Section 2.7

1. $[A|I_2] = \left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{array}\right]$ $R_2 \to R_2 + 3R_1$ $\left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 0 & 13 & 3 & 1 \end{array}\right]$

$R_2 \to \frac{1}{13}R_2$ $\left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 0 & 1 & 3/13 & 1/13 \end{array}\right]$ $R_1 \to R_1 - 4R_2$ $\left[\begin{array}{rr|rr} 1 & 0 & 1/13 & -4/13 \\ 0 & 1 & 3/13 & 1/13 \end{array}\right]$.

Hence A is non–singular and $A^{-1} = \begin{bmatrix} 1/13 & -4/13 \\ 3/13 & 1/13 \end{bmatrix}$.

Moreover

$$E_{12}(-4)E_2(1/13)E_{21}(3)A = I_2,$$

so

$$A^{-1} = E_{12}(-4)E_2(1/13)E_{21}(3).$$

Hence

$$A = \{E_{21}(3)\}^{-1}\{E_2(1/13)\}^{-1}\{E_{12}(-4)\}^{-1} = E_{21}(-3)E_2(13)E_{12}(4).$$

2. Let $D = [d_{ij}]$ be an $m \times m$ diagonal matrix and let $A = [a_{jk}]$ be an $m \times n$ matrix. Then

$$(DA)_{ik} = \sum_{j=1}^{n} d_{ij}a_{jk} = d_{ii}a_{ik},$$

as $d_{ij} = 0$ if $i \neq j$. It follows that the ith row of DA is obtained by multiplying the ith row of A by d_{ii}.

Similarly, post–multiplication of a matrix by a diagonal matrix D results in a matrix whose columns are those of A, multiplied by the respective diagonal elements of D.

In particular,

$$\mathrm{diag}\,(a_1, \ldots, a_n)\mathrm{diag}\,(b_1, \ldots, b_n) = \mathrm{diag}\,(a_1b_1, \ldots, a_nb_n),$$

as the left–hand side can be regarded as pre–multiplication of the matrix $\mathrm{diag}\,(b_1, \ldots, b_n)$ by the diagonal matrix $\mathrm{diag}\,(a_1, \ldots, a_n)$.

Finally, suppose that each of $a_1, \ldots, a_n$ is non–zero. Then $a_1^{-1}, \ldots, a_n^{-1}$ all exist and we have

$$\begin{aligned} \mathrm{diag}\,(a_1, \ldots, a_n)\mathrm{diag}\,(a_1^{-1}, \ldots, a_n^{-1}) &= \mathrm{diag}\,(a_1a_1^{-1}, \ldots, a_na_n^{-1}) \\ &= \mathrm{diag}\,(1, \ldots, 1) = I_n. \end{aligned}$$

Hence diag $(a_1, \ldots, a_n)$ is non–singular and its inverse is diag $(a_1^{-1}, \ldots, a_n^{-1})$.

Next suppose that $a_i = 0$. Then diag $(a_1, \ldots, a_n)$ is row–equivalent to a matix containing a zero row and is hence singular.

3. $[A|I_3] = \left[\begin{array}{ccc|ccc} 0 & 0 & 2 & 1 & 0 & 0 \\ 1 & 2 & 6 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{cccccc} 1 & 2 & 6 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right]$

$$R_3 \to R_3 - 3R_1 \left[\begin{array}{cccccc} 1 & 2 & 6 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -9 & 0 & -3 & 1 \end{array}\right] \quad R_2 \leftrightarrow R_3 \left[\begin{array}{cccccc} 1 & 2 & 6 & 0 & 1 & 0 \\ 0 & 1 & -9 & 0 & -3 & 1 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{array}\right]$$

$$R_3 \to \tfrac{1}{2}R_3 \left[\begin{array}{cccccc} 1 & 2 & 6 & 0 & 1 & 0 \\ 0 & 1 & -9 & 0 & -3 & 1 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 - 2R_2 \left[\begin{array}{cccccc} 1 & 0 & 24 & 0 & 7 & -2 \\ 0 & 1 & -9 & 0 & -3 & 1 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 \to R_1 - 24R_3 \\ R_2 \to R_2 + 9R_3 \end{array} \left[\begin{array}{cccccc} 1 & 0 & 0 & -12 & 7 & -2 \\ 0 & 1 & 0 & 9/2 & -3 & 1 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \end{array}\right].$$

Hence A is non–singular and $A^{-1} = \left[\begin{array}{ccc} -12 & 7 & -2 \\ 9/2 & -3 & 1 \\ 1/2 & 0 & 0 \end{array}\right]$.

Also

$$E_{23}(9)E_{13}(-24)E_{12}(-2)E_3(1/2)E_{23}E_{31}(-3)E_{12}A = I_3.$$

Hence

$$A^{-1} = E_{23}(9)E_{13}(-24)E_{12}(-2)E_3(1/2)E_{23}E_{31}(-3)E_{12},$$

so

$$A = E_{12}E_{31}(3)E_{23}E_3(2)E_{12}(2)E_{13}(24)E_{23}(-9).$$

4.

$$A = \left[\begin{array}{ccc} 1 & 2 & k \\ 3 & -1 & 1 \\ 5 & 3 & -5 \end{array}\right] \to \left[\begin{array}{ccc} 1 & 2 & k \\ 0 & -7 & 1-3k \\ 0 & -7 & -5-5k \end{array}\right] \to \left[\begin{array}{ccc} 1 & 2 & k \\ 0 & -7 & 1-3k \\ 0 & 0 & -6-2k \end{array}\right] = B.$$

Hence if $-6 - 2k \neq 0$, i.e. if $k \neq -3$, we see that B can be reduced to I_3 and hence A is non–singular.

If $k = -3$, then $B = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -7 & 10 \\ 0 & 0 & 0 \end{bmatrix} = B$ and consequently A is singular, as it is row–equivalent to a matrix containing a zero row.

5. $E_{21}(2)\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Hence, as in the previous question, $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$ is singular.

6. Starting from the equation $A^2 - 2A + 13I_2 = 0$, we deduce

$$A(A - 2I_2) = -13I_2 = (A - 2I_2)A.$$

Hence $AB = BA = I_2$, where $B = \frac{-1}{13}(A - 2I_2)$. Consequently A is non–singular and $A^{-1} = B$.

7. We assume the equation $A^3 = 3A^2 - 3A + I_3$.

$$\text{(ii)} \quad \begin{aligned} A^4 &= A^3A = (3A^2 - 3A + I_3)A = 3A^3 - 3A^2 + A \\ &= 3(3A^2 - 3A + I_3) - 3A^2 + A = 6A^2 - 8A + 3I_3. \end{aligned}$$

(iii) $A^3 - 3A^2 + 3A = I_3$. Hence

$$A(A^2 - 3A + 3I_3) = I_3 = (A^2 - 3A + 3I_3)A.$$

Hence A is non–singular and

$$\begin{aligned} A^{-1} &= A^2 - 3A + 3I_3 \\ &= \begin{bmatrix} -1 & -3 & 1 \\ 2 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

8. (i) If $B^3 = 0$ then

$$\begin{aligned} (I_n - B)(I_n + B + B^2) &= I_n(I_n + B + B^2) - B(I_n + B + B^2) \\ &= (I_n + B + B^2) - (B + B^2 + B^3) \\ &= I_n - B^3 = I_n - 0 = I_n. \end{aligned}$$

Similarly $(I_n + B + B^2)(I_n - B) = I_n$.

Hence $A = I_n - B$ is non–singular and $A^{-1} = I_n + B + B^2$.

It follows that the system $AX = b$ has the unique solution

$$X = A^{-1}b = (I_n + B + B^2)b = b + Bb + B^2b.$$

(ii) Let $B = \begin{bmatrix} 0 & r & s \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}$. Then $B^2 = \begin{bmatrix} 0 & 0 & rt \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B^3 = 0$. Hence from the preceding question

$$\begin{aligned} (I_3 - B)^{-1} &= I_3 + B + B^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & r & s \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & rt \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & r & s + rt \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

9. (i) Suppose that $A^2 = 0$. Then if A^{-1} exists, we deduce that $A^{-1}(AA) = A^{-1}0$, which gives $A = 0$ and this is a contradiction, as the zero matrix is singular. We conclude that A does not have an inverse.

(ii). Suppose that $A^2 = A$ and that A^{-1} exists. Then

$$A^{-1}(AA) = A^{-1}A,$$

which gives $A = I_n$. Equivalently, if $A^2 = A$ and $A \neq I_n$, then A does not have an inverse.

10. The system of linear equations

$$\begin{aligned} x + y - z &= a \\ z &= b \\ 2x + y + 2z &= c \end{aligned}$$

is equivalent to the matrix equation $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

By Question 7, A^{-1} exists and hence the system has the unique solution

$$X = \begin{bmatrix} -1 & -3 & 1 \\ 2 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a - 3b + c \\ 2a + 4b - c \\ b \end{bmatrix}.$$

Hence $x = -a - 3b + c, \ y = 2a + 4b - c, \ z = b$.

12.

$$\begin{aligned} A &= E_3(2)E_{14}E_{42}(3) = E_3(2)E_{14} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix} \\ &= E_3(2) \begin{bmatrix} 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Also

$$\begin{aligned} A^{-1} &= (E_3(2)E_{14}E_{42}(3))^{-1} \\ &= (E_{42}(3))^{-1}E_{14}^{-1}(E_3(2))^{-1} \\ &= E_{42}(-3)E_{14}E_3(1/2) \\ &= E_{42}(-3)E_{14} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= E_{42}(-3) \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 1 & -3 & 0 & 0 \end{bmatrix}. \end{aligned}$$

13. (All matrices in this question are over $\mathbb{Z}_2$.)

(a)
$$\left[\begin{array}{cccc|cccc} 1&1&0&1&1&0&0&0\\ 0&0&1&1&0&1&0&0\\ 1&1&1&1&0&0&1&0\\ 1&0&0&1&0&0&0&1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1&1&0&1&1&0&0&0\\ 0&0&1&1&0&1&0&0\\ 0&0&1&0&1&0&1&0\\ 0&1&0&0&1&0&0&1 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{cccc|cccc} 1&1&0&1&1&0&0&0\\ 0&1&0&0&1&0&0&1\\ 0&0&1&0&1&0&1&0\\ 0&0&1&1&0&1&0&0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1&0&0&1&0&0&0&1\\ 0&1&0&0&1&0&0&1\\ 0&0&1&0&1&0&1&0\\ 0&0&0&1&1&1&1&0 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{cccc|cccc} 1&0&0&0&1&1&1&1\\ 0&1&0&0&1&0&0&1\\ 0&0&1&0&1&0&1&0\\ 0&0&0&1&1&1&1&0 \end{array}\right].$$

Hence A is non–singular and
$$A^{-1} = \begin{bmatrix} 1&1&1&1\\ 1&0&0&1\\ 1&0&1&0\\ 1&1&1&0 \end{bmatrix}.$$

(b) $A = \begin{bmatrix} 1&1&0&1\\ 0&1&1&1\\ 1&0&1&0\\ 1&1&0&1 \end{bmatrix}$ $\quad R_4 \rightarrow R_4 + R_1 \quad$ $\begin{bmatrix} 1&1&0&1\\ 0&1&1&1\\ 1&0&1&0\\ 0&0&0&0 \end{bmatrix}$, so A is singular.

14.

(a)
$$\left[\begin{array}{ccc|ccc} 1&1&1&1&0&0\\ -1&1&0&0&1&0\\ 2&0&0&0&0&1 \end{array}\right] \begin{array}{c} R_3 \rightarrow \frac{1}{2}R_3\\ R_1 \rightarrow R_1 - R_3\\ R_2 \rightarrow R_2 + R_3\\ R_1 \leftrightarrow R_3 \end{array} \left[\begin{array}{ccc|ccc} 1&0&0&0&0&1/2\\ 0&1&0&0&1&1/2\\ 0&1&1&1&0&-1/2 \end{array}\right]$$
$$R_3 \rightarrow R_3 - R_2 \quad \left[\begin{array}{ccc|ccc} 1&0&0&0&0&1/2\\ 0&1&0&0&1&1/2\\ 0&0&1&1&-1&-1 \end{array}\right].$$

Hence A^{-1} exists and
$$A^{-1} = \begin{bmatrix} 0&0&1/2\\ 0&1&1/2\\ 1&-1&-1 \end{bmatrix}.$$

(b) $\left[\begin{array}{ccc|ccc} 2 & 2 & 4 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 \to R_1 - 2R_2 \\ R_1 \leftrightarrow R_2 \\ R_2 \leftrightarrow R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 1 & -2 & 0 \end{array}\right]$

$$R_3 \to R_3 - 2R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & -2 & -2 \end{array}\right]$$

$$R_3 \to \tfrac{1}{2}R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & -1 & -1 \end{array}\right]$$

$$R_1 \to R_1 - R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & -1 & -1 \end{array}\right].$$

Hence A^{-1} exists and

$$A^{-1} = \begin{bmatrix} -1/2 & 2 & 1 \\ 0 & 0 & 1 \\ 1/2 & -1 & -1 \end{bmatrix}.$$

(c) $\begin{bmatrix} 4 & 6 & -3 \\ 0 & 0 & 7 \\ 0 & 0 & 5 \end{bmatrix} \begin{array}{c} R_2 \to \frac{1}{7}R_2 \\ R_3 \to \frac{1}{5}R_3 \end{array} \begin{bmatrix} 4 & 6 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} R_3 \to R_3 - R_2 \begin{bmatrix} 4 & 6 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$

Hence A is singular by virtue of the zero row.

(d) $\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 \to \frac{1}{2}R_1 \\ R_2 \to \frac{-1}{5}R_2 \\ R_3 \to \frac{1}{7}R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/7 \end{array}\right].$

Hence A^{-1} exists and $A^{-1} = \mathrm{diag}\,(1/2, -1/5, 1/7)$.

(Of course this was also immediate from Question 2.)

(e) $\left[\begin{array}{cccc|cccc} 1 & 2 & 4 & 6 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right] R_1 \to R_1 - 2R_2 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 6 & 1 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$

$$R_2 \to R_2 - 2R_3 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 6 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1 \to R_1 - 3R_4 \\ R_2 \to R_2 + 2R_4 \\ R_3 \to R_3 - R_4 \\ R_4 \to \frac{1}{2}R_4 \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 \end{array}\right].$$

Hence A^{-1} exists and

$$A^{-1} = \left[\begin{array}{rrrr} 1 & -2 & 0 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1/2 \end{array}\right].$$

(f)

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{array}\right] \begin{array}{c} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 5R_1 \end{array} \left[\begin{array}{rrr} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -3 & -6 \end{array}\right] \; R_3 \to R_3 - R_2 \; \left[\begin{array}{rrr} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{array}\right].$$

Hence A is singular by virtue of the zero row.

15. Suppose that A is non–singular. Then

$$AA^{-1} = I_n = A^{-1}A.$$

Taking transposes throughout gives

$$\begin{aligned} (AA^{-1})^t &= I_n^t = (A^{-1}A)^t \\ (A^{-1})^t A^t &= I_n = A^t(A^{-1})^t, \end{aligned}$$

so A^t is non–singular and $(A^t)^{-1} = (A^{-1})^t$.

16. Let $A = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$, where $ad - bc = 0$. Then the equation

$$A^2 - (a+d)A + (ad-bc)I_2 = 0$$

reduces to $A^2 - (a+d)A = 0$ and hence $A^2 = (a+d)A$. From the last equation, if A^{-1} exists, we deduce that $A = (a+d)I_2$, or

$$\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = \left[\begin{array}{cc} a+d & 0 \\ 0 & a+d \end{array}\right].$$

Hence $a = a+d$, $b = 0$, $c = 0$, $d = a+d$ and $a = b = c = d = 0$, which contradicts the assumption that A is non–singular.

17.

$$A = \begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix} \qquad \begin{array}{l} R_2 \to R_2 + aR_1 \\ R_3 \to R_3 + bR_1 \end{array} \begin{bmatrix} 1 & a & b \\ 0 & 1+a^2 & c+ab \\ 0 & ab-c & 1+b^2 \end{bmatrix}$$

$$R_2 \to \tfrac{1}{1+a^2} R_2 \begin{bmatrix} 1 & a & b \\ 0 & 1 & \frac{c+ab}{1+a^2} \\ 0 & ab-c & 1+b^2 \end{bmatrix}$$

$$R_3 \to R_3 - (ab-c)R_2 \begin{bmatrix} 1 & a & b \\ 0 & 1 & \frac{c+ab}{1+a^2} \\ 0 & 0 & 1+b^2+\frac{(c-ab)(c+ab)}{1+a^2} \end{bmatrix} = B.$$

Now

$$\begin{aligned} 1+b^2+\frac{(c-ab)(c+ab)}{1+a^2} &= 1+b^2+\frac{c^2-(ab)^2}{1+a^2} \\ &= \frac{1+a^2+b^2+c^2}{1+a^2} \neq 0. \end{aligned}$$

Hence B can be reduced to I_3 using four more row operations and consequently A is non–singular.

18. The proposition is clearly true when $n = 1$. So let $n \geq 1$ and assume $(P^{-1}AP)^n = P^{-1}A^nP$. Then

$$\begin{aligned} (P^{-1}AP)^{n+1} &= (P^{-1}AP)^n(P^{-1}AP) \\ &= (P^{-1}A^nP)(P^{-1}AP) \\ &= P^{-1}A^n(PP^{-1})AP \\ &= P^{-1}A^nIAP \\ &= P^{-1}(A^nA)P \\ &= P^{-1}A^{n+1}P \end{aligned}$$

and the induction goes through.

19. Let $A = \begin{bmatrix} 2/3 & 1/4 \\ 1/3 & 3/4 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$. Then $P^{-1} = \frac{1}{7}\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix}$.

We then verify that $P^{-1}AP = \begin{bmatrix} 5/12 & 0 \\ 0 & 1 \end{bmatrix}$. Then from the previous question,

$$P^{-1}A^nP = (P^{-1}AP)^n = \begin{bmatrix} 5/12 & 0 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} (5/12)^n & 0 \\ 0 & 1^n \end{bmatrix} = \begin{bmatrix} (5/12)^n & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence

$$\begin{aligned}
A^n &= P\begin{bmatrix} (5/12)^n & 0 \\ 0 & 1 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} (5/12)^n & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{7}\begin{bmatrix} (5/12)^n & 3 \\ -(5/12)^n & 4 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{7}\begin{bmatrix} 4(5/12)^n + 3 & (-3)(5/12)^n + 3 \\ -4(5/12)^n + 4 & 3(5/12)^n + 4 \end{bmatrix} \\
&= \frac{1}{7}\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} + \frac{1}{7}(5/12)^n\begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix}.
\end{aligned}$$

Notice that $A^n \to \frac{1}{7}\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$ as $n \to \infty$. This problem is a special case of a more general result about Markov matrices.

20. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix whose elements are non–negative real numbers satisfying

$$a \geq 0,\ b \geq 0,\ c \geq 0,\ d \geq 0,\ a + c = 1 = b + d.$$

Also let $P = \begin{bmatrix} b & 1 \\ c & -1 \end{bmatrix}$ and suppose that $A \neq I_2$.

(i) $\det P = -b - c = -(b + c)$. Now $b + c \geq 0$. Also if $b + c = 0$, then we would have $b = c = 0$ and hence $d = a = 1$, resulting in $A = I_2$. Hence $\det P < 0$ and P is non–singular.

Next,

$$\begin{aligned}
P^{-1}AP &= \frac{-1}{b+c}\begin{bmatrix} -1 & -1 \\ -c & b \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} b & 1 \\ c & -1 \end{bmatrix} \\
&= \frac{-1}{b+c}\begin{bmatrix} -a-c & -b-d \\ -ac+bc & -cb+bd \end{bmatrix}\begin{bmatrix} b & 1 \\ c & -1 \end{bmatrix} \\
&= \frac{-1}{b+c}\begin{bmatrix} -1 & -1 \\ -ac+bc & -cb+bd \end{bmatrix}\begin{bmatrix} b & 1 \\ c & -1 \end{bmatrix} \\
&= \frac{-1}{b+c}\begin{bmatrix} -b-c & 0 \\ (-ac+bc)b + (-cb+bd)c & -ac+bc+cb-bd \end{bmatrix}.
\end{aligned}$$

Now

$$\begin{aligned} -acb + b^2c - c^2b + bdc &= -cb(a+c) + bc(b+d) \\ &= -cb + bc = 0. \end{aligned}$$

Also

$$\begin{aligned} -(a+d-1)(b+c) &= -ab - ac - db - dc + b + c \\ &= -ac + b(1-a) + c(1-d) - bd \\ &= -ac + bc + cb - bd. \end{aligned}$$

Hence

$$P^{-1}AP = \frac{-1}{b+c}\left[\begin{array}{cc} -(b+c) & 0 \\ 0 & -(a+d-1)(b+c) \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & a+d-1 \end{array}\right].$$

(ii) We next prove that if we impose the extra restriction that $A \neq \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$, then $|a+d-1| < 1$. This will then have the following consequence:

$$\begin{aligned} A &= P\left[\begin{array}{cc} 1 & 0 \\ 0 & a+d-1 \end{array}\right]P^{-1} \\ A^n &= P\left[\begin{array}{cc} 1 & 0 \\ 0 & a+d-1 \end{array}\right]^n P^{-1} \\ &= P\left[\begin{array}{cc} 1 & 0 \\ 0 & (a+d-1)^n \end{array}\right]P^{-1} \\ &\to P\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right]P^{-1} \\ &= \left[\begin{array}{cc} b & 1 \\ c & -1 \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right]\frac{-1}{b+c}\left[\begin{array}{cc} -1 & -1 \\ -c & b \end{array}\right] \\ &= \frac{-1}{b+c}\left[\begin{array}{cc} b & 0 \\ c & 0 \end{array}\right]\left[\begin{array}{cc} -1 & -1 \\ -c & b \end{array}\right] \\ &= \frac{-1}{b+c}\left[\begin{array}{cc} -b & -b \\ -c & -c \end{array}\right] \\ &= \frac{1}{b+c}\left[\begin{array}{cc} b & b \\ c & c \end{array}\right], \end{aligned}$$

where we have used the fact that $(a+d-1)^n \to 0$ as $n \to \infty$.

We first prove the inequality $|a+d-1| \le 1$:

$$\begin{aligned} a+d-1 &\le 1+d-1 = d \le 1 \\ a+d-1 &\ge 0+0-1 = -1. \end{aligned}$$

Next, if $a+d-1=1$, we have $a+d=2$; so $a=1=d$ and hence $c=0=b$, contradicting our assumption that $A \ne I_2$. Also if $a+d-1=-1$, then $a+d=0$; so $a=0=d$ and hence $c=1=b$ and hence $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

22. The system is inconsistent: We work towards reducing the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 1 & 5 \\ 3 & 5 & 12 \end{array}\right] \quad \begin{array}{c} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array} \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{array}\right]$$

$$R_3 \to R_3 - R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{array}\right].$$

The last row reveals inconsistency.

The system in matrix form is $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 5 \\ 12 \end{bmatrix}.$$

The normal equations are given by the matrix equation

$$A^t AX = A^t B.$$

Now

$$\begin{aligned} A^t A &= \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 11 & 18 \\ 18 & 30 \end{bmatrix} \\ A^t B &= \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 12 \end{bmatrix} = \begin{bmatrix} 45 \\ 73 \end{bmatrix}. \end{aligned}$$

Hence the normal equations are

$$\begin{aligned} 11x + 18y &= 45 \\ 18x + 30y &= 73. \end{aligned}$$

These may be solved, for example, by Cramer's rule:

$$\begin{aligned} x &= \frac{\begin{vmatrix} 45 & 18 \\ 73 & 30 \end{vmatrix}}{\begin{vmatrix} 11 & 18 \\ 18 & 30 \end{vmatrix}} = \frac{36}{6} = 6 \\ y &= \frac{\begin{vmatrix} 11 & 45 \\ 18 & 73 \end{vmatrix}}{\begin{vmatrix} 11 & 18 \\ 18 & 30 \end{vmatrix}} = \frac{-7}{6}. \end{aligned}$$

23. Substituting the coordinates of the five points into the parabola equation gives the following equations:

$$\begin{aligned} a &= 0 \\ a + b + c &= 0 \\ a + 2b + 4c &= -1 \\ a + 3b + 9c &= 4 \\ a + 4b + 16c &= 8. \end{aligned}$$

The associated normal equations are given by

$$\begin{bmatrix} 5 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 11 \\ 42 \\ 160 \end{bmatrix},$$

which have the solution $a = 1/5,\ b = -2,\ c = 1$.

24. Suppose that A is symmetric, i.e. $A^t = A$ and that AB is defined. Then

$$(B^tAB)^t = B^tA^t(B^t)^t = B^tAB,$$

so B^tAB is also symmetric.

25. Let A be $m \times n$ and B be $n \times m$, where $m > n$. Then the homogeneous system $BX = 0$ has a non–trivial solution X_0, as the number of unknowns is greater than the number of equations. Then

$$(AB)X_0 = A(BX_0) = A0 = 0$$

and the $m \times m$ matrix AB is therefore singular, as $X_0 \neq 0$.

26. (i) Let B be a singular $n \times n$ matrix. Then $BX = 0$ for some non–zero column vector X. Then $(AB)X = A(BX) = A0 = 0$ and hence AB is also singular.

(ii) Suppose A is a singular $n \times n$ matrix. Then A^t is also singular and hence by (i) so is $B^tA^t = (AB)^t$. Consequently AB is also singular