Section 3.6

1. (a) Let S be the set of vectors $[x,\, y]$ satisfying $x = 2y$. Then S is a vector subspace of $\mathbb{R}^2$. For

(i) $[0,\, 0] \in S$ as $x = 2y$ holds with $x = 0$ and $y = 0$.

(ii) S is closed under addition. For let $[x_1,\, y_1]$ and $[x_2,\, y_2]$ belong to S. Then $x_1 = 2y_1$ and $x_2 = 2y_2$. Hence

$$x_1 + x_2 = 2y_1 + 2y_2 = 2(y_1 + y_2)$$

and hence

$$[x_1 + x_2,\, y_1 + y_2] = [x_1,\, y_1] + [x_2,\, y_2]$$

belongs to S.

(iii) S is closed under scalar multiplication. For let $[x,\, y] \in S$ and $t \in \mathbb{R}$. Then $x = 2y$ and hence $tx = 2(ty)$. Consequently

$$[tx,\, ty] = t[x,\, y] \in S.$$

(b) Let S be the set of vectors $[x,\, y]$ satisfying $x = 2y$ and $2x = y$. Then S is a subspace of $\mathbb{R}^2$. This can be proved in the same way as (a), or alternatively we see that $x = 2y$ and $2x = y$ imply $x = 4x$ and hence $x = 0 = y$. Hence $S = \{[0,\, 0]\}$, the set consisting of the zero vector. This is always a subspace.

(c) Let S be the set of vectors $[x,\, y]$ satisfying $x = 2y + 1$. Then S doesn't contain the zero vector and consequently fails to be a vector subspace.

(d) Let S be the set of vectors $[x,\, y]$ satisfying $xy = 0$. Then S is not closed under addition of vectors. For example $[1,\, 0] \in S$ and $[0,\, 1] \in S$, but $[1,\, 0] + [0,\, 1] = [1,\, 1] \notin S$.

(e) Let S be the set of vectors $[x,\, y]$ satisfying $x \geq 0$ and $y \geq 0$. Then S is not closed under scalar multiplication. For example $[1,\, 0] \in S$ and $-1 \in \mathbb{R}$, but $(-1)[1,\, 0] = [-1,\, 0] \notin S$.

2. Let X, Y, Z be vectors in $\mathbb{R}^n$. Then by Lemma 3.2.1

$$\langle X + Y,\, X + Z,\, Y + Z\rangle \subseteq \langle X,\, Y,\, Z\rangle,$$

as each of $X + Y$, $X + Z$, $Y + Z$ is a linear combination of X, Y, Z.

Also

$$\begin{aligned}
X &= \frac{1}{2}(X+Y)+\frac{1}{2}(X+Z)-\frac{1}{2}(Y+Z),\\
Y &= \frac{1}{2}(X+Y)-\frac{1}{2}(X+Z)+\frac{1}{2}(Y+Z),\\
Z &= \frac{-1}{2}(X+Y)+\frac{1}{2}(X+Z)+\frac{1}{2}(Y+Z),
\end{aligned}$$

so

$$\langle X, Y, Z\rangle \subseteq \langle X+Y, X+Z, Y+Z\rangle.$$

Hence

$$\langle X, Y, Z\rangle = \langle X+Y, X+Z, Y+Z\rangle.$$

3. Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$. We have to decide if X_1, X_2, X_3 are linearly independent, that is if the equation $xX_1 + yX_2 + zX_3 = 0$ has only the trivial solution. This equation is equivalent to the folowing homogeneous system

$$\begin{aligned}
x + 0y + z &= 0\\
0x + y + z &= 0\\
x + y + z &= 0\\
2x + 2y + 3z &= 0.
\end{aligned}$$

We reduce the coefficient matrix to reduced row–echelon form:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and consequently the system has only the trivial solution $x = 0$, $y = 0$, $z = 0$. Hence the given vectors are linearly independent.

4. The vectors

$$X_1 = \begin{bmatrix} \lambda \\ -1 \\ -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ \lambda \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -1 \\ -1 \\ \lambda \end{bmatrix}$$

are linearly dependent for precisely those values of λ for which the equation $xX_1 + yX_2 + zX_3 = 0$ has a non–trivial solution. This equation is equivalent to the system of homogeneous equations

$$\begin{aligned} \lambda x - y - z &= 0 \\ -x + \lambda y - z &= 0 \\ -x - y + \lambda z &= 0. \end{aligned}$$

Now the coefficient determinant of this system is

$$\begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} = (\lambda+1)^2(\lambda-2).$$

So the values of λ which make X_1, X_2, X_3 linearly independent are those λ satisfying $\lambda \neq -1$ and $\lambda \neq 2$.

5. Let A be the following matrix of rationals:

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 5 & 0 & 3 \\ 0 & 0 & 0 & 1 & 3 \\ 8 & 11 & 19 & 0 & 11 \end{bmatrix}.$$

Then A has reduced row–echelon form

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

From B we read off the following:

(a) The rows of B form a basis for $R(A)$. (Consequently the rows of A also form a basis for $R(A)$.)

(b) The first four columns of A form a basis for $C(A)$.

(c) To find a basis for $N(A)$, we solve $AX = 0$ and equivalently $BX = 0$. From B we see that the solution is

$$\begin{aligned} x_1 &= x_5 \\ x_2 &= 0 \\ x_3 &= -x_5 \\ x_4 &= -3x_5, \end{aligned}$$

with x_5 arbitrary. Then

$$X = \begin{bmatrix} x_5 \\ 0 \\ -x_5 \\ -3x_5 \\ x_5 \end{bmatrix} = x_5 \begin{bmatrix} 1 \\ 0 \\ -1 \\ -3 \\ 1 \end{bmatrix},$$

so $[1, 0, -1, -3, 1]^t$ is a basis for $N(A)$.

6. In Section 1.6, problem 12, we found that the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

has reduced row–echelon form

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From B we read off the following:

(a) The three non–zero rows of B form a basis for $R(A)$.

(b) The first three columns of A form a basis for $C(A)$.

(c) To find a basis for $N(A)$, we solve $AX = 0$ and equivalently $BX = 0$. From B we see that the solution is

$$\begin{aligned} x_1 &= -x_4 - x_5 = x_4 + x_5 \\ x_2 &= -x_4 - x_5 = x_4 + x_5 \\ x_3 &= -x_4 = x_4, \end{aligned}$$

with x_4 and x_5 arbitrary elements of $\mathbb{Z}_2$. Hence

$$X = \begin{bmatrix} x_4 + x_5 \\ x_4 + x_5 \\ x_4 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence $[1, 1, 1, 1, 0]^t$ and $[1, 1, 0, 0, 1]^t$ form a basis for $N(A)$.

7. Let A be the following matrix over $\mathbb{Z}_5$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 3 \\ 2 & 1 & 4 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 & 3 & 2 \end{bmatrix}.$$

We find that A has reduced row–echelon form B:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 & 4 & 4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{bmatrix}.$$

From B we read off the following:

(a) The four rows of B form a basis for $R(A)$. (Consequently the rows of A also form a basis for $R(A)$.

(b) The first four columns of A form a basis for $C(A)$.

(c) To find a basis for $N(A)$, we solve $AX = 0$ and equivalently $BX = 0$. From B we see that the solution is

$$\begin{aligned} x_1 &= -2x_5 - 4x_6 = 3x_5 + x_6 \\ x_2 &= -4x_5 - 4x_6 = x_5 + x_6 \\ x_3 &= 0 \\ x_4 &= -3x_5 = 2x_5, \end{aligned}$$

where x_5 and x_6 are arbitrary elements of $\mathbb{Z}_5$. Hence

$$X = x_5 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so $[3, 1, 0, 2, 1, 0]^t$ and $[1, 1, 0, 0, 0, 1]^t$ form a basis for $R(A)$.

8. Let $F = \{0, 1, a, b\}$ be a field and let A be the following matrix over F:

$$A = \begin{bmatrix} 1 & a & b & a \\ a & b & b & 1 \\ 1 & 1 & 1 & a \end{bmatrix}.$$

In Section 1.6, problem 17, we found that A had reduced row–echelon form

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

From B we read off the following:

(a) The rows of B form a basis for $R(A)$. (Consequently the rows of A also form a basis for $R(A)$.

(b) The first three columns of A form a basis for $C(A)$.

(c) To find a basis for $N(A)$, we solve $AX = 0$ and equivalently $BX = 0$. From B we see that the solution is

$$\begin{aligned} x_1 &= 0 \\ x_2 &= -bx_4 = bx_4 \\ x_3 &= -x_4 = x_4, \end{aligned}$$

where x_4 is an arbitrary element of F. Hence

$$X = x_4 \begin{bmatrix} 0 \\ b \\ 1 \\ 1 \end{bmatrix},$$

so $[0, b, 1, 1]^t$ is a basis for $N(A)$.

9. Suppose that $X_1, \ldots, X_m$ form a basis for a subspace S. We have to prove that

$$X_1,\, X_1 + X_2, \ldots, X_1 + \cdots + X_m$$

also form a basis for S.

First we prove the independence of the family: Suppose

$$x_1X_1 + x_2(X_1 + X_2) + \cdots + x_m(X_1 + \cdots + X_m) = 0.$$

Then

$$(x_1 + x_2 + \cdots + x_m)X_1 + \cdots + x_m X_m = 0.$$

Then the linear independence of $X_1, \ldots, X_m$ gives

$$x_1 + x_2 + \cdots + x_m = 0, \ldots, x_m = 0,$$

form which we deduce that $x_1 = 0, \ldots, x_m = 0$.

Secondly we have to prove that every vector of S is expressible as a linear combination of $X_1,\ X_1 + X_2, \ldots, X_1 + \cdots + X_m$. Suppose $X \in S$. Then

$$X = a_1 X_1 + \cdots + a_m X_m.$$

We have to find $x_1, \ldots, x_m$ such that

$$\begin{aligned} X &= x_1 X_1 + x_2(X_1 + X_2) + \cdots + x_m(X_1 + \cdots + X_m) \\ &= (x_1 + x_2 + \cdots + x_m)X_1 + \cdots + x_m X_m. \end{aligned}$$

Then

$$a_1 X_1 + \cdots + a_m X_m = (x_1 + x_2 + \cdots + x_m)X_1 + \cdots + x_m X_m.$$

So if we can solve the system

$$x_1 + x_2 + \cdots + x_m = a_1, \ldots, x_m = a_m,$$

we are finished. Clearly these equations have the unique solution

$$x_1 = a_1 - a_2, \ldots, x_{m-1} = a_m - a_{m-1},\ x_m = a_m.$$

10. Let $A = \begin{bmatrix} a & b & c \\ 1 & 1 & 1 \end{bmatrix}$. If $[a,\ b,\ c]$ is a multiple of $[1,\ 1,\ 1]$, (that is, $a = b = c$), then rank $A = 1$. For if

$$[a,\ b,\ c] = t[1,\ 1,\ 1],$$

then

$$R(A) = \langle [a,\ b,\ c],\ [1,\ 1,\ 1] \rangle = \langle t[1,\ 1,\ 1],\ [1,\ 1,\ 1] \rangle = \langle [1,\ 1,\ 1] \rangle,$$

so $[1,\ 1,\ 1]$ is a basis for $R(A)$.

However if $[a,\ b,\ c]$ is not a multiple of $[1,\ 1,\ 1]$, (that is at least two of $a,\ b,\ c$ are distinct), then the left–to–right test shows that $[a,\ b,\ c]$ and

$[1,\ 1,\ 1]$ are linearly independent and hence form a basis for $R(A)$. Consequently $\operatorname{rank} A = 2$ in this case.

11. Let S be a subspace of F^n with $\dim S = m$. Also suppose that $X_1, \ldots, X_m$ are vectors in S such that $S = \langle X_1, \ldots, X_m \rangle$. We have to prove that $X_1, \ldots, X_m$ form a basis for S; in other words, we must prove that $X_1, \ldots, X_m$ are linearly independent.

However if $X_1, \ldots, X_m$ were linearly dependent, then one of these vectors would be a linear combination of the remaining vectors. Consequently S would be spanned by $m-1$ vectors. But there exist a family of m linearly independent vectors in S. Then by Theorem 3.3.2, we would have the contradiction $m \leq m-1$.

12. Let $[x,\ y,\ z]^t \in S$. Then $x + 2y + 3z = 0$. Hence $x = -2y - 3z$ and

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y-3z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Hence $[-2,\ 1,\ 0]^t$ and $[-3,\ 0,\ 1]^t$ form a basis for S.

Next $(-1) + 2(-1) + 3(1) = 0$, so $[-1,\ -1,\ 1]^t \in S$.

To find a basis for S which includes $[-1,\ -1,\ 1]^t$, we note that $[-2,\ 1,\ 0]^t$ is not a multiple of $[-1,\ -1,\ 1]^t$. Hence we have found a linearly independent family of two vectors in S, a subspace of dimension equal to 2. Consequently these two vectors form a basis for S.

13. Without loss of generality, suppose that $X_1 = X_2$. Then we have the non–trivial dependency relation:

$$1X_1 + (-1)X_2 + 0X_3 + \cdots + 0X_m = 0.$$

14. (a) Suppose that X_{m+1} is a linear combination of $X_1, \ldots, X_m$. Then

$$\langle X_1, \ldots, X_m,\ X_{m+1} \rangle = \langle X_1, \ldots, X_m \rangle$$

and hence

$$\dim \langle X_1, \ldots, X_m,\ X_{m+1} \rangle = \dim \langle X_1, \ldots, X_m \rangle.$$

(b) Suppose that X_{m+1} is not a linear combination of $X_1, \ldots, X_m$. If not all of $X_1, \ldots, X_m$ are zero, there will be a subfamily $X_{c_1}, \ldots, X_{c_r}$ which is a basis for $\langle X_1, \ldots, X_m \rangle$.

Then as X_{m+1} is not a linear combination of $X_{c_1}, \ldots, X_{c_r}$, it follows that $X_{c_1}, \ldots, X_{c_r}, X_{m+1}$ are linearly independent. Also

$$\langle X_1, \ldots, X_m, X_{m+1}\rangle = \langle X_{c_1}, \ldots, X_{c_r}, X_{m+1}\rangle.$$

Consequently

$$\dim \langle X_1, \ldots, X_m, X_{m+1}\rangle = r + 1 = \dim \langle X_1, \ldots, X_m\rangle + 1.$$

Our result can be rephrased in a form suitable for the second part of the problem:

$$\dim \langle X_1, \ldots, X_m, X_{m+1}\rangle = \dim \langle X_1, \ldots, X_m\rangle$$

if and only if X_{m+1} is a linear combination of $X_1, \ldots, X_m$.

If $X = [x_1, \ldots, x_n]^t$, then $AX = B$ is equivalent to

$$B = x_1 A_{*1} + \cdots + x_n A_{*n}.$$

So $AX = B$ is soluble for X if and only if B is a linear combination of the columns of A, that is $B \in C(A)$. However by the first part of this question, $B \in C(A)$ if and only if $\dim C([A|B]) = \dim C(A)$, that is, $\operatorname{rank}[A|B] = \operatorname{rank} A$.

15. Let $a_1, \ldots, a_n$ be elements of F, not all zero. Let S denote the set of vectors $[x_1, \ldots, x_n]^t$, where $x_1, \ldots, x_n$ satisfy

$$a_1 x_1 + \cdots + a_n x_n = 0.$$

Then $S = N(A)$, where A is the row matrix $[a_1, \ldots, a_n]$. Now $\operatorname{rank} A = 1$ as $A \neq 0$. So by the "rank + nullity" theorem, noting that the number of columns of A equals n, we have

$$\dim N(A) = \text{nullity}\,(A) = n - \operatorname{rank} A = n - 1.$$

16. (a) (Proof of Lemma 3.2.1) Suppose that each of $X_1, \ldots, X_r$ is a linear combination of $Y_1, \ldots, Y_s$. Then

$$X_i = \sum_{j=1}^{s} a_{ij} Y_j, \quad (1 \leq i \leq r).$$

Now let $X = \sum_{i=1}^{r} x_i X_i$ be a linear combination of $X_1, \ldots, X_r$. Then

$$\begin{aligned} X &= x_1(a_{11}Y_1 + \cdots + a_{1s}Y_s) \\ &+ \cdots \\ &+ x_r(a_{r1}Y_1 + \cdots + a_{rs}Y_s) \\ &= y_1Y_1 + \cdots + y_sY_s, \end{aligned}$$

where $y_j = a_{1j}x_1 + \cdots + a_{rj}x_r$. Hence X is a linear combination of $Y_1, \ldots, Y_s$.

Another way of stating Lemma 3.2.1 is

$$\langle X_1, \ldots, X_r \rangle \subseteq \langle Y_1, \ldots, Y_s \rangle, \quad (1)$$

if each of $X_1, \ldots, X_r$ is a linear combination of $Y_1, \ldots, Y_s$.

(b) (Proof of Theorem 3.2.1) Suppose that each of $X_1, \ldots, X_r$ is a linear combination of $Y_1, \ldots, Y_s$ and that each of $Y_1, \ldots, Y_s$ is a linear combination of $X_1, \ldots, X_r$. Then by (a) equation (1) above

$$\langle X_1, \ldots, X_r \rangle \subseteq \langle Y_1, \ldots, Y_s \rangle$$

and

$$\langle Y_1, \ldots, Y_s \rangle \subseteq \langle X_1, \ldots, X_r \rangle.$$

Hence

$$\langle X_1, \ldots, X_r \rangle = \langle Y_1, \ldots, Y_s \rangle.$$

(c) (Proof of Corollary 3.2.1) Suppose that each of $Z_1, \ldots, Z_t$ is a linear combination of $X_1, \ldots, X_r$. Then each of $X_1, \ldots, X_r$, $Z_1, \ldots, Z_t$ is a linear combination of $X_1, \ldots, X_r$.

Also each of $X_1, \ldots, X_r$ is a linear combination of $X_1, \ldots, X_r$, $Z_1, \ldots, Z_t$, so by Theorem 3.2.1

$$\langle X_1, \ldots, X_r,\ Z_1, \ldots, Z_t \rangle = \langle X_1, \ldots, X_r \rangle.$$

(d) (Proof of Theorem 3.3.2) Let $Y_1, \ldots, Y_s$ be vectors in $\langle X_1, \ldots, X_r \rangle$ and assume that $s > r$. We have to prove that $Y_1, \ldots, Y_s$ are linearly dependent. So we consider the equation

$$x_1Y_1 + \cdots + x_sY_s = 0.$$

Now $Y_i = \sum_{j=1}^{r} a_{ij}X_j$, for $1 \le i \le s$. Hence

$$\begin{aligned} x_1Y_1 + \cdots + x_sY_s &= x_1(a_{11}X_1 + \cdots + a_{1r}X_r) \\ &+ \cdots \\ &+ x_r(a_{s1}X_1 + \cdots + a_{sr}X_r). \\ &= y_1X_1 + \cdots + y_rX_r, \quad (1) \end{aligned}$$

where $y_j = a_{1j}x_1 + \cdots + a_{sj}x_s$. However the homogeneous system

$$y_1 = 0, \ \cdots, \ y_r = 0$$

has a non–trivial solution $x_1, \ldots, x_s$, as $s > r$ and from (1), this results in a non–trivial solution of the equation

$$x_1Y_1 + \cdots + x_sY_s = 0.$$

Hence $Y_1, \ldots, Y_s$ are linearly dependent.

17. Let R and S be subspaces of F^n, with $R \subseteq S$. We first prove

$$\dim R \le \dim S.$$

Let $X_1, \ldots, X_r$ be a basis for R. Now by Theorem 3.5.2, because $X_1, \ldots, X_r$ form a linearly independent family lying in S, this family can be extended to a basis $X_1, \ldots, X_r, \ldots, X_s$ for S. Then

$$\dim S = s \ge r = \dim R.$$

Next suppose that $\dim R = \dim S$. Let $X_1, \ldots, X_r$ be a basis for R. Then because $X_1, \ldots, X_r$ form a linearly independent family in S and S is a subspace whose dimension is r, it follows from Theorem 3.4.3 that $X_1, \ldots, X_r$ form a basis for S. Then

$$S = \langle X_1, \ldots, X_r \rangle = R.$$

18. Suppose that R and S are subspaces of F^n with the property that $R \cup S$ is also a subspace of F^n. We have to prove that $R \subseteq S$ or $S \subseteq R$. We argue by contradiction: Suppose that $R \not\subseteq S$ and $S \not\subseteq R$. Then there exist vectors u and v such that

$$u \in R \text{ and } v \notin S, \quad v \in S \text{ and } v \notin R.$$

Consider the vector $u+v$. As we are assuming $R \cup S$ is a subspace, $R \cup S$ is closed under addition. Hence $u+v \in R \cup S$ and so $u+v \in R$ or $u+v \in S$. However if $u+v \in R$, then $v = (u+v) - u \in R$, which is a contradiction; similarly if $u+v \in S$.

Hence we have derived a contradiction on the asumption that $R \not\subseteq S$ and $S \not\subseteq R$. Consequently at least one of these must be false. In other words $R \subseteq S$ or $S \subseteq R$.

19. Let $X_1, \ldots, X_r$ be a basis for S.

(i) First let

$$\begin{aligned} Y_1 &= a_{11}X_1 + \cdots + a_{1r}X_r \\ &\vdots \\ Y_r &= a_{r1}X_1 + \cdots + a_{rr}X_r, \end{aligned} \tag{2}$$

where $A = [a_{ij}]$ is non–singular. Then the above system of equations can be solved for $X_1, \ldots, X_r$ in terms of $Y_1, \ldots, Y_r$. Consequently by Theorem 3.2.1

$$\langle Y_1, \ldots, Y_r \rangle = \langle X_1, \ldots, X_r \rangle = S.$$

It follows from problem 11 that $Y_1, \ldots, Y_r$ is a basis for S.

(ii) We show that *all* bases for S are given by equations 2. So suppose that $Y_1, \ldots, Y_r$ forms a basis for S. Then because $X_1, \ldots, X_r$ form a basis for S, we can express $Y_1, \ldots, Y_r$ in terms of $X_1, \ldots, X_r$ as in 2, for some matrix $A = [a_{ij}]$. We show A is non–singular by demonstrating that the linear independence of $Y_1, \ldots, Y_r$ implies that the rows of A are linearly independent.

So assume

$$x_1[a_{11}, \ldots, a_{1r}] + \cdots + x_r[a_{r1}, \ldots, a_{rr}] = [0, \ldots, 0].$$

Then on equating components, we have

$$\begin{aligned} a_{11}x_1 + \cdots + a_{r1}x_r &= 0 \\ &\vdots \\ a_{1r}x_1 + \cdots + a_{rr}x_r &= 0. \end{aligned}$$

Hence

$$\begin{aligned} x_1Y_1 + \cdots + x_rY_r &= x_1(a_{11}X_1 + \cdots + a_{1r}X_r) + \cdots + x_r(a_{r1}X_1 + \cdots + a_{rr}X_r) \\ &= (a_{11}x_1 + \cdots + a_{r1}x_r)X_1 + \cdots + (a_{1r}x_1 + \cdots + a_{rr}x_r)X_r \\ &= 0X_1 + \cdots + 0X_r = 0. \end{aligned}$$

Then the linear independence of $Y_1, \ldots, Y_r$ implies $x_1 = 0, \ldots, x_r = 0$.

(We mention that the last argument is reversible and provides an alternative proof of part (i).)