Section 5.8

1.

$$\begin{aligned}
\text{(i) } (-3+i)(14-2i) &= (-3)(14-2i) + i(14-2i)\\
&= \{(-3)14 - (-3)(2i)\} + i(14) - i(2i)\\
&= (-42+6i) + (14i+2) = -40+20i.
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } \frac{2+3i}{1-4i} &= \frac{(2+3i)(1+4i)}{(1-4i)(1+4i)}\\
&= \frac{((2+3i)+(2+3i)(4i)}{1^2+4^2}\\
&= \frac{-10+11i}{17} = \frac{-10}{17} + \frac{11}{17}i.
\end{aligned}$$

$$\begin{aligned}
\text{(iii) } \frac{(1+2i)^2}{1-i} &= \frac{1+4i+(2i)^2}{1-i}\\
&= \frac{1+4i-4}{1-i} = \frac{-3+4i}{1-i}\\
&= \frac{(-3+4i)(1+i)}{2} = \frac{-7+i}{2} = -\frac{7}{2} + \frac{1}{2}i.
\end{aligned}$$

2. (i)

$$\begin{aligned}
iz + (2-10i)z = 3z + 2i \quad &\Leftrightarrow \quad z(i+2-10i-3) = 2i\\
&= \Leftrightarrow \quad z(-1-9i) = 2i \Leftrightarrow z = \frac{-2i}{1+9i}\\
&= \quad \frac{-2i(1-9i)}{1+81} = \frac{-18-2i}{82} = \frac{-9-i}{41}.
\end{aligned}$$

(ii) The coefficient determinant is

$$\begin{vmatrix} 1+i & 2-i \\ 1+2i & 3+i \end{vmatrix} = (1+i)(3+i) - (2-i)(1+2i) = -2+i \neq 0.$$

Hence Cramer's rule applies: there is a unique solution given by

$$\begin{aligned}
z &= \frac{\begin{vmatrix} -3i & 2-i \\ 2+2i & 3+i \end{vmatrix}}{-2+i} = \frac{-3-11i}{-2+i} = -1+5i\\
w &= \frac{\begin{vmatrix} 1+i & -3i \\ 1+2i & 2+2i \end{vmatrix}}{-2+i} = \frac{-6+7i}{-2+i} = \frac{19-8i}{5}.
\end{aligned}$$

3.

$$\begin{aligned} 1+(1+i)+\cdots+(1+i)^{99} &= \frac{(1+i)^{100}-1}{(1+i)-1} \\ &= \frac{(1+i)^{100}-1}{i} = -i\left\{(1+i)^{100}-1\right\}. \end{aligned}$$

Now $(1+i)^2 = 2i$. Hence

$$(1+i)^{100} = (2i)^{50} = 2^{50}i^{50} = 2^{50}(-1)^{25} = -2^{50}.$$

Hence $-i\left\{(1+i)^{100}-1\right\} = -i(-2^{50}-1) = (2^{50}+1)i$.

4. (i) Let $z^2 = -8-6i$ and write z=x+iy, where x and y are real. Then

$$z^2 = x^2 - y^2 + 2xyi = -8-6i,$$

so $x^2 - y^2 = -8$ and $2xy = -6$. Hence

$$y = -3/x, \quad x^2 - \left(\frac{-3}{x}\right)^2 = -8,$$

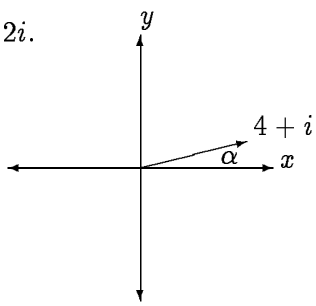
so $x^4 + 8x^2 - 9 = 0$. This is a quadratic in x^2. Hence $x^2 = 1$ or -9 and consequently $x^2 = 1$. Hence $x = 1$, $y = -3$ or $x = -1$ and $y = 3$. Hence $z = 1-3i$ or $z = -1+3i$.

(ii) $z^2 - (3+i)z + 4 + 3i = 0$ has the solutions $z = (3+i\pm d)/2$, where d is any complex number satisfying

$$d^2 = (3+i)^2 - 4(4+3i) = -8-6i.$$

Hence by part (i) we can take $d = 1-3i$. Consequently

$$z = \frac{3+i\pm(1-3i)}{2} = 2-i \quad \text{or} \quad 1+2i.$$

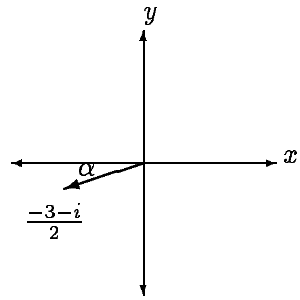


(i) The number lies in the first quadrant of the complex plane.

$$|4+i| = \sqrt{4^2+1^2} = \sqrt{17}.$$

Also $\operatorname{Arg}(4+i) = \alpha$, where $\tan\alpha = 1/4$ and $0 < \alpha < \pi/2$. Hence $\alpha = \tan^{-1}(1/4)$.

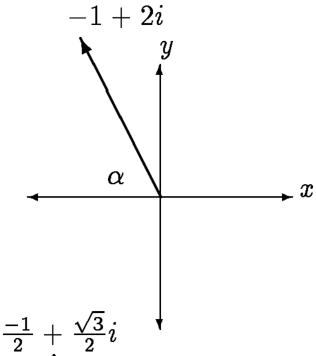
(ii) The number lies in the third quadrant of the complex plane.



$$\left|\frac{-3-i}{2}\right| = \frac{|-3-i|}{2}$$
$$= \frac{1}{2}\sqrt{(-3)^2+(-1)^2} = \frac{1}{2}\sqrt{9+1} = \frac{\sqrt{10}}{2}.$$

Also $\mathrm{Arg}\,(\frac{-3-i}{2}) = -\pi + \alpha$, where $\tan\alpha = \frac{1}{2}/\frac{3}{2} = 1/3$ and $0 < \alpha < \pi/2$. Hence $\alpha = \tan^{-1}(1/3)$.

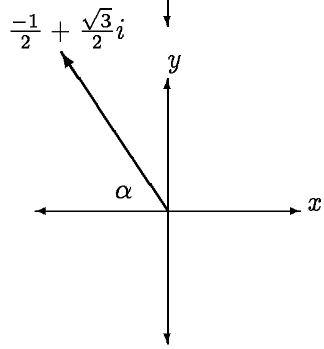
(iii) The number lies in the second quadrant of the complex plane.



$$|-1+2i| = \sqrt{(-1)^2+2^2} = \sqrt{5}.$$

Also $\mathrm{Arg}\,(-1+2i) = \pi - \alpha$, where $\tan\alpha = 2$ and $0 < \alpha < \pi/2$. Hence $\alpha = \tan^{-1}2$.

(iv) The number lies in the second quadrant of the complex plane.



$$\left|\frac{-1+i\sqrt{3}}{2}\right| = \frac{|-1+i\sqrt{3}|}{2}$$
$$= \frac{1}{2}\sqrt{(-1)^2+(\sqrt{3})^2} = \frac{1}{2}\sqrt{1+3} = 1.$$

Also $\mathrm{Arg}\,(\frac{-1}{2} + \frac{\sqrt{3}}{2}i) = \pi - \alpha$, where $\tan\alpha = \frac{\sqrt{3}}{2}/\frac{1}{2} = \sqrt{3}$ and $0 < \alpha < \pi/2$. Hence $\alpha = \pi/3$.

6. (i) Let $z = (1+i)(1+\sqrt{3}i)(\sqrt{3}-i)$. Then

$$\begin{aligned} |z| &= |1+i||1+\sqrt{3}i||\sqrt{3}-i| \\ &= \sqrt{1^2+1^2}\sqrt{1^2+(\sqrt{3})^2}\sqrt{(\sqrt{3})^2+(-1)^2} \\ &= \sqrt{2}\sqrt{4}\sqrt{4} = 4\sqrt{2}. \end{aligned}$$

$$\begin{aligned}
\operatorname{Arg} z &\equiv \operatorname{Arg}(1+i) + \operatorname{Arg}(1+\sqrt{3}) + \operatorname{Arg}(\sqrt{3}-i) \pmod{2\pi} \\
&\equiv \frac{\pi}{4} + \frac{\pi}{3} - \frac{\pi}{6} \equiv \frac{5}{12}.
\end{aligned}$$

Hence $\operatorname{Arg} z = \frac{5}{12}$ and the polar decomposition of z is

$$z = 4\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right).$$

(ii) Let $z = \frac{(1+i)^5(1-i\sqrt{3})^5}{(\sqrt{3}+i)^4}$. Then

$$|z| = \frac{|(1+i)|^5|(1-i\sqrt{3})|^5}{|(\sqrt{3}+i)|^4} = \frac{\left(\sqrt{2}\right)^5 2^5}{2^4} = 2^{7/2}.$$

$$\begin{aligned}
\operatorname{Arg} z &\equiv \operatorname{Arg}(1+i)^5 + \operatorname{Arg}(1-\sqrt{3}i)^5 - \operatorname{Arg}(\sqrt{3}+i)^4 \pmod{2\pi} \\
&\equiv 5\operatorname{Arg}(1+i) + 5\operatorname{Arg}(1-\sqrt{3}i) - 4\operatorname{Arg}(\sqrt{3}+i) \\
&\equiv 5\frac{\pi}{4} + 5\left(\frac{-\pi}{3}\right) - 4\frac{\pi}{6} \equiv \frac{-13\pi}{12} \equiv \frac{11\pi}{12}.
\end{aligned}$$

Hence $\operatorname{Arg} z = \frac{11\pi}{12}$ and the polar decomposition of z is

$$z = 2^{7/2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right).$$

7. (i) Let $z = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ and $w = 3(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$. (Both of these numbers are already in polar form.)

(a) $zw = 6(\cos(\frac{\pi}{4} + \frac{\pi}{6}) + i\sin(\frac{\pi}{4} + \frac{\pi}{6}))$
$= 6(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$.

(b) $\frac{z}{w} = \frac{2}{3}(\cos(\frac{\pi}{4} - \frac{\pi}{6}) + i\sin(\frac{\pi}{4} - \frac{\pi}{6}))$
$= \frac{2}{3}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$.

(c) $\frac{w}{z} = \frac{3}{2}(\cos(\frac{\pi}{6} - \frac{\pi}{4}) + i\sin(\frac{\pi}{6} - \frac{\pi}{4}))$
$= \frac{3}{2}(\cos(\frac{-\pi}{12}) + i\sin(\frac{-\pi}{12}))$.

(d) $\frac{z^5}{w^2} = \frac{2^5}{3^2}(\cos(\frac{5\pi}{4} - \frac{2\pi}{6}) + i\sin(\frac{5\pi}{4} - \frac{2\pi}{6}))$
$= \frac{32}{9}(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12})$.

(a) $(1+i)^2 = 2i$, so
$$(1+i)^{12} = (2i)^6 = 2^6 i^6 = 64(i^2)^3 = 64(-1)^3 = -64.$$

(b) $(\frac{1-i}{\sqrt{2}})^2 = -i$, so
$$\begin{aligned}\left(\frac{1-i}{\sqrt{2}}\right)^{-6} &= \left(\left(\frac{1-i}{\sqrt{2}}\right)^2\right)^{-3} \\ &= (-i)^{-3} = \frac{-1}{i^3} = \frac{-1}{-i} = \frac{1}{i} = -i.\end{aligned}$$

8. (i) To solve the equation $z^2 = 1+\sqrt{3}i$, we write $1+\sqrt{3}i$ in modulus–argument form:
$$1+\sqrt{3}i = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}).$$
Then the solutions are
$$z_k = \sqrt{2}\left(\cos\left(\frac{\frac{\pi}{3}+2k\pi}{2}\right) + i\sin\left(\frac{\frac{\pi}{3}+2k\pi}{2}\right)\right), \quad k = 0,\ 1.$$
Now $k=0$ gives the solution
$$z_0 = \sqrt{2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \frac{\sqrt{3}}{\sqrt{2}} + \frac{i}{\sqrt{2}}.$$
Clearly $z_1 = -z_0$.

(ii) To solve the equation $z^4 = i$, we write i in modulus–argument form:
$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}.$$
Then the solutions are
$$z_k = \cos\left(\frac{\frac{\pi}{2}+2k\pi}{4}\right) + i\sin\left(\frac{\frac{\pi}{2}+2k\pi}{4}\right), \quad k = 0, 1, 2, 3.$$
Now $\cos\left(\frac{\frac{\pi}{2}+2k\pi}{4}\right) = \cos\left(\frac{\pi}{8} + \frac{k\pi}{2}\right)$, so
$$\begin{aligned} z_k &= \cos\left(\frac{\pi}{8} + \frac{k\pi}{2}\right) + \sin\left(\frac{\pi}{8} + \frac{k\pi}{2}\right) \\ &= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^k (\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}) \\ &= i^k(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}).\end{aligned}$$

Geometrically, the solutions lie equi–spaced on the unit circle at arguments

$$\frac{\pi}{8},\ \frac{\pi}{8}+\frac{\pi}{2}=\frac{5\pi}{8},\ \frac{\pi}{8}+\pi=\frac{9\pi}{8},\ \frac{\pi}{8}+3\frac{\pi}{2}=\frac{13\pi}{8}.$$

Also $z_2 = -z_0$ and $z_3 = -z_1$.

(iii) To solve the equation $z^3 = -8i$, we rewrite the equation as

$$\left(\frac{z}{-2i}\right)^3 = 1.$$

Then

$$\left(\frac{z}{-2i}\right) = 1,\quad \frac{-1+\sqrt{3}i}{2},\quad \text{or}\quad \frac{-1-\sqrt{3}i}{2}.$$

Hence $z = -2i,\ \sqrt{3}+i$ or $-\sqrt{3}+i$.

Geometrically, the solutions lie equi–spaced on the circle $|z| = 2$, at arguments

$$\frac{\pi}{6},\ \frac{\pi}{6}+\frac{2\pi}{3}=\frac{5\pi}{6},\ \frac{\pi}{6}+2\frac{2\pi}{3}=\frac{3\pi}{2}.$$

(iv) To solve $z^4 = 2-2i$, we write $2-2i$ in modulus–argument form:

$$2-2i = 2^{3/2}\left(\cos\frac{-\pi}{4}+i\sin\frac{-\pi}{4}\right).$$

Hence the solutions are

$$z_k = 2^{3/8}\cos\left(\frac{\frac{-\pi}{4}+2k\pi}{4}\right)+i\sin\left(\frac{\frac{-\pi}{4}+2k\pi}{4}\right),\quad k=0,\,1,\,2,\,3.$$

We see the solutions can also be written as

$$\begin{aligned} z_k &= 2^{3/8}i^k\left(\cos\frac{-\pi}{16}+i\sin\frac{-\pi}{16}\right) \\ &= 2^{3/8}i^k\left(\cos\frac{\pi}{16}-i\sin\frac{\pi}{16}\right). \end{aligned}$$

Geometrically, the solutions lie equi–spaced on the circle $|z| = 2^{3/8}$, at arguments

$$\frac{-\pi}{16},\ \frac{-\pi}{16}+\frac{\pi}{2}=\frac{7\pi}{16},\ \frac{-\pi}{16}+2\frac{\pi}{2}=\frac{15\pi}{16},\ \frac{-\pi}{16}+3\frac{\pi}{2}=\frac{23\pi}{16}.$$

Also $z_2 = -z_0$ and $z_3 = -z_1$.

9.

$$\begin{bmatrix} 2+i & -1+2i & 2 \\ 1+i & -1+i & 1 \\ 1+2i & -2+i & 1+i \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{matrix} \begin{bmatrix} 1 & i & 1 \\ 1+i & -1+i & 1 \\ i & -1 & i \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 - (1+i)R_1 \\ R_3 \to R_3 - iR_1 \end{matrix} \begin{bmatrix} 1 & i & 1 \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{bmatrix} \; R_2 \to iR_2 \begin{bmatrix} 1 & i & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to R_1 - R_2 \begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The last matrix is in reduced row–echelon form.

10. (i) Let $p = l + im$ and $z = x + iy$. Then

$$\begin{aligned} \overline{p}z + p\overline{z} &= (l - im)(x + iy) + (l + im)(x - iy) \\ &= (lx + liy - imx + my) + (lx - liy + imx + my) \\ &= 2(lx + my). \end{aligned}$$

Hence $\overline{p}z + p\overline{z} = 2n \Leftrightarrow lx + my = n$.

(ii) Let w be the complex number which results from reflecting the complex number z in the line $lx + my = n$. Then because p is perpendicular to the given line, we have

$$w - z = tp, \quad t \in \mathbb{R}. \qquad (a)$$

Also the midpoint $\frac{w+z}{2}$ of the segment joining w and z lies on the given line, so

$$\begin{aligned} \overline{p}\left(\frac{w+z}{2}\right) + p\left(\overline{\frac{w+z}{2}}\right) &= n, \\ \overline{p}\left(\frac{w+z}{2}\right) + p\left(\frac{\overline{w}+\overline{z}}{2}\right) &= n. \qquad (b) \end{aligned}$$

Taking conjugates of equation (a) gives

$$\overline{w} - \overline{z} = t\overline{p}. \qquad (c)$$

Then substituting in (b), using (a) and (c), gives

$$\overline{p}\left(\frac{2w - tp}{2}\right) + p\left(\frac{2\overline{z} + t\overline{p}}{2}\right) = n$$

and hence

$$\overline{p}w + p\overline{z} = n.$$

(iii) Let $p = b - a$ and $n = |b|^2 - |a|^2$. Then

$$\begin{aligned} |z-a| = |z-b| \;\Leftrightarrow\;& |z-a|^2 = |z-b|^2 \\ \Leftrightarrow (z-a)(\overline{z-a}) &= (z-b)(\overline{z-b}) \\ \Leftrightarrow (z-a)(\overline{z}-\overline{a}) &= (z-b)(\overline{z}-\overline{b}) \\ \Leftrightarrow z\overline{z} - a\overline{z} - z\overline{a} + a\overline{a} &= z\overline{z} - b\overline{z} - z\overline{b} + b\overline{b} \\ \Leftrightarrow (\overline{b}-\overline{a})z + (b-a)\overline{z} &= |b|^2 - |a|^2 \\ \Leftrightarrow \overline{p}z + p\overline{z} &= n. \end{aligned}$$

Suppose z lies on the circle $\left|\frac{z-a}{z-b}\right|$ and let w be the reflection of z in the line $\overline{p}z + p\overline{z} = n$. Then by part (ii)

$$\overline{p}w + p\overline{z} = n.$$

Taking conjugates gives $p\overline{w} + \overline{p}z = n$ and hence

$$z = \frac{n - p\overline{w}}{\overline{p}} \qquad (a)$$

Substituting for z in the circle equation, using (a) gives

$$\lambda = \left|\frac{\frac{n-p\overline{w}}{\overline{p}} - a}{\frac{n-p\overline{w}}{\overline{p}} - b}\right| = \left|\frac{n - p\overline{w} - \overline{p}a}{n - p\overline{w} - \overline{p}b}\right|. \qquad (b)$$

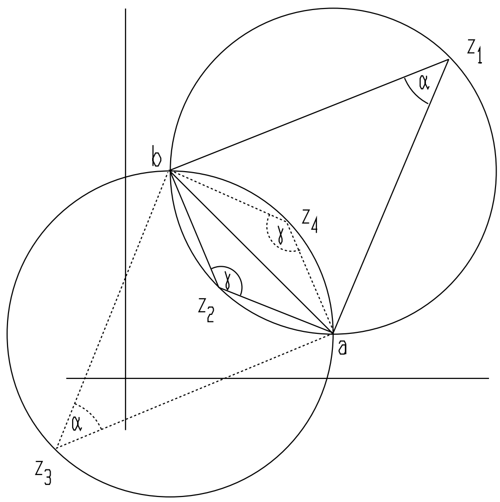
However

$$\begin{aligned} n - \overline{p}a &= |b|^2 - |a|^2 - (\overline{b} - \overline{a})a \\ &= \overline{b}b - \overline{a}a - \overline{b}a + \overline{a}a \\ &= \overline{b}(b-a) = \overline{b}p. \end{aligned}$$

Similarly $n - \overline{p}b = \overline{a}p$. Consequently (b) simplifies to

$$\lambda = \left|\frac{\overline{b}p - p\overline{w}}{\overline{a}p - p\overline{w}}\right| = \left|\frac{\overline{b} - \overline{w}}{\overline{a} - \overline{w}}\right| = \left|\frac{w-b}{w-a}\right|,$$

which gives $\left|\frac{w-a}{w-b}\right| = \frac{1}{\lambda}$.



11. Let a and b be distinct complex numbers and $0 < \alpha < \pi$.

(i) When z_1 lies on the circular arc shown, it subtends a constant angle α. This angle is given by $\operatorname{Arg}(z_1 - a) - \operatorname{Arg}(z_1 - b)$. However

$$\begin{aligned}\operatorname{Arg}\left(\frac{z_1 - a}{z_1 - b}\right) &= \operatorname{Arg}(z_1 - a) - \operatorname{Arg}(z_1 - b) + 2k\pi \\ &= \alpha + 2k\pi.\end{aligned}$$

It follows that $k = 0$, as $0 < \alpha < \pi$ and $-\pi < \operatorname{Arg}\theta \le \pi$. Hence

$$\operatorname{Arg}\left(\frac{z_1 - a}{z_1 - b}\right) = \alpha.$$

Similarly if z_2 lies on the circular arc shown, then

$$\operatorname{Arg}\left(\frac{z_2 - a}{z_2 - b}\right) = -\gamma = -(\pi - \alpha) = \alpha - \pi.$$

Replacing α by $\pi - \alpha$, we deduce that if z_4 lies on the circular arc shown, then

$$\operatorname{Arg}\left(\frac{z_4 - a}{z_4 - b}\right) = \pi - \alpha,$$

while if z_3 lies on the circular arc shown, then

$$\operatorname{Arg}\left(\frac{z_3 - a}{z_3 - b}\right) = -\alpha.$$

The straight line through a and b has the equation

$$z = (1 - t)a + tb,$$

where t is real. Then $0 < t < 1$ describes the segment ab. Also

$$\frac{z-a}{z-b} = \frac{t}{t-1}.$$

Hence $\frac{z-a}{z-b}$ is real and negative if z is on the segment a, but is real and positive if z is on the remaining part of the line, with corresponding values

$$\operatorname{Arg}\left(\frac{z-a}{z-b}\right) = \pi,\ 0,$$

respectively.

(ii) Case (a) Suppose $z_1,\ z_2$ and z_3 are not collinear. Then these points determine a circle. Now z_1 and z_2 partition this circle into two arcs. If z_3 and z_4 lie on the same arc, then

$$\operatorname{Arg}\left(\frac{z_3-z_1}{z_3-z_2}\right) = \operatorname{Arg}\left(\frac{z_4-z_1}{z_4-z_2}\right);$$

whereas if z_3 and z_4 lie on opposite arcs, then

$$\operatorname{Arg}\left(\frac{z_3-z_1}{z_3-z_2}\right) = \alpha$$

and

$$\operatorname{Arg}\left(\frac{z_4-z_1}{z_4-z_2}\right) = \alpha - \pi.$$

Hence in both cases

$$\begin{aligned}\operatorname{Arg}\left(\frac{z_3-z_1}{z_3-z_2}\Big/\frac{z_4-z_1}{z_4-z_2}\right) &\equiv \operatorname{Arg}\left(\frac{z_3-z_1}{z_3-z_2}\right) - \operatorname{Arg}\left(\frac{z_4-z_1}{z_4-z_2}\right) \pmod{2\pi} \\ &\equiv 0 \text{ or } \pi.\end{aligned}$$

In other words, the cross–ratio

$$\frac{z_3-z_1}{z_3-z_2}\Big/\frac{z_4-z_1}{z_4-z_2}$$

is real.

(b) If $z_1,\ z_2$ and z_3 are collinear, then again the cross–ratio is real.

The argument is reversible.

(iii) Assume that A, B, C, D are distinct points such that the cross–ratio

$$r = \frac{z_3-z_1}{z_3-z_2}\Big/\frac{z_4-z_1}{z_4-z_2}$$

is real. Now r cannot be 0 or 1. Then there are three cases:

(i) $0 < r < 1$;

(ii) $r < 0$;

(iii) $r > 1$.

Case (i). Here $|r| + |1 - r| = 1$. So

$$\left|\frac{z_4 - z_1}{z_4 - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}\right| + \left|1 - \left(\frac{z_4 - z_1}{z_4 - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}\right)\right| = 1.$$

Multiplying both sides by the denominator $|z_4 - z_2||z_3 - z_1|$ gives after simplification

$$|z_4 - z_1||z_3 - z_2| + |z_2 - z_1||z_4 - z_3| = |z_4 - z_2||z_3 - z_1|,$$

or

$$\text{(a)} \quad AD \cdot BC + AB \cdot CD = BD \cdot AC.$$

Case (ii). Here $1 + |r| = |1 - r|$. This leads to the equation

$$\text{(b)} \quad BD \cdot AC + AD \cdot BC + = AB \cdot CD.$$

Case (iii). Here $1 + |1 - r| = |r|$. This leads to the equation

$$\text{(c)} \quad BD \cdot AC + AB \cdot CD = AD \cdot BC.$$

Conversely if (a), (b) or (c) hold, then we can reverse the argument to deduce that r is a complex number satisfying one of the equations

$$|r| + |1 - r| = 1, \quad 1 + |r| = |1 - r|, \quad 1 + |1 - r| = |r|,$$

from which we deduce that r is real.