbb{1}) - u_{\phi+su})\, d\mu| \le C\tau^i.$$

so the difference is bounded $C\tau^i\tau^j$. The summands corresponding to II and II are 0.

Finally we replace m by μ at the large integral. To simplify notation write $F = u - \mu u, G = v - \mu v$. Then, using $i \leq j$, we can write

$$|\int (F \circ T^i)(G \circ T^j)\, dm - \int (F \circ T^i)(G \circ T^j)\, d\mu| =$$

$$|\int (F \cdot (G \circ T^{j-i})) \circ T^i\, dm - \int (F \circ T^i)(G \circ T^j) d\mu| \leq$$

$$C\tau^i \int |F \cdot (G \circ T^{j-i})|\, dm \leq Cm(F)m(G)\tau^i\tau^{j-i} = C\,\tau^j$$

by Theorem 4.4.10 (exponential decay of correlations), the latter C depending again on the Hölder norms of $u, v, \phi + su,$. Summing over $0 \leq i \leq j < n$ gives a bound $C \sum_{j=0}^{n-1} j\tau^j$ which again disappears while dealing with n.

For $i > j$ we do the same replacements changing the roles of u and v. ♣

Exercises.

Exercise 1. Prove that (4.1.1) with an arbitrary $0 < \xi' \leq \xi$ in place of ξ implies (4.1.1) for every $0 < \xi' \leq \xi$ (with C depending on ξ') .