CHAPTER 3
DISTANCE EXPANDING MAPS

version Feb.1, 2000

We devote this Chapter to a closer topological study of distance expanding maps. Often however weaker assumptions will be sufficient. We always assume the maps are continuous on a compact metric space X and usually assume the maps are open, which means that open sets have open images. This is equivalent to saying that if $f(x) = y$ and $y_n \to y$ then there exist $x_n \to x$ such that $f(x_n) = y_n$ for n large enough.

In theorems with assertions of topological character the assumption that a map is only expansive gives in fact always the same as if we assumed that the map is expanding, in view of Sec.6. We shall prove in Sec.6 that for every expansive map there always exists a metric compatible with the topology on X given by an original metric, so that the map is distance expanding in it.

Recall that for (X, ρ) a compact metric space, a continuous mapping $T : X \to X$ is said to be *distance expanding* (with respect to the metric ρ) if there exist constants $\lambda > 1$ and $\eta > 0$ such that

$$\rho(x, y) \le 2\eta \implies \rho(T^n(x), T^n(y)) \ge \lambda\rho(x, y) \tag{3.1.1}$$

We say that T is *distance expanding at a set* $Y \subset X$ if the above holds for every $x, y \in B(z, \eta)$ for $z \in Y$.

In future we shall usually be able to assume that $n = 1$ i.e. that

$$\rho(x, y) \le 2\eta \implies \rho(T(x), T(y)) \ge \lambda\rho(x, y) \tag{3.1.2}$$

One can achieve this in two ways:

(1) If T is Lipschitz continuous (say with constant $L > 1$) replace the metric $\rho(x, y)$ by $\sum_{j=0}^{n-1} \rho(T^j(x), T^j(y))$. Of course then λ and η change. As an exercise you can check that the number $1 + (\lambda - 1)\left(\frac{L-1}{L^n-1}\right)$ can play the role of λ in (3.1.2).

(2) Consider T^n instead of T.

Sometimes we shall write for short *expanding*, instead of *distance expanding*.

§3.1 DISTANCE EXPANDING OPEN MAPS, BASIC PROPERTIES

Let us start with a lemma where we assume $T : X \to X$ is a continuous open map of a compact metric space X. We do not need to assume in this lemma that T is distance expanding.

Lemma 3.1.2. If $T : X \to X$ is a continuous open map, then for every $\eta > 0$ there exists $\xi > 0$ such that $T(B(x, \eta)) \supset B(T(x), \xi)$ for every $x \in X$.

Proof. For every $x \in X$ let

$$\xi(x) = \sup\{r > 0 : T(B(x, \eta)) \supset B(T(x), r)\}.$$

Since T is open, $\xi(x) > 0$. Since $T(B(x, \eta)) \supset B(T(x), \xi(x))$, it suffices to show that $\xi = \inf\{\xi(x) : x \in X\} > 0$. Suppose conversely that $\xi = 0$. Then there exists a sequence of points $x_n \in X$ such that

$$\xi(x_n) \to 0 \quad \text{as} \quad n \to \infty \tag{3.1.3}$$

and, as X is compact, we can assume that $x_n \to y$ for some $y \in X$. Hence $B(x_n, \eta) \supset B(y, \frac{1}{2}\eta)$ for all n large enough. Therefore

$$T(B(x_n, \eta)) \supset T\left(B(y, \frac{1}{2}\eta)\right) \supset B(T(y), \varepsilon) \supset B\left(T(x_n), \frac{1}{2}\varepsilon\right)$$

for some $\varepsilon > 0$ and again for every n large enough. The existence of ε such that the second inclusion holds follows from the openness of T. Consequently $\xi(x_n) \geq \frac{1}{2}\varepsilon$ for these n, which contradicts (3.1.3). ♣

If $T : X \to X$ is an open, expanding map, then by (3.1.1), for all $x \in X$, the restriction $T|_{B(x,\eta)}$ is injective and therefore it has an inverse map. The same holds for expanding at Y for all $x \in Y$. In view of Lemma 3.1.2 we can introduce the following definition.

Notation 3.1.3. If $T : X \to X$ is expanding then for all $x \in X$ the inverse of the map $T|_{B(x,\eta)}$ restricted to the ball $B(T(x), \xi)$ will be denoted by T_x^{-1}.

Observe that for every $y \in X$

$$T^{-1}(B(y, \xi)) = \bigcup_{x \in T^{-1}(y)} T_x^{-1}(B(y, \xi)) \tag{3.1.4}$$

Indeed, suppose that $y' = T(x') \in B(y, \xi)$. Then $y \in B(y', \xi)$. Let $x = T_{x'}^{-1}(y)$. As T_x^{-1} and $T_{x'}^{-1}$ coincide on y, they coincide on y' because they map y' into $B(x, \eta)$ and T is injective on $B(x, \eta)$. Thus $x' = T_x^{-1}(y')$.

A map T with the property that there exists ξ such that for each $B(x, \xi)$ (3.1.4) holds with the sets in the union disjoint from each other and T restricted to each of them being a homeomorphism, is called a *covering map*. So we proved that a continuous open locally injective map of a compact metric space is a covering map. This is well known but we gave the proof for the completeness of the exposition.

Immediately from Definition 3.1.3 we have

$$T_x^{-1}(B(T(x), \xi)) \subset B(x, \eta) \tag{3.1.5}$$

¿From now on throughout this section we assume also the expanding property, i.e. (3.1.2). We then get the following.

Lemma 3.1.4. If $x \in X$ and $y, z \in B(T(x), \xi)$ then

$$\rho(T_x^{-1}(y), T_x^{-1}(z)) \leq \lambda^{-1}\rho(y, z)$$

In particular $T_x^{-1}(B(T(x), \xi)) \subset B(x, \lambda^{-1}\xi) \subset B(x, \xi)$.

Definition 3.1.5. For every $x \in X$, every $n \geq 1$ and every $j = 0, 1, \ldots, n-1$ write $x_j = T^j(x)$. In view of Lemma 3.1.4 the composition $T_{x_0}^{-1} \circ T_{x_1}^{-1} \circ \ldots \circ T_{x_{n-1}}^{-1} : B(T^n(x), \xi) \to X$ is well-defined and will be denoted by T_x^{-n}.

Below we collect the basic elementary properties of maps T_x^{-n} following immediately from the above. For every $y \in X$

$$T^{-n}(B(y, \xi)) = \bigcup_{x \in T^{-n}(y)} T_x^{-n}(B(y, \xi)); \tag{3.1.6}$$

$$\rho(T_x^{-n}(y), T_x^{-n}(z)) \leq \lambda^{-n}\rho(y, z) \text{ for all } y, z \in B(T^n(x), \xi); \tag{3.1.7}$$

$$T_x^{-n}(B(T^n(x), r)) \subset B(x, \min\{\eta, \lambda^{-n}r\}) \text{ for every } r \leq \xi. \tag{3.1.8}$$

Remark. All these properties hold, and notation makes sense, also for open maps $T : X \to X$ expanding at $Y \subset X$, provided $x, T(x), ..., T^n(x) \in Y$.

§3.2 SHADOWING OF PSEUDOORBITS

We keep the notation of Section 3.1. We consider an open distance expanding map $T : X \to X$ with the constants η, λ, ξ.

Let n be a non-negative integer or ∞. Given $\alpha \geq 0$ a sequence $(x_i : i = 0, \dots, n)$ is said to be an *α-pseudo-orbit* for $T : X \to X$ if and only if for every $i = 0, \dots, n-1$

$$\rho(T(x_i), x_{i+1}) \leq \alpha \tag{3.2.1}$$

Of course every (real) orbit $(x, T(x), \dots, T^n(x))$, $x \in X$, is an α-pseudo-orbit for every $\alpha \geq 0$. We shall prove a kind of a converse fact, that in case of open, distance expanding maps, each "sufficiently good" pseudo-orbit can be approximated (shadowed) by a real orbit. To make this precise we proceed as follows. Let $\beta > 0$. We say that an orbit of $x \in X$, *β-shadows* the pseudo-orbit $(x_i : i = 0, ..., n)$ if and only if for every $i = 0, \dots, n$

$$\rho(T^i(x), x_i) \leq \beta \tag{3.2.2}$$

Definition 3.2.0. We say that a continuous map $T : X \to X$ has *shadowing property* if for every $\beta > 0$ there exists $\alpha > 0$ such that for every finite n every α-pseudo-orbit can be β-shadowed by an orbit.

Note that due to the compactness of X this property implies the same with $n = \infty$ included.

Below is a simple observation on the uniqueness of the shadowing. Assume only that T is expansive (cf. Section 2.2.).

Proposition 3.2.1. If 2β is less than an expansiveness constant of T (we do not need to assume here that T is expanding with respect to the metric ρ) and $n = \infty$ then there exists at most one point x whose orbit β-shadows the pseudo-orbit $(x_i)_{i=1}^{\infty}$.

Proof. Suppose the forward orbits of x and y shadow (x_i). Then for every $n \geq 0$ we have $\rho(T^n(x), T^n(y)) \leq 2\beta$. Then by the definition of the expansiveness $x = y$. ♣

We shall now prove some less trivial results, concerning the existence of β-shadowing orbits.

Lemma 3.2.2. Let $T : X \to X$ be an open distance expanding map. Let $0 < \beta < \xi$, $0 < \alpha \leq \min\{(\lambda - 1)\beta, \xi\}$. If $(x_i : i = 0, 1, \dots, n)$, $0 \leq n \leq \infty$, is an α-pseudo-orbit and $x'_i = T_{x_i}^{-1}(x_{i+1})$, then

(a) For all $i = 0, 1, 2, \dots, n-1$

$$T_{x'_i}^{-1}(\overline{B(x_{i+1}, \beta)}) \subset \overline{B(x_i, \beta)}$$

and consequently for all $i = 0, 1, \dots, n$ the compositions

$$T_i = T_{x'_0}^{-1} \circ T_{x'_1}^{-1} \circ \dots \circ T_{x'_{i-1}}^{-1} : \overline{B(x_i, \beta)} \to X$$

are well-defined.

(b) The sequence of closed sets $T_i(\overline{B(x_i, \beta)})$, $i = 0, 1, \ldots, n$, is decreasing in the sense of inclusion.

(c) The intersection

$$\bigcap_{i=0}^{n} T_i(\overline{B(x_i, \beta)}$$

is non-empty and the forward orbits (for times $0, 1, ..., n$) of all the points of this intersection β-shadow the pseudo-orbit $(x_i : i = 0, 1, \ldots, n)$.

Proof. In order to prove (a) observe that by (3.1.8) and (3.1.7) we have

$$T_{x_i'}^{-1}(\overline{B(T(x_{i+1}), \beta)}) \subset \overline{B(x_i', \lambda^{-1}\beta)} \subset \overline{B(x_i, \lambda^{-1}\beta + \lambda^{-1}\alpha)}$$

and $\lambda^{-1}\beta + \lambda^{-1}\alpha \leq \beta$. The statement (b) follows immediately from (a). The first part of (c) follows immediately from (b) and the compactness of the space X. To prove the second part call the intersection which appears in (c) by A. By the definition of A we have $T^i(A) \subset \overline{B(x_i, \beta)}$ for all $i = 0, 1, \ldots, n$. Thus the forward orbit for the times $0, 1, ..., n$ of every point in A, β shadows $(x_i : i = 0, 1, \ldots, n)$. The proof is finished. ♣

As an immediate consequence of Lemma 3.2.2 we get the following.

Corollary 3.2.3. (Shadowing lemma) Every open, distance expanding map satisfies the shadowing property. More precisely, for all $\beta > 0$ and $\alpha > 0$ as in Lemma 3.2.2 every α-pseudo-orbit $(x_i : i = 0, \ldots, n)$ can be β-shadowed by an orbit in X.

As a consequence of Corollary 3.2.3 we shall prove the following.

Corollary 3.2.4. (Closing lemma) Let $T : X \to X$ be an expansive map, satisfying the shadowing property. Then for every $\beta > 0$ there exists $\alpha > 0$ such that if $x \in X$ and $\rho(x, T^l(x)) \leq \alpha$ for some $l \geq 1$, then there exists a periodic point of period l whose orbit β-shadows the pseudo-orbit $(x, T(x), \ldots, T^{l-1}(x))$. The choices of α to β are the same as in the definition of shadowing, for 2β not exceeding the expansivness constant.

In particular the above holds for $T : X \to X$ open, expanding.

Proof. Since $\rho(x, T^l(x)) \leq \alpha$, the sequence made up as the infinite concatenation of the sequence $(x, T(x), \ldots, T^{l-1}(x))$ is an α-pseudo-orbit. Hence, by shadowing with $n = \infty$, there is a point $y \in X$ whose orbit β-shadows this pseudo-orbit. But note that then the orbit of the point $T^l(y)$ also does it and therefore, by Lemma 3.2.1, $T^l(y) = y$. The proof is finished. ♣

Note that the assumption T is expansive is substantial. The adding machine map, see Ch.0.3' ???, satisfies the shadowing property, whereas it has no periodic orbits at all. In fact the same proof yields the following *periodic shadowing.*

Definition 3.2.5. We say that a continuous map $T : X \to X$ satisfies *periodic shadowing* property if for every $\beta > 0$ there exists $\alpha > 0$ such that for every finite n and every *periodic α-pseudo-orbit* $x_0, ..., x_{n-1}$, that is a sequence of points $x_0, ..., x_{n-1}$ such that $\rho(T(x_i), x_{(i+1)(\mathrm{mod} n)}) \le \alpha$, there exists a point $y \in X$ of period n such that for all $0 \le i < n \;\; \rho(T^i(y), x_i) \le \beta$.

Note that shadowing and periodic shadowing can hold for the maps that are not expansive. One can just add artificially the missing periodic orbits, of periods 2^n to the adding machine space. This example appears in fact as the nonwandering set for any Feigenbaum-like map of the interval, see Ch ??? (dawny 4.6)

§3.3 SPECTRAL DECOMPOSITION. MIXING PROPERTIES.

Let us start with general observations concerning iterations of continuous mappings

Definition 3.3.1 We call a continuous mapping $T : X \to X$ for a compact metric space X *topologically transitive* if for all non-empty open sets $U, V \subset X$ there exists $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$. By the compactness of X topological transitivity implies that T maps X onto X.

Example 3.3.2 Consider a topological Markov chain Σ_A, or $\tilde{\Sigma}_A$ in a one-sided or two-sided shift space of d states, see Example 0.3. Observe that the left shift map s on the topological Markov chain is topologically transitive iff the matrix A is *irreducible* that is for each i, j there exists an $n > 0$ such that the i, j-th entry $A^n_{i,j}$ of the n-th composition matrix A^n is non-zero.
One can consider a directed graph consisting of d vertices such that there is an edge from a vertex v_i to v_j iff $A_{i,j} \neq 0$; then one can identify elements of the topological Markov chain with infinite paths in the graph (that is sequences of edges indexed by all integers or nonnegative integers depending as we consider the two-sided or one-sided case, such that each edge begins at the vertex where the preceding edge ends). Then it is easy to see that A is irreducible iff for every two vertices v_1, v_2 there exists a finite path from v_i to v_j.

A notion stronger than the topological transitivity, which makes a non-trivial sense only for f non-invertible, is the following

Definition 3.3.2 A continuous mapping $T : X \to X$ for a compact metric space X is called *topologically exact* (or *locally eventually onto*) if for every open set $U \subset X$ there exists $n > 0$ such that $T^n(U) = X$.

In Example 3.3.2 in the one-sided shift space case topological exactness is equivalent to the property that there exists $n > 0$ such that the matrix A^n has all entries positive. Such a matrix is called *aperiodic*.

In the two-sided case aperiodicity of the matrix is equivalent to *topological mixing* of the shift map. We say a continuous map is *topologically mixing* if for every non-empty open sets $U, V \subset X$ there exists $N > 0$ such that for every $n \geq N$ we have $T^n(U) \cap V \neq \emptyset$.

Proposition 3.3.3 The following 3 conditions are equivalent:

(1) $T : X \to X$ is topologically transitive.
(2) For every non-empty open sets $U, V \subset X$ and every $N \geq 0$ there exists $n \geq N$ such that $T^n(U) \cap V \neq \emptyset$.
(3) There exists a T-trajectory $(x_n, n = 0, 1, ...)$, such that every $x \in X$ is its ω-limit point, that is for every $N \geq 0$ the set $\{x_n : n > N\}$ is dense in X.

Proof. Let us prove first the implication (1)$\Rightarrow$(3). So, suppose $T : X \to X$ is topologically transitive. Then for every open non-empty set $V \subset X$, the set

$$K(V) := \{x \in X : \text{there exists } n \geq 0 \text{ such that } \ T^n(x) \in V\} = \bigcup_{n \geq 0} T^{-n}(V)$$

is open and dense in X. Let $\{V_k\}_{k \geq 1}$ be a countable basis of topology of X. By Baire's category theorem, the intersection

$$K := \bigcap_{k \geq 1} \bigcap_{N \geq 0} K(T^{-N}(V_k)$$

is a dense G_δ subset of X. In particular K is non-empty and by its definition the trajectory $\{T^n(x) : n \geq N\}$ is dense in X for every $x \in K$. Thus (1) implies (3).

Let us now prove that (3)$\Rightarrow$(2). Indeed, if x_n is a trajectory satisfying the condition (3), then for all non-empty open sets $U, V \subset X$ and $N \geq 0$, there exist $n \geq m > 0, n - m \geq N$ such that $x_m \in U$ and $x_n \in V$. Hence $T^{n-m}(U) \cap V \neq \emptyset$. Thus (3) implies (2). Since (2) implies (1) trivially the proof is complete. ♣

Definition 3.3.a. A point $x \in X$ is called wandering if there exists an open neighhbourhood V of x such that $V \cap T^n(V) = \emptyset$ for all $n \geq 1$. Otherwise x is called non-wandering. We denote the set of all non-wandering points for T by Ω or $\Omega(T)$.

Proposition 3.3.b For $T : X \to X$ satisfying the periodic shadowing property, the set of periodic points is dense in the set Ω of non-wandering points.

Proof. Take any $x \in \Omega(T)$ and given $\beta > 0$ its neighborhood V in X of diameter α chosen for β in the definition of periodic shadowing. Then by the definition of $\Omega(T)$ there exists $y \in V$ and $n > 0$ such that $T^n(y) \in V$. So $rho(y, T^n(y)) \leq \text{diam} V$ hence $(y, T(y), ..., T^n(y))$ can be β-shadowed by a periodic orbit. We can take β arbitrarily small hence we obtain the density of periodic points in $\Omega(T)$. ♣

Remark 3.3.c. It is not true that for every open, distance expanding map $T : X \to X$ we have $\overline{\text{Per}} = X$. Here is an example: Let $X = \{(1/2)^n : n = 0, 1, 2, ...\} \cup \{0\}$. Let $T((1/2)^n) = (1/2)^{(n-1)}$ for $n > 0$, $T(0) = 0, T(1) = 1$. Let the metric be the restriction to X of the standard metric on the real line. This T is distance expanding on X but $\Omega(T) = \text{Per}(T) = \{0\} \cup \{1\}$. See also Exercise 3.3.1.

Here is the main theorem of this section. Its assertion holds under the assumption that $T : X \to X$ is open, distance expanding and even under weaker assumptions below.

Theorem 3.3.4 (on the existence of Spectral Decomposition) Suppose that $T : X \to X$ is an open map which satisfies also the periodic shadowing property and is expanding at the set $\overline{\text{Per}(T)}$, the closure of the set of periodic points.

Then $\overline{\text{Per}(T)}$ is the union of finitely many disjoint compact sets $\Omega_j, j = 1, ..., J$ with

$$(T|_{\overline{\text{Per}(T)}})^{-1}(\Omega_j) = \Omega_j$$

and $T|_{\Omega_j}$ topologically transitive.

Each Ω_j is the union of $k(j)$ disjoint compact sets Ω_j^k which are cyclically permuted by T and such that $T^{k(j)}|_{\Omega_j^k}$ is topologically exact.

Proof of Theorem 3.3.4 Let us start with defining an equivalence relation $\sim$ on $\mathrm{Per}(T)$. For $x, y \in \mathrm{Per}(T)$ we write $x \sim_{\to} y$ if for every $\varepsilon > 0$ there exist $x' \in X$ and positive integer m such that $\rho(x, x') < \varepsilon$ and $T^m(x') = T^m(y)$. We write $x \sim y$ if $x \sim y$ and $y \sim x$. Of course for every $x \in \mathrm{Per}(T)$, $x \sim x$. Suppose that $x \sim y$ and $y \sim z$. Let k_y, k_z denote periods of y, z respectively.

Let x' be close to x and $T^n(x') = T^n(y) = y$; an integer n satisfiying the latter equality exists since we can take an integer so that the first equality holds and then take any larger integer divisible by k_y. Choose n divisible by $k_y k_z$. Next, since T is open, for y' close enough to y, with $T^m(y') = T^m(z) = z$ for m divisible by k_z, there exists x'' close to x' such that $T^m(x'') = y'$. Hence $T^{n+m}(x'') = T^m(y') = y' = T^{n+m}(y')$, since both m and n are divisible by k_z. Thus $x \sim z$. This proof is illustrated at Fig 3.1.a.

Fig.3.1.a

Fig.3.1.b

Fig.3.1.b illustrates the transitivity for hyperbolic sets $\overline{\mathrm{Per}(T)}$ (see Exercises or [KH] ???), where $x \sim y$ if the unstable manifold of x intersects transversally the stable manifold of y. In our expanding case the role of transversality is played by the openness of T.

Till this point we did not use the expanding assumption.

Observe now that for every $x, y \in \mathrm{Per}(T)$, $\rho(x, y) \leq \xi$ implies $x \sim y$. Indeed, we can take $x' = T_x^{-nk_x k_y}(y)$ for n arbitrarily large. Then x' is arbitrarily close to x and $T^{nk_x k_y}(x') = y = T^{nk_x k_y}(y)$. Hence the number of equivalence classes of $\sim$, denote them $P_1, ..., P_N$, is finite. Moreover the sets $\overline{P}_1, ..., \overline{P}_N$ are pairwise disjoint and the distances between them are at least ξ. We have $T(\mathrm{Per}(T)) = \mathrm{Per}(T)$, and if $x \sim y$ then $T(x) \sim T(y)$. The latter follows straight from the definition of $\sim$. So T permutes the sets P_i. This permutation decomposes into cyclic permutations we were looking for. More precisely: consider the partition of $\overline{\mathrm{Per}(T)}$ into the sets of the form

$$\bigcup_{n=0}^{\infty} T^n(\overline{P}_i), \ i = 1, ..., N.$$

The unions are in fact over finite families. It does not matter in which place the closure is placed because X is compact so for every $A \subset X$ we have $T(\overline{A}) = \overline{T(A)}$. We consider this partition as a partition into Ω_j's we were looking for. Ω_j^k's are the summands $T^n(\overline{P}_i)$ in the unions.

Observe now that T is topologically transitive on each Ω_j.

Indeed, if x, y belong to the same Ω_j there exist $x' \in B(x, \xi)$ and $y' \in B(y, \xi)$ such that $T^n(x') = T^{n_0}(y)$ and $T^m(y') = T^{m_0}(x)$ for some natural numbers n, m and $n_0 \leq k_y, m_0 \leq k_x$. For an arbitrary $\beta > 0$ choose $\alpha > 0$ from the definition of periodic shadowing and consider x'', y'' such that $\rho(x'', x) \leq \alpha$, $\rho(y'', y) \leq \alpha$ and $T^{n_1}(x'') = x'$, $T^{m_1}(y'') = y'$ for some natural numbers n_1, m_1, existing by the expanding property at $\overline{\mathrm{Per}(T)}$. Then the sequence of points $T(x''), ..., T^{n_1+n+k_y-n_0}(x''), T(y''), ..., T^{m_1+m+k_x-m_0}(y'')$ is a periodic α-pseudo-orbit, of period $n_1+n+k_y-n_0+m_1+m+k_x-m_0$, so it can be β-shadowed by a periodic orbit. Thus, there exists $z \in \mathrm{Per}(T)$ such that $\rho(z, x) \leq \beta$ and $\rho(T^N(z), y) \leq \beta$ for an integer $N > 0$. Now take arbitrary neighbourhoods $U \ni x$ and $V \ni y$ and take β such that $B(x, \beta) \subset U$ and $B(y, \beta) \subset V$. We find a periodic point z as above. Note that, provided $\beta \leq \xi$, $z \sim x$ and $T^N(z) \sim y$. We obtain $T^N(z) \in T^N(U \cap \Omega_j) \cap (V \cap \Omega_j)$ so this set is nonempty. This proves the topological transitivity.

Note that by the way we proved that the orbits $x'', ..., T^{n_1}(x'') = x', ..., T^n(x')$ with n_1, n arbitrarily large, can be arbitrarily well shadowed by parts of periodic orbits. This corresponds to the approximation of transversal cycles of heteroclinic orbits by periodic ones, in the hyperbolic theory (see also Exercise 3.3.3).

This analogy justifies the name *heteroclinic cycle points* for the points x' and y', or *heteroclinic cycle orbits* for their orbits discussed above. Thus we proved

Lemma 3.3.7'. Under the assumptions of Theorem 3.3.4 every heteroclinic cycle point is a limit of periodic points.

Now we can prove another fact interesting in itself:

Lemma 3.3.8 $T|_{\overline{\mathrm{Per}(T)}}$ is an open map.

Proof. Fix $x, y \in \overline{\mathrm{Per}(T)}$ and $\rho(T(x), y) \leq \varepsilon \leq \xi/3$. Since T is open, by Lemma 3.1.2, and due to the expanding property at $\overline{\mathrm{Per}(T)}$ there exists $\hat{y} = T_x^{-1}(y) \in B(x, \lambda^{-1}\xi/3)$ We want to prove that $\hat{y} \in \overline{\mathrm{Per}(T)}$.

There exist $z_1, z_2 \in \mathrm{Per}(T)$ such that $\rho(z_1, x) \leq \lambda^{-1}\xi/3$ and $\rho(z_2, y) \leq \xi/3$. Hence $\rho(T(z_1), z_2) \leq \xi$, hence $T(z_1) \sim z_2$. Then $T_x^{-1}(z_2)$ is a heteroclinic cycle point, so by Lemma 3.3.7' it is a limit of periodic points. ♣

We go back to Proof of Theorem 3.3.4. We can prove now the topological exactness of $T^{k(j)}|_{\Omega_j^k}$. So fix $\Omega_j^k = P_i$ with $T^{k(j)}(P_i) = P_i$. Let $\{x_s\}, s = 1, ...S$ be a $\xi'/2$-spanning set in P_i, where ξ' is a constant having the properties of ξ for the map $T|_{\overline{\mathrm{Per}}}$, existing by the openness of $T|_{\overline{\mathrm{Per}(T)}}$ (Lemma 3.1.2). Write $k(P_i) = \prod_{s=1}^{S} k_{x_s}$. Take an arbitrary open set $U \subset \overline{P}_i$. It contains a periodic point x.

Note that for every ball $B = B(y, r)$ in $\overline{\mathrm{Per}(T)}$ with the origin at $y \in \mathrm{Per}(T)$ and radius r less than η and $\lambda^{-k_y}\xi'$, we have $T^{k_y}(B) \supset B(y, \lambda^{k_y} r)$. Repeating this step by step we obtain $T^{nk(y)}(B) \supset B(y, \xi')$, (see (3.1.8).

Let us go back to U and consider $B_x = B(x, r) \subset U$ with $r \leq \lambda^{-k(P_i)}\xi'$. Then $T^{nk(P_i)}(B_x)$ is an increasing family of sets for $n = 0, 1, 2, ...$

By the definition of $\sim$, the set $\bigcup_{n \geq 0} T^{nk(P_i)}(B_x)$ contains $\{x_s :, s = 1, ...S\}$, because the points x_s are in the relation $\sim$ with x. This uses the fact proved above, see Lemma 3.3.7', that x' in the definition of $\sim$, such that $T^m(x') = T^m(x_s)$, belongs to $\overline{\mathrm{Per}(T)}$. It belongs even to P_i, since for $z \in \mathrm{Per}(T)$ close to x' we have $z \sim x_s$, with the use of the same

x' as one of a heteroclinic cycle points. Hence, by the observation above $\bigcup_{n\geq 0} T^{nk(P_i)}(B_x)$ contains the ball $B(x_s, \xi')$ for each s. So it contains $\overline{P}_i$. Since $T^{nk(P_i)}(B_x)$ is an increasing family of open sets in $\overline{\mathrm{Per}(T)}$ that is compact, just one of these sets covers $\overline{\mathrm{Per}(T)}$. The topological exactness is proved. ♣

Remark. It is easy to see that if T is a covering map than the assumption of periodic shadowing can be skipped. We used it only to approximate heteroclinic cycle points by periodic ones. See also Exercise 3.3.2.

As a corollary we obtain the following two theorems.

Theorem 3.3.9. Let $T : X \to X$ be an open distance expanding map, or expanding at the set $\overline{\mathrm{Per}(T)}$ satisfying the periodic shadowing property. Then, if T is topologically transitive, or is surjective and its spectral decomposition consists of just one set $\Omega_1 = \bigcup_{k=1}^{k(1)} \Omega_1^k$, the following properties hold:

1. The set of periodic points is dense in X, which is thus equal to Ω_1.
2. For every open $U \subset X$ there exists $N = N(U)$ such that $\bigcup_{j=0}^{N} T^j(U) = X$.
3. $(\forall r > 0)(\exists N)(\forall x \in X) \bigcup_{j=0}^{N} T^j(B(x, r)) = X$.
4. The following *specification property* holds: For every $\beta > 0$ there exists a positive integer N such that for every n and every T-orbit $(x_0, ...x_n)$ there exists a periodic point y of period not larger than $n + N$ whose orbit for the times $0, ..., n$ β-shadows $(x_0, ...x_n)$.

Proof. By the topological transitivity for all open U there exist $n \geq 1$ such that $T^n(U) \cap U \neq \emptyset$, (use the condition (2) in Proposition 3.3.3 for $N = 1$). Hence for the set Ω of the non-wandering points we have $\Omega = X$. This gives the density of $\mathrm{Per}(T)$ by Proposition 3.3.b.

If we assume only that there is one $\Omega_1 (= \Omega = \overline{\mathrm{Per}(T)})$ in the Spectral Decomposition, then for an arbitrary $z \in X$ we find by the surjectivity a backward orbit z_{-n} of z and notice that $z_{-n} \to \Omega$ and $T^n(z) \to \Omega$, that follows easily from the definition of Ω. So for every $\alpha > 0$ there exist $w_1, w_2 \in \mathrm{Per}(T)$ and natural numbers k, n such that $T^k(w_2) \sim w_1$, $\rho(w_1, z_{-n}) \leq \alpha$ and $\rho(w_2, T^n(z)) \leq \alpha$. This allows to find a periodic point in $B(z, \beta)$, where $\beta > 0$ is arbitrarily small and α chosen for β from the periodic shadowing property.

We conclude that $X = \bigcup_{j=1}^{J} \Omega_j$, each Ω_j is T-invariant, closed, and also open since Ω_j's are at least ξ-distant from each other. So $J = 1$. Otherwise, by the topological transitivity, for $j \neq i$ there existed n such that $T^n(\Omega_j) \cap \Omega_i \neq \emptyset$, what would contradict the T-invariance of Ω_j.

Thus $X = \bigcup_{k=1}^{k(1)} (\Omega_1^k)$ and the assertion 2. follows immediately from the exactness of $T^{k(1)}$ on each Ω_1^k, $k = 1, ..., k(1)$.

The property 3. follows from 2. where given r we choose $N = \max\{N(U)\}$ where we consider a finite covering of X by sets U of diameter not exceeding $r/2$. Indeed, then for every $B(x, r)$ the set U containing x is a subset of$B(x, r)$.

Now let us prove the specification property. By the property 3. for every $\alpha > 0$ there exists $N = N(\alpha)$ such that for every $v, w \in X$ there exists $m \leq N$ and $z \in B(v, \alpha)$ such that $T^m(z) \in B(w, \alpha)$. Consider any T-orbit $x_0, ...x_n$. Then consider an α-pseudo-orbit $x_0, ...x_{n-1}, z, ..., T^{m-1}(z)$ with $m \leq N$ and $z \in B(x_n, \alpha, T^m(z) \in B(x_0, \alpha)$. By Corollary 3.2.4 we can β-shadow it by a periodic orbit of period $n + m \leq n + N$. ♣

The same proof yields

Theorem 3.3.10. Let T satisfies the assumptions of Theorem 3.3.9, and be also topologically mixing, i.e. $k(1) = 1$. Then

1. T is topologically exact, i.e. for every open $U \subset X$ there exists $N = N(U)$ such that $T^N(U) = X$.

2. $(\forall r > 0)(\exists N)(\forall x \in X) \;\; T^N(B(x, r)) = X$.

§3.4 HÖLDER CONTINUOUS FUNCTIONS

For distance expanding maps, Hölder continuous functions play a special role. Recall that a function $\phi : X \to \mathbb{C}$ (or $\mathbb{R}$) is said to be *Hölder continuous* with an exponent $0 < \alpha \le 1$ if and only if there exists $C > 0$ such that

$$|\phi(y) - \phi(x)| \le C\rho(y,x)^\alpha$$

for all $x, y \in X$. All Hölder continuous functions are continuous, if $\alpha = 1$ they are usually called Lipschitz continuous.

Let $C(X)$ denote as in the previous chapters the space of all continuous, real or complex-valued functions defined on a compact space X and for $\psi : X \to \mathbb{C}$ we write $\|\psi\|_\infty := \sup\{|\psi(x)| : x \in X\}$ for its supremum norm. For any $\alpha > 0$ let $\mathcal{H}_\alpha(X)$ denote the space of all Hölder continuous functions with exponent $\alpha > 0$. If $\psi \in \mathcal{H}_\alpha(X)$ let

$$\vartheta_{\alpha,\xi}(\psi) = \sup\left\{ \frac{|\psi(y) - \psi(x)|}{\rho(y,x)^\alpha} : x, y \in X,\ x \ne y \text{ and } \rho(x,y) \le \xi \right\}$$

and

$$\vartheta_\alpha(\psi) = \sup\left\{ \frac{|\psi(y) - \psi(x)|}{\rho(y,x)^\alpha} : x, y \in X, x \ne y \right\}.$$

Note that

$$\vartheta_\alpha(\psi) \le \max\left\{ \frac{2||\psi||_\infty}{\xi^\alpha}, \vartheta_{\alpha,\xi}(\psi) \right\}.$$

The reader will check easily that $\mathcal{H}_\alpha(X)$ becomes a Banach space when equipped with the norm

$$\|\psi\|_{\mathcal{H}_\alpha} = \vartheta_\alpha(\psi) + \|\psi\|_\infty.$$

Thus, to estimate in future $||\psi||_{\mathcal{H}_\alpha}$ it is enough to estimate $\vartheta_{\alpha,\xi}(\psi)$ and $||\psi||_\infty$. The following result is a straightforward consequence of Arzela-Ascoli theorem.

Theorem 3.4.1. Any bounded subset of the Banach space $\mathcal{H}_\alpha(X)$ with the norm $\|\cdot\|_{\mathcal{H}_\alpha}$ is relatively compact as a subset of the Banach space $C(X)$ with the supremum norm $\|\cdot\|_\infty$. Moreover if $\{\psi_n : n = 1, 2, \ldots\}$ is a sequence of continuous functions in $\mathcal{H}_\alpha(X)$ such that $\|x_n\|_{\mathcal{H}_\alpha} \le C$ for all $n \ge 1$ and some constant C and if $\lim_{n\to\infty} \|\psi_n - \psi\|_\infty = 0$ for some $\psi \in C(X)$, then $\psi \in \mathcal{H}_\alpha(X)$ and $\|\psi\|_{\mathcal{H}_\alpha} \le C$.

Now let us formulate a simple but very basic lemma in which you will see a coherence of the expanding property of T and the Hölder continuity property of a function.

Lemma 3.4.2 (pre-Bounded Distortion Lemma for Iteration). Let $T : X \to X$ be a distance expanding map and $\phi : X \to \mathbb{C}$ be a Hölder continuous function with the exponent α. Then for every positive integer n and $x, y \in X$ such that

$$\rho(T^j(x), T^j(y)) < 2\eta \quad \text{for every} \quad j = 0, 1, ..., n-1 \tag{3.4.1}$$

we have

$$(3.4.2) \qquad |S_n\phi(x) - S_n\phi(y)| \le \rho(T^n(x), T^n(y))^\alpha \Big(\frac{\vartheta_\alpha(\phi)}{1-\lambda^{-\alpha}}\Big).$$

If T is open we can assume $x, y \in T_z^{-n}(B(T^n(z), \xi)$ for a point $z \in X$, instead of (3.4.1). Then in (3.4.2) we can replace ϑ_α by $\vartheta_{\alpha,\xi}$.

The sense of (3.4.2) is that the coefficient $\frac{\vartheta_\alpha(\phi)}{1-\lambda^{-\alpha}}$ does not depend on x, y, n).

Proof. By (3.1.2) we have $\rho(T^j(x), T^j(y)) \le \lambda^{-(n-j)}\rho(T^n(y), T^n(z))$ for every $0 \le j \le n$. Hence

$$|\phi(T^j(y)) - \phi(T^j(z))| \le \vartheta_\alpha(\phi)\lambda^{-(n-j)\alpha}\rho(T^n(y), T^n(z))^\alpha$$

Thus

$$\begin{aligned}|S_n\phi(y) - S_n\phi(z)| &\le \vartheta_\alpha(\phi)\rho(T^n(y), T^n(z))^\alpha \sum_{j=0}^{n-1} \lambda^{-(n-j)\alpha} \\ &\le \vartheta_\alpha(\phi)\rho(T^n(y), T^n(z))^\alpha \sum_{j=0}^{\infty} \lambda^{-j\alpha} = \frac{\vartheta_\alpha(\phi)}{1-\lambda^{-\alpha}}\rho(T^n(y), T^n(z))^\alpha\end{aligned}$$

The proof is finished. ♣

For an open distance expanding topologically transitive map we can replace topological pressure defined in Chapter 2 by a corresponding notion related with a "tree" of pre-images of an arbitrary point (compare this with Exercise 4 ??? in Chapter 2).

Proposition 3.4.3. If $T : X \to X$ is a topologically transitive distance expanding map, then for every Hölder continuous potential $\phi : X \to I\!R$ and for every $x \in X$ there exists the limit

$$\mathrm{P}_x(T, \phi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\overline{x}\in T^{-n}(x)} \exp S_n\phi(\overline{x})$$

and it is equal to the topological pressure $\mathrm{P}(T, \phi)$. In addition, there exists a constant C such that for every $x, y \in X$ and every positive integer n

$$(3.4.3) \qquad \frac{\sum_{\overline{x}\in T^{-n}(x)} \exp S_n\phi(\overline{x})}{\sum_{\overline{y}\in T^{-n}(y)} \exp S_n\phi(\overline{y})} < C$$

Proof. If $\rho(x, y) < \xi$ then (3.4.3) follows immediately from Lemma 3.4.2 with some constant, say C_1. Now observe that by the topological transitivity of T there exists N (depending on ξ) such that for all $x, y \in X$ there exists $0 \le m < N$ such that $T^m(B(x, \xi)) \cap B(y, \xi) \ne \emptyset$. Indeed, for example by the condition 3) in Proposition 3.3.3 we can find two blocks of a trajectory of z with dense ω-limit set, say $T^k(z), ..., T^{k'}(z)$ and $T^l(z), ..., T^{l'}(z)$

with $l > k'$, each ξ-dense in X. Then we set $N = l' - k$. We can find t between k and k' and s between l and l' so that $T^t(z) \in B(x, \xi)$ and $T^s(z) \in B(y, \xi$. We have $m := s - t \leq N$.

Now fix arbitrary $x, y \in X$. So, there exists a point $y' \in T^{-m}(B(y, \xi)) \cap B(x, \xi)$. We then have

$$\begin{aligned}
\sum_{\overline{x} \in T^{-n}(x)} \exp S_n\phi(\overline{x}) \leq C_1 \sum_{\overline{y'} \in T^{-n}(y')} \exp S_n\phi(\overline{y'}) \\
&= C_1 \exp(-S_m\phi(T^m(y'))) \sum_{\overline{y'} \in T^{-n}(y')} \exp S_{n+m}\phi(\overline{y'}) \\
&\leq C_1 \exp(-m \inf \phi) \sum_{\overline{y'} \in T^{-(n+m)}(T^m(y'))} \exp S_{n+m}\phi(\overline{y'}) \\
&\leq C_1 \exp(-m \inf \phi) \sum_{\overline{y'} \in T^{-(n+m)}(T^m(y'))} \exp S_n\phi(T^m(\overline{y'})) \exp S_m\phi(\overline{y'}) \\
&\leq C_1 \exp(m \sup \phi - m \inf \phi) \sum_{\overline{y'} \in T^{-(n+m)}(T^m(y'))} \exp S_n\phi(T^m(\overline{y'})) \\
&\leq C_1 \exp(2N||\phi||_\infty) D^N \sum_{\overline{y'} \in T^{-n}(T^m(y'))} \exp S_n\phi(\overline{y'}) \\
&\leq C_1^2 \exp(2N||\phi||) \sum_{\overline{y} \in T^{-n}(y)} \exp S_n\phi(\overline{y}),
\end{aligned}$$

where $D = \sup\{\#(T^{-1}(z)) : z \in X\} < \infty$. This proves (3.4.3).

Observe that each set $T^{-n}(x)$ is $(n, 2\eta)$-separated, whence

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{\overline{x} \in T^{-n}(x)} \exp S_n\phi(\overline{x}) \leq \mathrm{P}(T, \phi),$$

by the characterization of pressure given in Theorem 2.2.10.

In order to prove the opposite inequality fix $\varepsilon < 2\xi$ and for every $n \geq 1$, an (n, ε)-separated set F_n. Cover X by finitely many balls $B(z_1, \varepsilon/2), B(z_2, \varepsilon/2), \dots, B(z_k, \varepsilon/2)$. Then $F_n = F_n \cap \left(\bigcup_{j=1}^k T^{-n}\big(B(z_j, \varepsilon/2)\big)\right)$ and therefore

$$\sum_{z \in F_n} \exp(S_n\phi(z)) = \sum_{j=1}^{k} \sum_{F_n \cap T^{-n}(B(z_j, \varepsilon/2))} \exp(S_n\phi(z)).$$

Given $y \in X$ choose as $j(y)$ an arbitrary j such that $y \in T^{-n}(B(z_{j(y)}, \varepsilon/2))$. Let $\overline{z_{j(y)}} \in T^{-n}(z)$ be defined by $y \in T^{-n}_{\overline{z_{j(y)}}}(B(z_{j(y)}, \varepsilon/2)$. We shall show that the function $y \mapsto \overline{z_{j(y)}}$ is injective. Indeed, suppose that $\overline{z_j} = \overline{z_{j(a)}} = \overline{z_{j(b)}}$ for some $a, b \in F_n \cap T^{-n}(B(z_j, \varepsilon/2))$. Then

$$\rho(T^l(a), T^l(b)) \leq \rho(T^l(a), T^l(\overline{z_j})) + \rho(T^l(\overline{z_j}), T^l(b)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for every $0 \leq l \leq n$. So, $a = b$ since F_n is (n, ε)-separated.

Hence, using (3.4.3), we obtain

$$\sum_{z \in F_n} \exp(S_n\phi(z)) \leq \sum_{j=1}^{k} C \sum_{\overline{z_j}} \exp(S_n\phi(\overline{z_j})) \leq kC^2 \sum_{\overline{x} \in T^{-n}(x)} \exp(S_n\phi(\overline{x}))$$

Letting $n \nearrow \infty$ and next $\varepsilon \to 0$, applying Theorem 2.2.10, we therefore get

$$\mathrm{P}(T, \phi) \leq \liminf_{n \to \infty} \frac{1}{n} \log \sum_{\overline{x} \in T^{-n}(x)} \exp S_n\phi(\overline{x}).$$

Thus

$$\liminf_{n \to \infty} \frac{1}{n} \log \sum_{\overline{x} \in T^{-n}(x)} \exp S_n\phi(\overline{x}) \geq \mathrm{P}(T, \phi) \geq \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\overline{x} \in T^{-n}(x)} \exp S_n\phi(\overline{x}).$$

So liminf=limsup above, the limit exists and is equal to $\mathrm{P}(T, \phi)$. ♣

Remark 3.4.4. It follows from Proposition 3.4.3, the proof of the Variational Principle Part II (see Section 2.3) and the expansiveness of T that for every $x \in X$ every weak limit of the measures $\frac{1}{n} \sum_{k=0}^{n-1} \mu_n \circ T^{-k}$ where

$$\mu_n = \frac{\sum_{\overline{x} \in T^{-n}(x)} \delta_x \exp S_n\phi(\overline{x})}{\sum_{\overline{x} \in T^{-n}(x)} \exp S_n\phi(\vee x)}$$

and δ_x denotes the Dirac measure concentrated at the point x, is an equilibrium state. In fact our very special situation allows to say a lot more about the measures involved. Chapter 3 will be devoted to this end.

Let us finish this section with one more very useful fact (compare Theorem 1.11.3.)

Proposition 3.4.5. Let $T : X \to X$ be an open, distance expanding, topologically transitive map. If $\phi, \psi \in \mathcal{H}_\alpha(X)$, then the following conditions are equivalent.

(1) If $x \in X$ is a periodic point of T and if n denotes its period, then $S_n\phi(x) - S_n\psi(x) = 0$.
(2) There exists a constant $C > 0$ such that for every $x \in X$ and integer $n \geq 0$, we have $|S_n\phi(x) - S_n\psi(x)| \leq C$.
(3) There exists a function $u \in \mathcal{H}_a$ such that $\phi - \psi = u \circ T - u$.

Proof. The implications (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) are very easy. The first one is obtained by summing up the equation in (3) along the orbit $x, T(x), ..., T^{n-1}(x)$ which gives $C = 2 \sup |\phi - \psi|$. The second one holds because otherwise, if $S_n\phi(x) - S_n\psi(x) = K \neq 0$ for x of period n, then we have $S_{jn}\phi(x) - S_{jn}\psi(x) = jK$ which contradicts (2) for j large enough. Now let us prove (1) $\Longrightarrow$ (3). Let $x \in X$ be a point such that for every $N \geq 0$ the orbit $(x_n : n = N, N+1, ...)$ is dense in X. Such x exists by topological transitivity of T, see Proposition 3.3.3. Write $\eta = \phi - \psi$. Define u on the forward orbit of x, the set $A = \{x_n : n = 0, 1, ...\}$ by $u(x_n) = S_n\eta(x)$. If x is periodic then X is just the orbit of x and the function u is well defined due to the equality in (1). So, suppose that x is not periodic. Then $x_n \neq x_m$ for $m \neq n$ hence u is well defined on A. We will show that it extends in a Hölder continuous manner to $\overline{A} = X$. Indeed, if we take points $x_m, x_n \in A$ such that $m < n$ and $\rho(x_m, x_n) < \varepsilon$ for ε small enough, then $x_m, ..., x_{n-1}$ can be β-shadowed by a periodic orbit $y, ..., T^{n-m-1}(y)$ of period $n - m$ by Corollary 3.2.4, where ε is related to β in the same way as α related to β in that Corollary. Then by the Lemma 3.4.2

$$\begin{aligned} |u(x_n) - u(x_m)| &= |S_n\eta(x) - S_m\eta(x)| = |S_{n-m}\eta(x_m)| \\ &= |S_{n-m}\eta(x_m) - S_{n-m}\eta(y)| \leq \vartheta(\phi)_\alpha\varepsilon^\alpha. \end{aligned}$$

In particular we proved that u is uniformly continuous on A which allows to extend u continuously to $\overline{A}$. By taking limits we see that this extension satisfies the same Hölder estimate on $\overline{A}$ as on A. Also the equality in (3) true on A, extends to $\overline{A}$ by the definition of u and by the continuity of η and u . The proof is finished. ♣

The equality in (3) is called *cohomology equation*, u is a solution of the equation, see Ch.1.11.2. Here the cohomology equation is solvable in the space $\mathcal{K} = \mathcal{H}_\alpha$. Note that proving 3) $\Longrightarrow$ 2) we used only the assumption that u is bounded. So, going through 2) $\Longrightarrow$ 1) $\Longrightarrow$ 3) we prove that if the cohomology equation is solvable with u bounded, then automatically $u \in \mathcal{H}_\alpha$. Later on ??? you will see that an assumption that u is finite measurable, for some probability T-invariant measure with support X, would be sufficient, even under assumptions on T weaker than expanding. Often u is forced to be as good as ϕ and ψ. This type of theorem is called *Livśic type theorem.*

§3.5 MARKOV PARTITIONS AND SYMBOLIC REPRESENTATION

We shall prove in this section that the topological Markov chains (Ch.0.3) describe quite precisely dynamics of general open expanding maps.

This can be done through so called Markov partitions of X. The sets of a partition will play the role of "cylinders" $\{i_0 = \text{ Const }\}$ in Σ_A.

Definition 3.5.1. A finite cover $\Re = \{R_1, \ldots, R_n\}$ of X is said to be a *Markov partition* of the space X for the mapping T if $\operatorname{diam}(\Re) < \min\{\eta, \xi\}$ and the following conditions are satisfied.

(a) $R = \overline{\operatorname{Int} R_i}$ for all $i = 1, 2, \ldots, d$
(b) $\operatorname{Int} R_i \cap \operatorname{Int} R_j = \emptyset$ for all $i \neq j$
(c) $\operatorname{Int} R_j \cap T(\operatorname{Int} R_i) \neq \emptyset \implies R_j \subset T(R_i)$ for all $i, j = 1, 2, \ldots, d$

Theorem 3.5.2. For the open, distance expanding mapping T there exist Markov partitions of arbitrarily small diameters.

Proof. Fix $\beta < \min\{\eta/4, \xi\}$ and let α be the number associated to β as in Lemma 3.2.2. Choose $0 < \gamma \leq \min\{\beta/2, \alpha/2\}$ so small that

$$\rho(x, y) \leq \gamma \implies \rho(T(x), T(y)) \leq \alpha/2 \tag{3.5.1}$$

and let $E = \{z_1, \ldots, z_r\}$ be a γ-spanning set of X. Define the space Ω putting

$$\Omega = \{q = (q_i) \in E^{\boldsymbol{Z}^+} : \rho(T(q_i), q_{i+1}) \leq \alpha \text{ for all } i \geq 0\}$$

By definition all elements of the space Ω are α-pseudo-orbits and therefore in view of Corollary 3.2.3 and Lemma 3.2.1 for every sequence $q \in \Omega$ there exists a unique point whose orbit for $n = 0, 1, \ldots$ β-shadows q. Denote this point by $\Theta(q)$. In this way we have defined a map $\Theta : \Omega \to X$. We will need some of its properties.

Let us show first that Θ is surjective. Indeed, since E is a γ spanning set, for every $x \in X$ and every $i \geq 0$ there exists $q_i \in E$ such that

$$\rho(T^i(x), q_i) < \gamma$$

and therefore, using also (3.5.1),

$$\rho(T(q_i), q_{i+1}) \leq \rho(T(q_i), T(T^i(x))) + \rho(T^{i+1}(x), q_{i+1}) < \alpha/2 + \gamma \leq \alpha/2 + \alpha/2 = \alpha$$

for all $i \geq 0$. Thus $q = (q_i : i = 0, 1, \ldots) \in \Omega$ and (as $\gamma < \beta$) $x = \Theta(q)$. The surjectivity of Θ is proved.

Now we shall show that Θ is continuous. For this aim we will need the following notation. If $q \in \Omega$ then we put

$$q(n) = \{p \in \Omega : p_i = q_i \text{ for every } i = 0, 1, \ldots, n\} \tag{3.5.2}$$

To prove continuity suppose now that $p, q \in \Omega$, $p(n) = q(n)$ with some $n \geq 0$ and denote $x = \Theta(q)$, $y = \Theta(p)$. Then for all $i = 0, 1, \ldots, n$

$$\rho(T^i(x), T^i(y)) \leq \rho(T^i(x), q_i) + \rho(p_i, T^i(y)) \leq \beta + \beta = 2\beta$$

As $\beta < \xi$, we therefore obtain by (3.1.2) that $\rho(T^{i+1}(x), T^{i+1}(y)) \geq \lambda \rho(T^i(x), T^i(y))$ for $i = 0, 1, \ldots, n-1$, (see (3.1.7)), and consequently $\rho(x, y) \leq \lambda^{-n} 2\beta$. The continuity of Θ is proved.

Now for every $k = 1, \ldots, r$ define the sets

$$P_k = \Theta(\{q \in \Omega : q_0 = z_k\})$$

Since Θ is continuous, Ω is a compact space, and the sets $\{q \in \Omega : q_0 = z_k\}$ are closed in Ω, all sets P_k are closed in X.

Denote

$$W(k) = \{l : \rho(T(z_k), z_l) \leq \alpha\}$$

We have the following basic property satisfied:

$$T(P_k) = \bigcup_{l \in W(k)} P_l \tag{3.5.3}$$

Indeed, if $x \in P_k$ then $x = \Theta(q)$ for $q \in \Omega$ with $q_0 = z_k$. By the definition of Ω we have $q_1 = z_l$ for some $l \in W(k)$. We obtain $T(x) \in P_l$.

Conversely, let $x \in P_l$ for $l \in W(k)$. It means that $x = \Theta(q)$ for some $q \in \Omega$ with $q_0 = z_l$. By the definition of $W(k)$ the concatenation $z_k q$ belongs to Ω and therefore the point $T(\Theta(z_k q))$ β-shadows q. Thus $T(\Theta(z_k q)) = \Theta(q) = x$ hence $x \in T(P_k)$.

Let now

$$Z = X \setminus \bigcup_{n=0}^{\infty} T^{-n}\Big(\bigcup_{k=1}^{r} \partial P_k\Big)$$

and for any $x \in Z$ denote

$$P(x) = \{k \in \{1, \ldots, r\} : x \in P_k\},$$

$$Q(x) = \{l \notin P(x) : P_l \cap \Big(\bigcup_{k \in P(x)} P_k\Big) \neq \emptyset\},$$

and

$$S(x) = \bigcap_{k \in P(x)} \mathrm{Int} P_k \setminus \Big(\bigcup_{k \in Q(x)} P_k\Big) = \bigcap_{k \in P(x)} \mathrm{Int} P_k \setminus \Big(\bigcup_{k \notin P(x)} P_k)\Big)$$

We shall show that the family $\{S(x) : x \in Z\}$ is in fact finite and moreover, that the family $\{\overline{S(x)} : x \in Z\}$ is a Markov partition of diameter not exceeding 2β.

Indeed, since $\operatorname{diam}(P_k) \leq 2\beta$ for every $k = 1, \dots, r$ we have

$$\operatorname{diam}(S(x)) \leq 2\beta \tag{3.5.4}$$

As the sets $S(x)$ are open, we have

$$\overline{\operatorname{Int}\overline{S(x)}} = \overline{S(x)} \tag{3.5.5}$$

for all $x \in Z$. This proves the property (a) of the Theorem.

We shall now show that for every $x \in Z$

$$T(S(x)) \supset S(Tx) \tag{3.5.6}$$

Note first that for $K(x) := \bigcup_{k \in P(x)} P_k \cup \bigcup_{l \in Q(x)} P_l$ we have $\operatorname{diam} K \leq 8\beta$ and therefore by the assumption $\beta < \eta/4$, the map T restricted to K is injective.

Consider $k \in P(x)$. Then there exists $l \in W(k)$ such that $T(x) \in P_l$ cf. (3.5.3), and using the definition of Z we get $T(x) \in \operatorname{Int}(P_l)$. Using the assumption that T is open and next (3.5.3) we obtain

$$T(\operatorname{Int} P_k) = \operatorname{Int}(T(P_k)) \supset \operatorname{Int} P_l \supset S(T(x))$$

and therefore

$$T\Big(\bigcap_{k \in P(x)} \operatorname{Int} P_k\Big) \supset S(T(x)) \tag{3.5.7)}$$

Now consider $k \in Q(x)$. We remind (3.5.3) and observe that by the injectivity of $T|_K$ the assumption $x \notin P_k$ implies $T(x) \notin P_l$, $l \in W(k)$.

Thus

$$T(P_k) \subset \bigcup_{l \notin P(T(x))} P_l$$

hence

$$T\Big(\bigcup_{l \in Q(x)} P_l\Big) \cap S(T(x)) = \emptyset$$

Combining this and (3.5.7) gives

$$T\Big(\bigcap_{k \in P(x)} \operatorname{Int} P_k \setminus \Big(\bigcup_{k \in Q(x)} P_k\Big)\Big) \supset S(T(x))$$

which exactly means that formula (3.5.6) is satisfied and therefore

$$T(\overline{S(x)}) \supset \overline{S(Tx)} \tag{3.5.8}$$

We shall now prove the following claim.

Claim. If $x, y \in Z$ then either $S(x) = S(y)$ or $S(x) \cap S(y) = \emptyset$.

Indeed, if $P(x) = P(y)$ then also $Q(x) = Q(y)$ and consequently $S(x) = S(y)$. If $P(x) \neq P(y)$ then there exists $k \in P(x) \div P(y)$, say $k \in P(x) \setminus P(y)$. Hence $S(x) \subset \mathrm{Int} P_k$ and $S(y) \subset X \setminus P_k$. Therefore $S(x) \cap S(y) = \emptyset$ and the Claim is proved.

(One can write the family $S(x)$ as $\bigvee_{k=1,\dots,r} \{\mathrm{Int} P_k, X \setminus P_k\}$, compare notation in Ch.1. Then the assertion of the Claim is immediate.)

Since the family $\{P(x) : x \in Z\}$ is finite so is the family $\{S(x) : x \in Z\}$. Note that $S(x) \cap S(y) = \emptyset$ implies $\mathrm{Int}\overline{S(x)} \cap \mathrm{Int}\overline{S(y)} = \emptyset$. This is a general property of pairs of open sets, $U \cap V = \emptyset$ implies $\overline{U} \cap V = \emptyset$ implies $\mathrm{Int}\overline{U} \cap V = \emptyset$ implies $\mathrm{Int}\overline{U} \cap \overline{V} = \emptyset$ implies $\mathrm{Int}\overline{U} \cap \mathrm{Int}\overline{V} = \emptyset$.

In view of Baire's theorem the set Z is dense in X. Since $\bigcup_{x \in Z} S(x) \supset Z$, we thus have $\bigcup_{x \in Z} \overline{S(x)} = X$. That the family $\{\overline{S(x)} : x \in Z\}$ is a Markov partition for T of diameter not exceeding 2β follows now from (3.5.5), (3.5.6), (3.5.4) and from the claim. The proof is finished. ♣

Each Markov partition allows to introduce a coding (symbolic representation) of $T : X \to X$ as follows.

Theorem 3.5.3. Let $T : X \to X$ be an open, distance expanding map. Let $\{R_1, \dots, R_d\}$ be a Markov partition. Let $A = (a_{i,j})$ be a $d \times d$ matrix with $a_{i,j} = 0$ or 1, $a_{i,j} = 1$ iff $T(\mathrm{Int} R_i) \cap \mathrm{Int} R_j \neq \emptyset$. Then consider the one-sided topological Markov chain Σ_A with the left shift σ, see Ch.0.3. Define a mapping $\pi : \Sigma_A \to X$ by

$$\pi((i_0, i_1, \dots)) = \bigcap_{n=0}^{\infty} T^{-n}(R_{i_n}).$$

Then π is well defined Hölder continuous mapping onto X and $T\pi = \pi\sigma$. Moreover $\pi|_{\pi^{-1}(X \setminus \bigcup_{n=0}^{\infty} T^{-n}(\bigcup_i \partial R_i))}$ is injective.

Proof. For an arbitrary sequence $(i_0, i_1, \dots) \in \Sigma_A$, $a_{i,j} = 1$ implies $T(R_{i_n}) \supset R_{i_{n+1}}$. Since $\mathrm{diam} R_{i_n} < 2\eta$, T is injective on R_{i_n}, hence there exists an inverse branch $T^{-1}_{R_{i_n}}$ on $R_{i_{n+1}}$ The subscript R_{i_n} indicates that we take the branch leading to R_{i_n}, compare notation from Ch.3.1. Thus, $T^{-1}_{R_{i_n}}(R_{i_{n+1}}) \subset R_{i_n}$. Hence

$$T^{-1}_{R_{i_0}} T^{-1}_{R_{i_1}} \dots T^{-1}_{R_{i_n}}(R_{i_{n+1}}) \subset T^{-1}_{R_{i_0}} T^{-1}_{R_{i_1}} \dots T^{-1}_{R_{i_{n-1}}}(R_{i_n}).$$

So $\bigcap_{n \geq 0} T^{-n}(R_{i_n}) \neq \emptyset$, as the intersection of the above decreasing family of compact sets. We have used here

$$T^{-1}_{R_{i_0}} \dots T^{-1}_{R_{i_{n-1}}}(R_{i_n}) = T^{-1}_{R_{i_0}} \dots T^{-1}_{R_{i_{n-2}}}(T^{-1}(R_{i_n}) \cap R_{i_{n-1}})$$

$$= T^{-1}_{R_{i_0}} \dots T^{-1}_{R_{i_{n-3}}}(T^{-2}(R_{i_n}) \cap T^{-1} R_{i_{n-1}} \cap R_{i_{n-2}}) = \dots = \bigcap_{k=0}^{n} T^{-k}(R_{i_k})$$

following from $T_{R_{i_k}}^{-1}(A) = T^{-1}(A) \cap R_{i_k}$ for every $A \subset R_{i_{k+1}}$, $k = 0, ..., n-1$.

Our infinite intersection consists of only one point, since $\mathrm{diam} R_i$ are less than the expansivness constant.

Let us prove now that π is Hölder continuous. Indeed, $\mathrm{dist}((i_n), (i'_n)) \le \lambda_1^{-N}$ implies $i_n = i'_n$ for all $n = 0, ..., N-1$, where we consider distance in the metric ρ' in Ch.0.3, with the factor λ'. Then, for $x = \pi((i_n))$, $y = \pi((i'_n))$ and every $n : 0 \le n < N$ we have $T^n(x), T^n(y) \in R_{i_n}$, hence $\mathrm{dist}(T^n(x), T^n(y)) \le \mathrm{diam} R_{i_n} \le \xi$, hence $\mathrm{dist}(x, y) \le \lambda^{-(N-1)}\xi$. Therefore π is Hölder with exponent $\min\{1, \log\lambda' / \log\lambda\}$.

Finally let us deal with the injectivity. If $x = \pi((i_n))$ and $T^n(x) \in \mathrm{Int} R_{i_n}$ for all $n = 0, 1, ...$, then $T^n(x) \notin R_j$ for all $j \ne i_n$. So, if $x \in \bigcap_n T^{-n}(R_{i'_n})$, then all $i'_n = i_n$. ♣

Remark. One would not think that π is always injective on the whole Σ_A. Consider for example the mapping of the unit interval $T(x) = 2x (\mathrm{mod}\ 1)$, compare Ch.0.3. Then dyadic expansion of x is not unique for $x \in \bigcup_{n=0}^{\infty} T^{-n}(\{\frac{1}{2}\})$. Dyadic expansion is the inverse, π^{-1}, of the coding obtained from the Markov partition $[0, 1] = \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$.

Remind finally that $\sigma : \Sigma_A \to \Sigma_A$ is an open, distance expanding map. The partition into the cylinders $C_i := \{(i_n) : i_0 = i\}$ for $i = 1, ..., d$, is a Markov partition into closed-open sets. The corresponding coding π is just the identity.

Another fact concerning a similarity between (Σ_A, σ) and (X, T) is the following

Theorem 3.5.4. For every Hölder continuous function $\phi : X \to I\!R$ the function $\phi \circ \pi$ is Hölder continuous on Σ_A and the pressures coincide, $\mathrm{P}(T, \phi) = \mathrm{P}(\sigma, \phi \circ \pi)$.

Proof. The functio $\pi \circ \Phi$ is Hölder as a composition of Hölder continuous functions. Consider next an arbitrary $x \in X \setminus \bigcup_{n=0}^{\infty} T^{-n}(\bigcup_i \partial R_i)$. Then, using Proposition 3.4.3 for T and σ we obtain

$$\mathrm{P}(T, \phi) = \mathrm{P}_x(T, \phi) = \mathrm{P}_{\pi^{-1}(x)}(\sigma, \phi \circ \pi) = \mathrm{P}(\sigma, \phi \circ \pi).$$

The middle equality follows directly from the definitions. ♣

Finally we shall prove that π is injective in a measure-theoretic sense.

Theorem 3.5.5. For every ergodic, invariant under the shift σ, probability Borel measure μ on Σ_A, positive on open sets, the mapping π yields an isomorphism between μ and the measure $\mu \circ \pi^{-1}$ on the Borel sets in X.

Proof. The set $\partial = \bigcup_{i=1}^{d} \partial(R_i)$, and hence $\pi^{-1}(\partial)$, have non-empty open complements in Σ_A. We have also $T(\partial) \subset \partial$ hence $\sigma(\pi^{-1}(\partial)) \subset \pi^{-1}(\partial)$. Hence, by the σ-invariance of μ we get $\mu(\pi^{-1}(\partial)) = \mu(\sigma(\pi^{-1}(\partial)))$, equal to 0 or 1 by the ergodicity. But the complement of $\pi^{-1}(\partial)$, as a non-empty open set, has positive measure μ. Hence $\mu(\pi^{-1}(\partial)) = 0$. Hence $\mu(E) = 0$ for $E := \bigcup_{n=0}^{\infty} T^{-n}(\pi^{-1}(\partial))$ and by Theorem 3.5.3 π is injective on $\Sigma_A \setminus E$. This proves that π is the required isomorphism. ♣

§3.6 EXPANSIVE MAPS ARE EXPANDING IN SOME METRIC

Theorem 3.1.1 says that distance expanding maps are expansive. In this section we prove the following much more difficult result which can be considered as a sort of the converse statement and which provides an additional strong justification to explore expanding maps.

Theorem 3.6.1. If a continuous map $T : X \to X$ of a compact metric space X is (positively) expansive then there exists a metric on X, compatible with the topology, such that the mapping T is distance expanding with respect to this metric.

The proof of Theorem 3.6.1 given here relies heavily on the old topological result of Frink (see [Frn], comp.[K, p.185]) which we state below without proof.

Lemma 3.6.2. (The Metrization Lemma of Frink) Let $\{U_n : n \geq 0\}$ be a sequence of open neighborhoods of the diagonal $\Delta \subset X \times X$ such that $U_0 = X \times X$,

$$\bigcup_{n=1}^{\infty} U_n = \Delta, \tag{3.6.1}$$

and for every $n \geq 1$

$$U_n \circ U_n \circ U_n \subset U_{n-1} \tag{3.6.2}$$

Then there exists a metric ρ, compatible with the topology on X, such that for every $n \geq 1$

$$U_n \subset \{(x, y) : \rho(x, y) < 2^{-n}\} \subset U_{n-1}. \tag{3.6.3}$$

We will also need the following almost obvious result.

Lemma 3.6.3. If $T : X \to X$ is a continuous map of a compact metric space X and T^n is distance expanding for some $n \geq 1$ then T is distance expanding with respect to some some metric compatible with the topology on X.

Proof. Let ρ be a compatible metric with respect to which T is distance expanding and let $\lambda > 1$ and $\eta > 0$ be constants such that

$$\rho(T^n(x), T^n(y)) \geq \lambda \rho(x, y)$$

whenever $\rho(x, y) < \eta$. Put $\xi = \lambda^{\frac{1}{n}}$ and define the new metric ρ' setting

$$\rho'(x, y) = \rho(x, y) + \frac{1}{\xi}\rho(T(x), T(y)) + \ldots + \frac{1}{\xi^{n-1}}\rho(T^{n-1}(x), T^{n-1}(y))$$

Then ρ' is a metric on X compatible with the topology and $\rho'(T^(x), T^(y)) \geq \xi \rho'(x, y)$ whenever $l\rho'(x, y) < \eta$. ♣

Now we can pass to the proof of Theorem 3.6.1.

Proof of Theorem 3.6.1. Let d be a metric on X compatible with the topology, and let $3\theta > 0$ be an expansive constant associated to T which does not exceed the constant η claimed in Proposition 2.4.9. For any $n \geq 1$ and $\gamma > 0$ let

$$V_n(\gamma) = \{(x,y) \in (X \times X) : d(T^j(x), T^j(y)) < \gamma \quad \text{for every } j = 0, \dots, n\}.$$

Then in view of Proposition 2.4.9 there exists $M \geq 1$ such that

$$V_M(3\theta) \subset \{(x,y) : d(x,y) < \theta\}. \tag{3.6.4}$$

Define $U_0 = X \times X$ and $U_n = V_{Mn}(\theta)$ for every $n \geq 1$. We will check that the sequence $\{U_n : n \geq 0\}$ satisfies the assumptions of Lemma 3.6.2. Indeed, (3.6.1) follows immediately from expasiveness of T and condition (3.6.2) will be proved by induction. For $n = 1$ nothing has to be proved. Suppose that (3.6.2) holds for some $n \geq 1$ and let (x,u), (u,v), $(v,y) \in U_{n+1}$. Then by the triangle inequality

$$d(T^j(y), T^j(x)) < 3\theta \qquad \text{for every } j = 0, \dots, (n+1)M.$$

Therefore, using (3.6.4), we conclude that

$$d(T^j(y), T^j(x)) < \theta \qquad \text{for every } j = 0, \dots, Mn$$

Equivalently $(x,y) \in V_{Mn}(\theta) = U_n$ which finishes the proof of (3.6.2).

So we have shown that the assumptions of Lemma 3.6.2 are satisfied, and therefore we obtain a compatible metric ρ on X satisfying (3.6.3). In view of Lemma 3.6.3 it sufficies to show that T^{3M} is expanding with respect to the metric ρ. So suppose that $0 < \rho(x,y) < \frac{1}{16}$. Then by (3.6.1) there exists an $n \geq 0$ such that

$$(x,y) \in U_n \setminus U_{n+1}. \tag{3.6.5}$$

As $0 < \rho(x,y) < \frac{1}{16}$, this and (3.6.3) imply that $n \geq 3$. It follows from (3.6.5) and the definitions of U_n and $V_{Mn}(\theta)$, that there exists $Mn < j \leq (n+1)M$ such that $d(T^j(y), T^j(x)) \geq \theta$. Since $3 \leq n$ we conclude that $d(T^i(T^{3M}(x)), T^i(T^{3M}(y))) \geq \theta$ for some $0 \leq i \leq (n-2)M$ and therefore $(T^{3M}(x), T^{3M}(y)) \notin U_{n-2}$. Consequently, by (3.6.3) and (3.6.5) we obtain that

$$\rho(T^{3M}(x), T^{3M}(y)) \geq 2^{-(n-1)} = 2 \cdot 2^{-n} > 2\rho(x,y).$$

The proof is finished. ♣

Exercises.

Exercise 3.2.1. Prove the following *Shadowing Theorem* generalizing Corollary 3.2.3 (Shadowing lemma) and Corollary 3.2.4 (Closing lemma):

Let $T : X \to X$ be an open map, expanding at a compact $Y \subset X$. Then, for every $\beta > 0$ there exists $\alpha > 0$ such that for every map $\Gamma : Z \to Z$ for a set Z and a map $\Phi : Z \to B(Y, \alpha)$ satisfying $\rho(T\Phi(z), \Phi\Gamma(z)) \le \alpha$ for every $z \in Z$, there exists a map $\Psi : Z \to X$ satisfying $T\Phi = \Phi\Gamma$, hence $T(Y') \subset Y'$ for $Y' = \Psi(Z)$, and such that for every $z \in Z$, $\rho(\Psi(z), \Phi(z)) \le \beta$. If Z is a metric space and Γ, Φ are continuous, then Ψ is continuous. If $T(Y) \subset Y$ and the map $T|_Y : Y \to Y$ be open, then $Y' \subset Y$.

(Hint: see Ch.5.1.)

Exercise 3.2.2. Prove the following *structural stability theorem.*

Let $T : X \to X$ be an open map with a compact $Y \subset X$ such that $T(Y) \subset Y$. Then for every $\lambda > 1$ and $\beta > 0$ there exists $\alpha > 0$ such that if $S : X \to X$ is distance expanding at Y with the expansion factor λ and for all $y \in Y$ $\rho(S(y), T(y)) \le \alpha$ then there exists a continuous mapping $h : Y \to X$ such that $Sh|_Y = hT|_Y$, in particular $S(Y') \subset Y'$ for $Y' = h(Y)$, and $\rho(h(z), z) \le \beta$.

(Hint: apply the previous exercise for $Z = Y, \Gamma = T|_Y, \Phi = \mathrm{id}$, $T = S$ and $Y = Y$. Compare also Ch.5.1.)

Exercise 3.3.1. Prove that every $T : X \to X$ open, distance expanding, for X compact connected, is topologically exact.

Exercise 3.3.2. Prove Lemma 3.3.7' and hence Theorem 3.3.4 (Spectral Decomposition) without the assumption of periodic shadowing, assuming that T is a branched covering of the Riemann sphere.

Exercise 3.3.3. Prove the existence of stable and unstable manifolds for hyperbolic sets and *Smale's Spectral Decomposition Theorem for Axiom A diffeomorphisms.*

An invariant set Λ for a diffeomorphism T is called *hyperbolic* if there exist constants $\lambda > 1$ and $C > 0$ such that the tangent bundle on X restricted to tangent spaces over points in Λ, $T_\Lambda X$ decomposes into DT-invariant subbundles $T_\Lambda X = T^u_\Lambda X \oplus T^s_\Lambda X$ such that $||DT^n(v)|| \ge C\lambda^n$ for all $v \in T^u_\Lambda X$ and $n \ge 0$ and $||DT^n(v)|| \ge C\lambda^n$ for all $v \in T^s_\Lambda X$ and $n \le 0$.

Prove that for every $x \in \Lambda$ the sets $W^u(x) = \{y \in X : \rho(T^n(x), T^n(y)) \to 0 \text{ as} n \to -\infty\}$, and $W^s(x) = \{y \in X : \rho(T^n(x), T^n(y)) \to 0 \text{ as} n \to \infty\}$ are immersed manifolds. (They are called *unstable* and *stable manifolds.*)

Assume next that a diffeomorphism $T : X \to X$ satisfies *Smale's Axiom A condition*, that is the set of non-wandering points Ω is hyperbolic and $\Omega = \overline{\mathrm{Per}}$.

Then the relation between periodic points is as follows. $x \sim y$ if there are points $z \in W^u(x) \cap W^s(y)$ and $z' \in W^u(y) \cap W^s(x)$ where $W^u(x)$a and $W^s(y)$, and $W^u(y)$a and $W^s(x)$ respectively, intersect transversally, that is the tangent spaces to these manifolds at z and z'

span the whole tangent spaces. Prove that this relation yields Spectral Decomposition, as in Theorem 3.3.4, with topological transitivity assertion rather than topological exactness of course.

As one of the steps prove a lemma corresponding to Lemma 3.3.7' about approximation of a transversal heteroclinic cycle points by periodic ones. That is assume that $x_1, x_2, ..., x_n$ are hyperbolic periodic points (i.e. their orbits are hyperbolic sets) for a diffeomorphism, and $W^u_{x_i}$ has a point p_i of transversal intersection with $W^s_{x_{(i+1) \bmod n}}$ for each $i = 1, ..., n$. Then $p_i \in \overline{\mathrm{Per}}$.

(For the theory of hyperbolic sets for diffeomorphisms see for example [KH].)

Exercise 3.4.1. Prove directly that 1) $\Longrightarrow$ 2) in Proposition 3.4.5, using the specification property, Theorem 3.2.9.

***Exercise 3.5.1.** Suppose $T : X \to X$ is a distance expanding map on a closed surface. Prove that there exist a Markov partition for an iterate T^N compatible with a cell complex structure. That is elements R_i of the partitions are topological discs, the 1-dimensional "skeleton" $\bigcup_i \partial R_i$ is a graph consisting of a finite number of continuous curves "edges" intersecting one another only at end points, called "vertices". Intersection of each two R_i is empty or one vertex or one edge, each vertex is contained in 2 or 3 edges.

(Hint: Start with any cellular partition, with R_i being nice topological discs and correct it by adding or subtracting components of $T^{-N}(R_i), T^{-2N}(R_i)$, etc. See [FJ1] for details.)

***Exercise 3.5.2.** Prove that if T is an expanding map of the 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$, a factor map of a linear map of $\mathbb{R}^2$ given by an integer matrix with two irrational eigenvalues of different moduli (for example $\left(\begin{smallmatrix} 0 & 11 \\ -1 & 7 \end{smallmatrix}\right)$ but not $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)$), then ∂R_i cannot be differentiable.

(Hint: Smooth curves $T^n(\partial R_i)$ become more and more dense in $\mathbb{R}^2/\mathbb{Z}^2$ as $n \to \infty$, stretching in the direction of the eigenspace corresponding to the eigenvalue with a larger modulus. So they cannot omit $\mathrm{Int} R_i$.

The same argument, looking backward, says that the components of $T^{-n}(\mathrm{Int} R_i)$ are dense and very distorted, since the eigenvalues have different moduli. The curve ∂R_i must manouver between them, so it is "fractal". See [PU] for more details.)

Historical and Bibliographical Notes.

For Shadowing Lemma in the hyperbolic setting see [Anosov], [Bowen] and [Kushnirenko] or [KH] (for the variant as in Exercise 3.2.1. For the expanding case see [Shub], where structural stability was proved for X a differentiable manifold, T being C^1. D. Sullivan introduced in [Sullivan] the notion *telescope* for the sequence $T^{-1}_{x'_i}(\overline{B(x_{i+1}, \beta)}) \subset$

$\overline{B(x_i,\beta)}$ to capture a shadowing orbit, hence to prove stability of expanding repellers, compare Ch.5.1. This stability was also proved in [Przytycki, 1977]. Recently a comprehensive monography on shadowing by S. Yu. Pilyugin [Pilyugin] appeared.

The existence of Spectral Decomposition in the sense of Theorem 3.3.4 (see Exercise 3.3.3.) was first proved by S. Smale [S] for diffeomorphisms which he called *Axiom A*, that is the set of non-wandering points Ω is hyperbolic and $\Omega = \overline{\text{Per}}$, see also [KH] and further historical informations therein. In a topological setting this was considered by Bowen [B2], called *Axiom A** and for *Axiom A endomorphisms*, covering the diffeomorphisms and expanding (smooth) cases, in [Przytycki, 1977]. For open, distance expanding maps $\Omega = \overline{\text{Per}}$ (Proposition 3.3.b.) corresponds to the analogous fact for Anosov diffeomorphisms. $\Omega = X$ is not known for Anosov diffeomorphisms. It is not true for some distance expanding endomorphisms (Remark 3.3.c), but true for X connected (Exercise 3.3.1), see [Shub] in the smooth case.

The construction of Markov partition in Sec. 5 is similar to the construction for basic sets of Axiom A diffeomorphisms in [Bowen, 1975]. For a general theory of cellular Markov partitions, including Exercise 3.5.1, see [FJ2]. The fact that Hausdorff dimension of the boundaries of 2-dimensional cells is greater than 1, in particular their non-differentiability, Exercise 3.5.2, follows from [PU].

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