Chapter 8

THREE–DIMENSIONAL GEOMETRY

8.1 Introduction

In this chapter we present a vector–algebra approach to three–dimensional geometry. The aim is to present standard properties of lines and planes, with minimum use of complicated three–dimensional diagrams such as those involving similar triangles. We summarize the chapter:

Points are defined as ordered triples of real numbers and the *distance* between points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is defined by the formula

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Directed line segments $\overrightarrow{AB}$ are introduced as three–dimensional column vectors: If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$, then

$$\overrightarrow{AB} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}.$$

If P is a point, we let $\mathbf{P} = \overrightarrow{OP}$ and call $\mathbf{P}$ the *position vector* of P.

With suitable definitions of *lines*, *parallel lines*, there are important geometrical interpretations of equality, addition and scalar multiplication of vectors.

(i) Equality of vectors: Suppose A, B, C, D are distinct points such that no three are collinear. Then $\overrightarrow{AB} = \overrightarrow{CD}$ if and only if $\overrightarrow{AB} \parallel \overrightarrow{CD}$ and $\overrightarrow{AC} \parallel \overrightarrow{BD}$ (See Figure 8.1.)

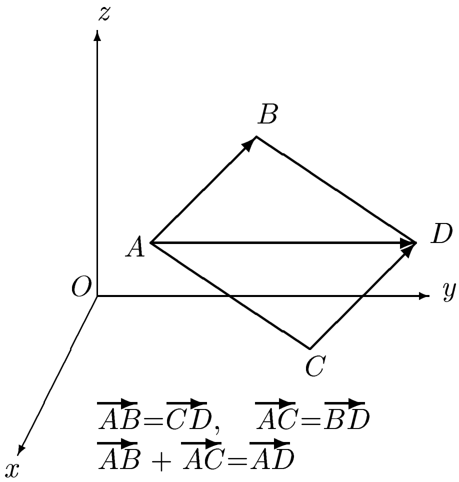

Figure 8.1: Equality and addition of vectors.

(ii) Addition of vectors obeys the *parallelogram law*: Let A, B, C be non–collinear. Then

$$\overrightarrow{AB}+\overrightarrow{AC}=\overrightarrow{AD},$$

where D is the point such that $\overrightarrow{AB} \parallel \overrightarrow{CD}$ and $\overrightarrow{AC} \parallel \overrightarrow{BD}$. (See Figure 8.1.)

(iii) Scalar multiplication of vectors: Let $\overrightarrow{AP}= t\ \overrightarrow{AB}$, where A and B are distinct points. Let $\overrightarrow{AP}= t\ \overrightarrow{AB}$, where A and B are distinct points. Then P is on the line AB,

$$\frac{AP}{AB} = |t|$$

and

(a) $P = A$ if $t = 0$, $P = B$ if $t = 1$;

(b) P is between A and B if $0 < t < 1$;

(c) B is between A and P if $1 < t$;

(d) A is between P and B if $t < 0$.

(See Figure 8.2.)

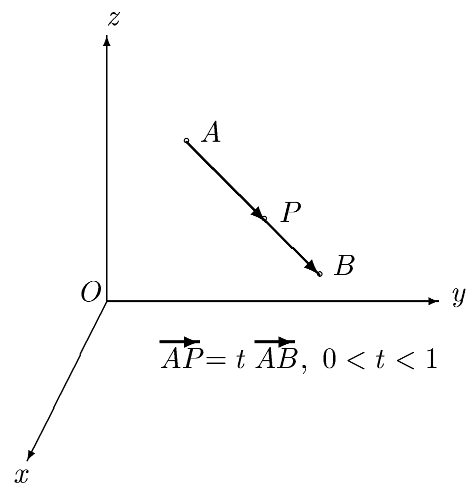

Figure 8.2: Scalar multiplication of vectors.

The *dot product* $X \cdot Y$ of vectors $X = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $Y = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$, is defined by

$$X \cdot Y = a_1a_2 + b_1b_2 + c_1c_2.$$

The *length* $||X||$ of a vector X is defined by

$$||X|| = (X \cdot X)^{1/2}$$

and the *Cauchy–Schwarz inequality* holds:

$$|X \cdot Y| \leq ||X|| \cdot ||Y||.$$

the *triangle inequality* for vector length now follows as a simple deduction:

$$||X + Y|| \leq ||X|| + ||Y||.$$

Using the equation

$$AB = ||\ \overrightarrow{AB}\ ||,$$

we deduce the corresponding familiar *triangle inequality* for distance:

$$AB \leq AC + CB.$$

The *angle* θ between two non–zero vectors X and Y is then defined by

$$\cos\theta = \frac{X \cdot Y}{||X|| \cdot ||Y||}, \quad 0 \leq \theta \leq \pi.$$

This definition makes sense. For by the Cauchy–Schwarz inequality,

$$-1 \leq \frac{X \cdot Y}{||X|| \cdot ||Y||} \leq 1.$$

Vectors X and Y are said to be *perpendicular* or *orthogonal* if $X \cdot Y = 0$.

Vectors of unit length are called *unit* vectors. The vectors

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are unit vectors and every vector is a linear combination of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

Non–zero vectors X and Y are *parallel* or *proportional* if the angle between X and Y equals 0 or π; equivalently if $X = tY$ for some real number t. Vectors X and Y are then said to have the same or opposite direction, according as $t > 0$ or $t < 0$.

We are then led to study straight lines. If A and B are distinct points, it is easy to show that $AP + PB = AB$ holds if and only if

$$\overrightarrow{AP} = t\,\overrightarrow{AB}, \text{ where } 0 \leq t \leq 1.$$

A *line* is defined as a set consisting of all points P satisfying

$$\mathbf{P} = \mathbf{P_0} + tX, \quad t \in \mathbb{R} \quad \text{or equivalently} \quad \overrightarrow{P_0P} = tX,$$

for some fixed point P_0 and fixed non–zero vector X called a *direction vector* for the line.

Equivalently, in terms of coordinates,

$$x = x_0 + ta,\ y = y_0 + tb,\ z = z_0 + tc,$$

where $P_0 = (x_0, y_0, z_0)$ and not all of a, b, c are zero.

There is then one and only one line passing passing through two distinct points A and B. It consists of the points P satisfying

$$\overrightarrow{AP} = t\,\overrightarrow{AB},$$

where t is a real number.

The *cross–product* $X \times Y$ provides us with a vector which is perpendicular to both X and Y. It is defined in terms of the components of X and Y:

Let $X = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $Y = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. Then

$$X \times Y = a\mathbf{i} + b\mathbf{j} + c\mathbf{k},$$

where

$$a = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad b = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \quad c = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

The cross–product enables us to derive elegant formulae for the distance from a point to a line, the area of a triangle and the distance between two skew lines.

Finally we turn to the geometrical concept of a plane in three–dimensional space.

A *plane* is a set of points P satisfying an equation of the form

$$\mathbf{P} = \mathbf{P}_0 + sX + tY, \; s, t \in \mathbb{R}, \tag{8.1}$$

where X and Y are non–zero, non–parallel vectors.

In terms of coordinates, equation 8.1 takes the form

$$\begin{aligned} x &= x_0 + sa_1 + ta_2 \\ y &= y_0 + sb_1 + tb_2 \\ z &= z_0 + sc_1 + tc_2, \end{aligned}$$

where $P_0 = (x_0, y_0, z_0)$.

There is then one and only one plane passing passing through three non–collinear points A, B, C. It consists of the points P satisfying

$$\overrightarrow{AP} = s\,\overrightarrow{AB} + t\,\overrightarrow{AC},$$

where s and t are real numbers.

The cross–product enables us to derive a concise equation for the plane through three non–collinear points A, B, C, namely

$$\overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0.$$

When expanded, this equation has the form

$$ax + by + cz = d,$$

where $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a non–zero vector which is perpendicular to $\overrightarrow{P_1P_2}$ for all points P_1, P_2 lying in the plane. Any vector with this property is said to be a *normal* to the plane.

It is then easy to prove that two planes with non–parallel normal vectors must intersect in a line.

We conclude the chapter by deriving a formula for the distance from a point to a plane.

8.2 Three–dimensional space

DEFINITION 8.2.1 *Three–dimensional space* is the set E^3 of ordered triples (x, y, z), where x, y, z are real numbers. The triple (x, y, z) is called a point P in E^3 and we write $P = (x, y, z)$. The numbers x, y, z are called, respectively, the x, y, z coordinates of P.

The *coordinate axes* are the sets of points:

$\{(x, 0, 0)\}$ (x–axis), $\{(0, y, 0)\}$ (y–axis), $\{(0, 0, z)\}$ (z–axis).

The only point common to all three axes is the *origin* $O = (0, 0, 0)$.

The *coordinate planes* are the sets of points:

$\{(x, y, 0)\}$ (xy–plane), $\{(0, y, z)\}$ (yz–plane), $\{(x, 0, z)\}$ (xz–plane).

The *positive octant* consists of the points (x, y, z), where $x > 0$, $y > 0$, $z > 0$.

We think of the points (x, y, z) with $z > 0$ as lying above the xy–plane, and those with $z < 0$ as lying beneath the xy–plane. A point $P = (x, y, z)$ will be represented as in Figure 8.3. The point illustrated lies in the positive octant.

DEFINITION 8.2.2 The *distance* P_1P_2 between points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is defined by the formula

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

For example, if $P = (x, y, z)$,

$$OP = \sqrt{x^2 + y^2 + z^2}.$$

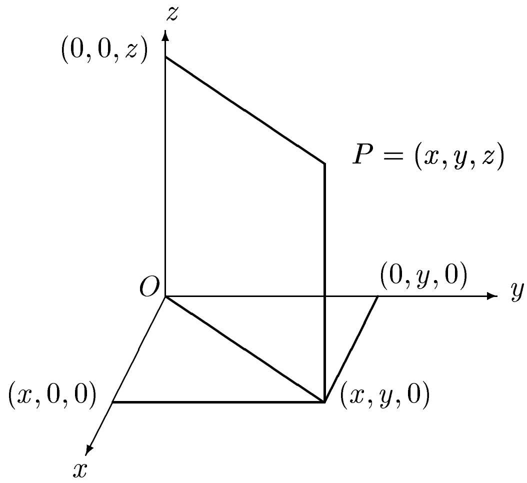

Figure 8.3: Representation of three-dimensional space.

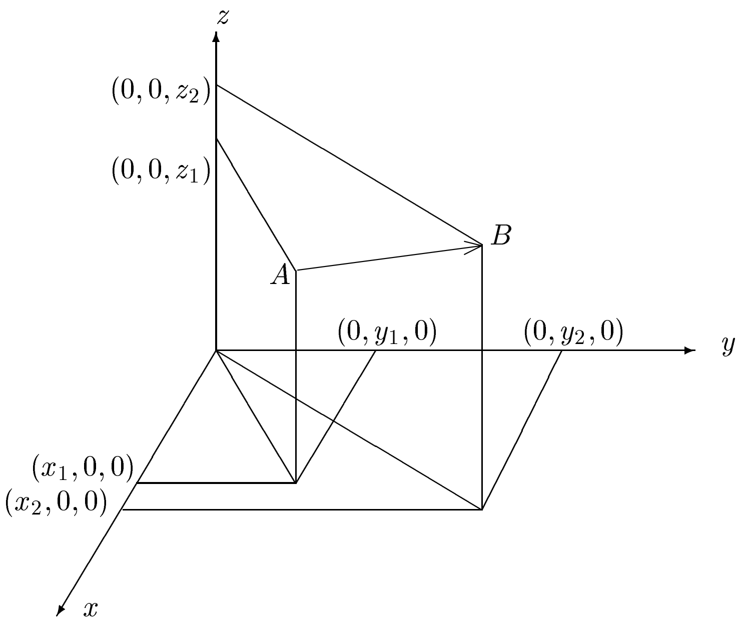

Figure 8.4: The vector $\overrightarrow{AB}$.

DEFINITION 8.2.3 If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ we define the symbol $\overrightarrow{AB}$ to be the column vector

$$\overrightarrow{AB} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}.$$

We let $\mathbf{P} = \overrightarrow{OP}$ and call $\mathbf{P}$ the *position vector* of P.

The components of $\overrightarrow{AB}$ are the coordinates of B when the axes are translated to A as origin of coordinates.

We think of $\overrightarrow{AB}$ as being represented by the directed line segment from A to B and think of it as an arrow whose tail is at A and whose head is at B. (See Figure 8.4.)

Some mathematicians think of $\overrightarrow{AB}$ as representing the translation of space which takes A into B.

The following simple properties of $\overrightarrow{AB}$ are easily verified and correspond to how we intuitively think of directed line segments:

(i) $\overrightarrow{AB} = 0 \Leftrightarrow A = B$;

(ii) $\overrightarrow{BA} = -\overrightarrow{AB}$;

(iii) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (the triangle law);

(iv) $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = \mathbf{C} - \mathbf{B}$;

(v) if X is a vector and A a point, there is exactly one point B such that $\overrightarrow{AB} = X$, namely that defined by $\mathbf{B} = \mathbf{A} + X$.

To derive properties of the distance function and the vector function $\overrightarrow{P_1P_2}$, we need to introduce the *dot product* of two vectors in $\mathbb{R}^3$.

8.3 Dot product

DEFINITION 8.3.1 If $X = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $Y = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$, then $X \cdot Y$, the *dot product* of X and Y, is defined by

$$X \cdot Y = a_1a_2 + b_1b_2 + c_1c_2.$$

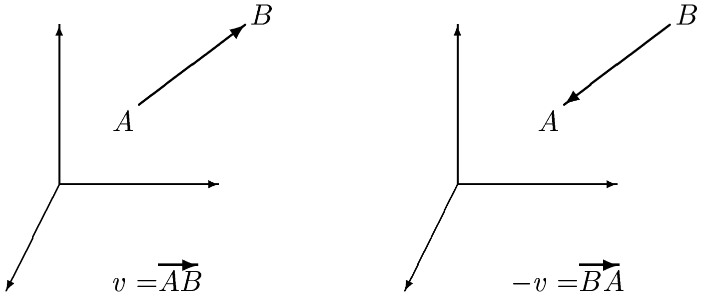

Figure 8.5: The negative of a vector.

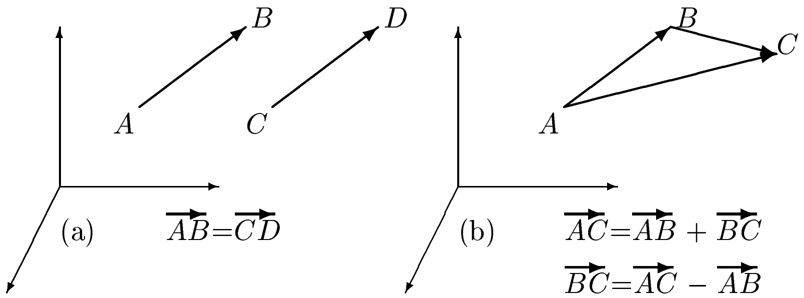

Figure 8.6: (a) Equality of vectors; (b) Addition and subtraction of vectors.

The dot product has the following properties:

(i) $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$;

(ii) $X \cdot Y = Y \cdot X$;

(iii) $(tX) \cdot Y = t(X \cdot Y)$;

(iv) $X \cdot X = a^2 + b^2 + c^2$ if $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$;

(v) $X \cdot Y = X^t Y$;

(vi) $X \cdot X = 0$ if and only if $X = 0$.

The *length* of X is defined by

$$||X|| = \sqrt{a^2 + b^2 + c^2} = (X \cdot X)^{1/2}.$$

We see that $||\mathbf{P}|| = OP$ and more generally $||\ \overrightarrow{P_1P_2}\ || = P_1P_2$, the distance between P_1 and P_2.

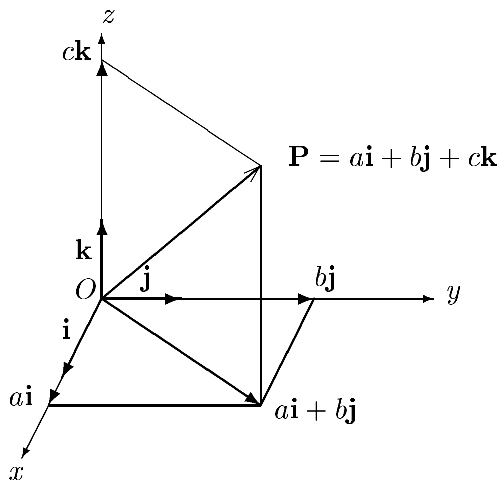

Figure 8.7: Position vector as a linear combination of **i**, **j** and **k**.

Vectors having unit length are called *unit* vectors.
The vectors

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are unit vectors. Every vector is a linear combination of **i**, **j** and **k**:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

(See Figure 8.7.)

It is easy to prove that

$$||tX|| = |t| \cdot ||X||,$$

if t is a real number. Hence if X is a non–zero vector, the vectors

$$\pm \frac{1}{||X||} X$$

are unit vectors.

A useful property of the length of a vector is

$$||X \pm Y||^2 = ||X||^2 \pm 2X \cdot Y + ||Y||^2. \tag{8.2}$$

The following important property of the dot product is widely used in mathematics:

THEOREM 8.3.1 (The Cauchy–Schwarz inequality)
If X and Y are vectors in $\mathbb{R}^3$, then

$$|X \cdot Y| \leq ||X|| \cdot ||Y||. \tag{8.3}$$

Moreover if $X \neq 0$ and $Y \neq 0$, then

$$\begin{array}{rcl} X \cdot Y = ||X|| \cdot ||Y|| & \Leftrightarrow & Y = tX,\, t > 0, \\ X \cdot Y = -||X|| \cdot ||Y|| & \Leftrightarrow & Y = tX,\, t < 0. \end{array}$$

Proof. If $X = 0$, then inequality 8.3 is trivially true. So assume $X \neq 0$. Now if t is any real number, by equation 8.2,

$$\begin{array}{rcl} 0 \leq ||tX - Y||^2 & = & ||tX||^2 - 2(tX) \cdot Y + ||Y||^2 \\ & = & t^2||X||^2 - 2(X \cdot Y)t + ||Y||^2 \\ & = & at^2 - 2bt + c, \end{array}$$

where $a = ||X||^2 > 0$, $b = X \cdot Y$, $c = ||Y||^2$.

Hence

$$\begin{array}{rl} a(t^2 - \frac{2b}{a}t + \frac{c}{a}) & \geq 0 \\ \left(t - \frac{b}{a}\right)^2 + \frac{ca - b^2}{a^2} & \geq 0 \quad . \end{array}$$

Substituting $t = b/a$ in the last inequality then gives

$$\frac{ac - b^2}{a^2} \geq 0,$$

so

$$|b| \leq \sqrt{ac} = \sqrt{a}\sqrt{c}$$

and hence inequality 8.3 follows.

To discuss equality in the Cauchy–Schwarz inequality, assume $X \neq 0$ and $Y \neq 0$.

Then if $X \cdot Y = ||X|| \cdot ||Y||$, we have for all t

$$\begin{array}{rcl} ||tX - Y||^2 & = & t^2||X||^2 - 2tX \cdot Y + ||Y||^2 \\ & = & t^2||X||^2 - 2t||X|| \cdot ||Y|| + ||Y||^2 \\ & = & ||tX - Y||^2. \end{array}$$

Taking $t = ||X||/||Y||$ then gives $||tX - Y||^2 = 0$ and hence $tX - Y = 0$. Hence $Y = tX$, where $t > 0$. The case $X \cdot Y = -||X|| \cdot ||Y$ is proved similarly.

COROLLARY 8.3.1 (The triangle inequality for vectors)
If X and Y are vectors, then

$$||X+Y|| \leq ||X|| + ||Y||. \tag{8.4}$$

Moreover if $X \neq 0$ and $Y \neq 0$, then equality occurs in inequality 8.4 if and only if $Y = tX$, where $t > 0$.

Proof.

$$\begin{aligned} ||X+Y||^2 &= ||X||^2 + 2X \cdot Y + ||Y||^2 \\ &\leq ||X||^2 + 2||X|| \cdot ||Y|| + ||Y||^2 \\ &= (||X|| + ||Y||)^2 \end{aligned}$$

and inequality 8.4 follows.

If $||X+Y|| = ||X|| + ||Y||$, then the above proof shows that

$$X \cdot Y = ||X|| \cdot ||Y||.$$

Hence if $X \neq 0$ and $Y \neq 0$, the first case of equality in the Cauchy–Schwarz inequality shows that $Y = tX$ with $t > 0$.

The triangle inequality for vectors gives rise to a corresponding inequality for the distance function:

THEOREM 8.3.2 (The triangle inequality for distance)
If A, B, C are points, then

$$AC \leq AB + BC. \tag{8.5}$$

Moreover if $B \neq A$ and $B \neq C$, then equality occurs in inequality 8.5 if and only if $\overrightarrow{AB} = r\ \overrightarrow{AC}$, where $0 < r < 1$.

Proof.

$$\begin{aligned} AC = ||\ \overrightarrow{AC}\ || &= ||\ \overrightarrow{AB} + \overrightarrow{BC}\ || \\ &\leq ||\ \overrightarrow{AB}\ || + ||\ \overrightarrow{BC}\ || \\ &= AB + BC. \end{aligned}$$

Moreover if equality occurs in inequality 8.5 and $B \neq A$ and $B \neq C$, then $X = \overrightarrow{AB} \neq 0$ and $Y = \overrightarrow{BC} \neq 0$ and the equation $AC = AB + BC$ becomes

$||X+Y|| = ||X|| + ||Y||$. Hence the case of equality in the vector triangle inequality gives

$$Y = \overrightarrow{BC} = tX = t\,\overrightarrow{AB}, \text{ where } t > 0.$$

Then

$$\begin{aligned}
\overrightarrow{BC} &= \overrightarrow{AC} - \overrightarrow{AB} = t\,\overrightarrow{AB} \\
\overrightarrow{AC} &= (1+t)\,\overrightarrow{AB} \\
\overrightarrow{AB} &= r\,\overrightarrow{AC},
\end{aligned}$$

where $r = 1/(t+1)$ satisfies $0 < r < 1$.

8.4 Lines

DEFINITION 8.4.1 A line in E^3 is the set $\mathcal{L}(P_0, X)$ consisting of all points P satisfying

$$\mathbf{P} = \mathbf{P_0} + tX, \quad t \in \mathbb{R} \quad \text{or equivalently} \quad \overrightarrow{P_0P} = tX, \tag{8.6}$$

for some fixed point P_0 and fixed non–zero vector X. (See Figure 8.8.)

Equivalently, in terms of coordinates, equation 8.6 becomes

$$x = x_0 + ta,\ y = y_0 + tb,\ z = z_0 + tc,$$

where not all of a, b, c are zero.

The following familiar property of straight lines is easily verified.

THEOREM 8.4.1 If A and B are distinct points, there is one and only one line containing A and B, namely $\mathcal{L}(A, \overrightarrow{AB})$ or more explicitly the line defined by $\overrightarrow{AP} = t\,\overrightarrow{AB}$, or equivalently, in terms of position vectors:

$$\mathbf{P} = (1-t)\mathbf{A} + t\mathbf{B} \quad \text{or} \quad \mathbf{P} = \mathbf{A} + t\,\overrightarrow{AB}\,. \tag{8.7}$$

Equations 8.7 may be expressed in terms of coordinates: if $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$, then

$$x = (1-t)x_1 + tx_2,\ y = (1-t)y_1 + ty_2,\ z = (1-t)z_1 + tz_2.$$

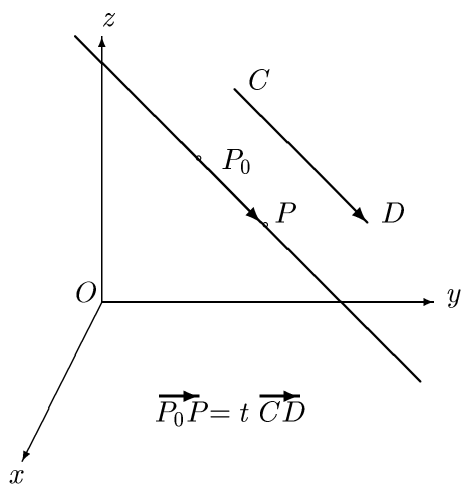

Figure 8.8: Representation of a line.

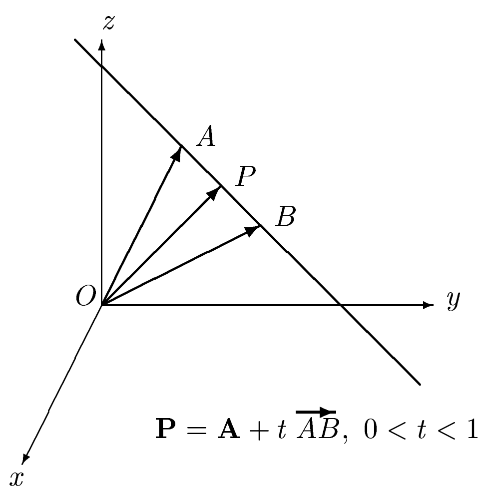

Figure 8.9: The line segment AB.

There is an important geometric significance in the number t of the above equation of the line through A and B. The proof is left as an exercise:

THEOREM 8.4.2 (Joachimsthal's ratio formulae)
If t is the parameter occurring in theorem 8.4.1, then

$$\text{(i)} \quad |t| = \frac{AP}{AB}; \quad \text{(ii)} \quad \left|\frac{t}{1-t}\right| = \frac{AP}{PB} \quad \text{if } P \neq B.$$

Also

(iii) P is between A and B if $0 < t < 1$;

(iv) B is between A and P if $1 < t$;

(v) A is between P and B if $t < 0$.

(See Figure 8.9.)

For example, $t = \frac{1}{2}$ gives the mid–point P of the segment AB:

$$\mathbf{P} = \frac{1}{2}(\mathbf{A} + \mathbf{B}).$$

EXAMPLE 8.4.1 $\mathcal{L}$ is the line AB, where $A = (-4,\ 3,\ 1)$, $B = (1,\ 1,\ 0)$; $\mathcal{M}$ is the line CD, where $C = (2,\ 0,\ 2)$, $D = (-1,\ 3,\ -2)$; $\mathcal{N}$ is the line EF, where $E = (1,\ 4,\ 7)$, $F = (-4,\ -3,\ -13)$. Find which pairs of lines intersect and also the points of intersection.

Solution. In fact only $\mathcal{L}$ and $\mathcal{N}$ intersect, in the point $(-\frac{2}{3},\ \frac{5}{3},\ \frac{1}{3})$. For example, to determine if $\mathcal{L}$ and $\mathcal{N}$ meet, we start with vector equations for $\mathcal{L}$ and $\mathcal{N}$:

$$\mathbf{P} = \mathbf{A} + t\,\overrightarrow{AB}, \quad \mathbf{Q} = \mathbf{E} + s\,\overrightarrow{EF},$$

equate $\mathbf{P}$ and $\mathbf{Q}$ and solve for s and t:

$$(-4\mathbf{i} + 3\mathbf{j} + k) + t(5\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = (\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}) + s(-5\mathbf{i} - 7\mathbf{j} - 20\mathbf{k}),$$

which on simplifying, gives

$$\begin{aligned} 5t + 5s &= 5 \\ -2t + 7s &= 1 \\ -t + 20s &= 6 \end{aligned}$$

This system has the unique solution $t = \frac{2}{3}$, $s = \frac{1}{3}$ and this determines a corresponding point P where the lines meet, namely $P = (-\frac{2}{3},\ \frac{5}{3},\ \frac{1}{3})$.

The same method yields inconsistent systems when applied to the other pairs of lines.

EXAMPLE 8.4.2 If $A = (5, 0, 7)$ and $B = (2, -3, 6)$, find the points P on the line AB which satisfy $AP/PB = 3$.

Solution. Use the formulae

$$\mathbf{P} = \mathbf{A} + t\,\overrightarrow{AB} \quad \text{and} \quad \left|\frac{t}{1-t}\right| = \frac{AP}{PB} = 3.$$

Then

$$\frac{t}{1-t} = 3 \text{ or } -3,$$

so $t = \frac{3}{4}$ or $t = \frac{3}{2}$. The corresponding points are $(\frac{11}{4}, \frac{9}{4}, \frac{25}{4})$ and $(\frac{1}{2}, \frac{9}{2}, \frac{11}{2})$.

DEFINITION 8.4.2 Let X and Y be non–zero vectors. Then X is *parallel* or *proportional* to Y if $X = tY$ for some $t \in \mathbb{R}$. We write $X \| Y$ if X is parallel to Y. If $X = tY$, we say that X and Y have the *same* or *opposite direction*, according as $t > 0$ or $t < 0$.

DEFINITION 8.4.3 if A and B are distinct points on a line $\mathcal{L}$, the non–zero vector $\overrightarrow{AB}$ is called a *direction vector* for $\mathcal{L}$.

It is easy to prove that any two direction vectors for a line are parallel.

DEFINITION 8.4.4 Let $\mathcal{L}$ and $\mathcal{M}$ be lines having direction vectors X and Y, respectively. Then $\mathcal{L}$ is *parallel* to $\mathcal{M}$ if X is parallel to Y. Clearly any line is parallel to itself.

It is easy to prove that the line through a given point A and parallel to a given line CD has an equation $\mathbf{P} = \mathbf{A} + t\,\overrightarrow{CD}$.

THEOREM 8.4.3 Let $X = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $Y = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ be non–zero vectors. Then X is parallel to Y if and only if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = 0. \tag{8.8}$$

Proof. The case of equality in the Cauchy–Schwarz inequality (theorem 8.3.1) shows that X and Y are parallel if and only if

$$|X \cdot Y| = ||X|| \cdot ||Y||.$$

Squaring gives the equivalent equality

$$(a_1a_2 + b_1b_2 + c_1c_2)^2 = (a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2),$$

which simplifies to

$$(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (a_1c_2 - a_2c_1)^2 = 0,$$

which is equivalent to

$$a_1b_2 - a_2b_1 = 0,\ b_1c_2 - b_2c_1 = 0,\ a_1c_2 - a_2c_1 = 0,$$

which is equation 8.8.

Equality of geometrical vectors has a fundamental geometrical interpretation:

THEOREM 8.4.4 Suppose A, B, C, D are distinct points such that no three are collinear. Then $\overrightarrow{AB}=\overrightarrow{CD}$ if and only if $\overrightarrow{AB} \parallel \overrightarrow{CD}$ and $\overrightarrow{AC} \parallel \overrightarrow{BD}$ (See Figure 8.1.)

Proof. If $\overrightarrow{AB}=\overrightarrow{CD}$ then

$$\begin{aligned} \mathbf{B} - \mathbf{A} &= \mathbf{D} - \mathbf{C}, \\ \mathbf{C} - \mathbf{A} &= \mathbf{D} - \mathbf{B} \end{aligned}$$

and so $\overrightarrow{AC}=\overrightarrow{BD}$. Hence $\overrightarrow{AB} \parallel \overrightarrow{CD}$ and $\overrightarrow{AC} \parallel \overrightarrow{BD}$.

Conversely, suppose that $\overrightarrow{AB} \parallel \overrightarrow{CD}$ and $\overrightarrow{AC} \parallel \overrightarrow{BD}$. Then

$$\overrightarrow{AB}= s\ \overrightarrow{CD} \quad \text{and} \quad \overrightarrow{AC}= t\ \overrightarrow{BD},$$

or

$$\mathbf{B} - \mathbf{A} = s(\mathbf{D} - \mathbf{C}) \quad \text{and} \quad \mathbf{C} - \mathbf{A} = t\mathbf{D} - \mathbf{B}.$$

We have to prove $s = 1$ or equivalently, $t = 1$.

Now subtracting the second equation above from the first, gives

$$\mathbf{B} - \mathbf{C} = s(\mathbf{D} - \mathbf{C}) - t(\mathbf{D} - \mathbf{B}),$$

so

$$(1 - t)\mathbf{B} = (1 - s)\mathbf{C} + (s - t)\mathbf{D}.$$

If $t \neq 1$, then

$$\mathbf{B} = \frac{1 - s}{1 - t}\mathbf{C} + \frac{s - t}{1 - t}\mathbf{D}$$

and B would lie on the line CD. Hence $t = 1$.

8.5 The angle between two vectors

DEFINITION 8.5.1 Let X and Y be non–zero vectors. Then the *angle* between X and Y is the unique value of θ defined by

$$\cos\theta = \frac{X \cdot Y}{||X|| \cdot ||Y||}, \quad 0 \le \theta \le \pi.$$

REMARK 8.5.1 By Cauchy's inequality, we have

$$-1 \le \frac{X \cdot Y}{||X|| \cdot ||Y||} \le 1,$$

so the above equation does define an angle θ.

In terms of components, if $X = [a_1, b_1, c_1]^t$ and $Y = [a_2, b_2, c_2]^t$, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}. \tag{8.9}$$

The next result is the well-known *cosine rule* for a triangle.

THEOREM 8.5.1 (Cosine rule) If A, B, C are points with $A \ne B$ and $A \ne C$, then the angle θ between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ satifies

$$\cos\theta = \frac{AB^2 + AC^2 - BC^2}{2AB \cdot AC}, \tag{8.10}$$

or equivalently

$$BC^2 = AB^2 + AC^2 - 2AB \cdot AC\cos\theta.$$

(See Figure 8.10.)

Proof. Let $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$, $C = (x_3, y_3, z_3)$. Then

$$\begin{aligned}
\overrightarrow{AB} &= a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k} \\
\overrightarrow{AC} &= a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k} \\
\overrightarrow{BC} &= (a_2 - a_1)\mathbf{i} + (b_2 - b_1)\mathbf{j} + (c_2 - c_1)\mathbf{k},
\end{aligned}$$

where

$$a_i = x_{i+1} - x_1,\ b_i = y_{i+1} - y_1,\ c_i = z_{i+1} - z_1,\ i = 1, 2.$$

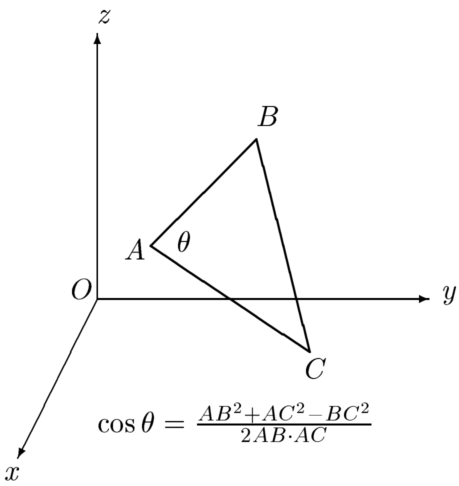

Figure 8.10: The cosine rule for a triangle.

Now by equation 8.9,

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{AB \cdot AC}.$$

Also

$$\begin{aligned} AB^2 + AC^2 - BC^2 &= (a_1^2 + b_1^2 + c_1^2) + (a_2^2 + b_2^2 + c_2^2) \\ &\quad - ((a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2) \\ &= 2a_1a_2 + 2b_1b_2 + c_1c_2. \end{aligned}$$

Equation 8.10 now follows, since

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = a_1a_2 + b_1b_2 + c_1c_2.$$

EXAMPLE 8.5.1 Let $A = (2, 1, 0)$, $B = (3, 2, 0)$, $C = (5, 0, 1)$. Find the angle θ between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Solution.

$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{AB \cdot AC}.$$

Now

$$\overrightarrow{AB} = \mathbf{i} + \mathbf{j} \quad \text{and} \quad \overrightarrow{AC} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

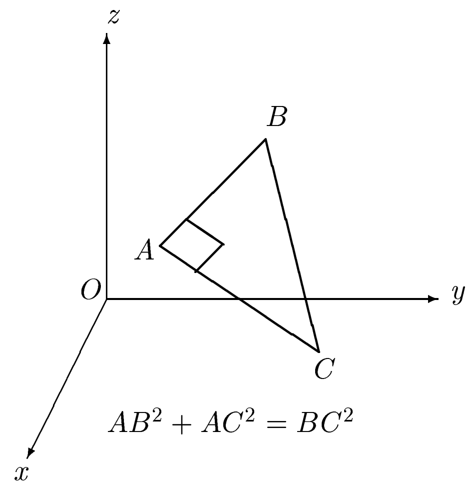

Figure 8.11: Pythagoras' theorem for a right–angled triangle.

Hence

$$\cos\theta = \frac{1\times 3 + 1\times(-1) + 0\times 1}{\sqrt{1^2+1^2+0^2}\sqrt{3^2+(-1)^2+1^2}} = \frac{2}{\sqrt{2}\sqrt{11}} = \frac{\sqrt{2}}{\sqrt{11}}.$$

Hence $\theta = \cos^{-1}\frac{\sqrt{2}}{\sqrt{11}}$.

DEFINITION 8.5.2 If X and Y are vectors satisfying $X \cdot Y = 0$, we say X is *orthogonal* or *perpendicular* to Y.

REMARK 8.5.2 If A, B, C are points forming a triangle and $\overrightarrow{AB}$ is orthogonal to $\overrightarrow{AC}$, then the angle θ between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ satisfies $\cos\theta = 0$ and hence $\theta = \frac{\pi}{2}$ and the triangle is *right–angled* at A.

Then we have *Pythagoras' theorem*:

$$BC^2 = AB^2 + AC^2. \tag{8.11}$$

We also note that $BC \geq AB$ and $BC \geq AC$ follow from equation 8.11. (See Figure 8.11.)

EXAMPLE 8.5.2 Let $A = (2, 9, 8)$, $B = (6, 4, -2)$, $C = (7, 15, 7)$. Show that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular and find the point D such that $ABDC$ forms a rectangle.

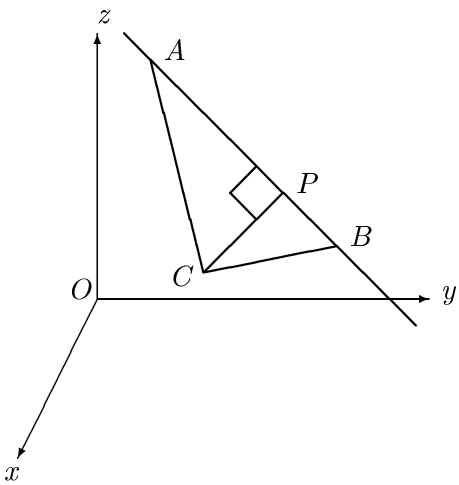

Figure 8.12: Distance from a point to a line.

Solution.

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = (4\mathbf{i} - 5\mathbf{j} - 10\mathbf{k}) \cdot (5\mathbf{i} + 6\mathbf{j} - \mathbf{k}) = 20 - 30 + 10 = 0.$$

Hence $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular. Also, the required fourth point D clearly has to satisfy the equation

$$\overrightarrow{BD} = \overrightarrow{AC}, \text{ or equivalently } \mathbf{D} - \mathbf{B} = \overrightarrow{AC}.$$

Hence

$$\mathbf{D} = \mathbf{B} + \overrightarrow{AC} = (6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) + (5\mathbf{i} + 6\mathbf{j} - \mathbf{k}) = 11\mathbf{i} + 10\mathbf{j} - 3\mathbf{k},$$

so $D = (11,\ 10,\ -3)$.

THEOREM 8.5.2 (Distance from a point to a line) If C is a point and $\mathcal{L}$ is the line through A and B, then there is exactly one point P on $\mathcal{L}$ such that $\overrightarrow{CP}$ is perpendicular to $\overrightarrow{AB}$, namely

$$\mathbf{P} = \mathbf{A} + t\,\overrightarrow{AB}, \quad t = \frac{\overrightarrow{AC} \cdot \overrightarrow{AB}}{AB^2}. \tag{8.12}$$

Moreover if Q is any point on $\mathcal{L}$, then $CQ \geq CP$ and hence P is the point on $\mathcal{L}$ closest to C.

The shortest distance CP is given by

$$CP = \frac{\sqrt{AC^2 AB^2 - (\overrightarrow{AC} \cdot \overrightarrow{AB})^2}}{AB}. \tag{8.13}$$

(See Figure 8.12.)

Proof. Let $\mathbf{P} = \mathbf{A} + t\,\overrightarrow{AB}$ and assume that $\overrightarrow{CP}$ is perpendicular to $\overrightarrow{AB}$. Then

$$\begin{aligned}
\overrightarrow{CP} \cdot \overrightarrow{AB} &= 0\\
(\mathbf{P} - \mathbf{C})\cdot \overrightarrow{AB} &= 0\\
(\mathbf{A} + t\,\overrightarrow{AB} - \mathbf{C})\cdot \overrightarrow{AB} &= 0\\
(\overrightarrow{CA} + t\,\overrightarrow{AB})\cdot \overrightarrow{AB} &= 0\\
\overrightarrow{CA} \cdot \overrightarrow{AB} + t(\overrightarrow{AB} \cdot \overrightarrow{AB}) &= 0\\
-\overrightarrow{AC} \cdot \overrightarrow{AB} + t(\overrightarrow{AB} \cdot \overrightarrow{AB}) &= 0,
\end{aligned}$$

so equation 8.12 follows.

The inequality $CQ \geq CP$, where Q is any point on $\mathcal{L}$, is a consequence of Pythagoras' theorem.

Finally, as $\overrightarrow{CP}$ and $\overrightarrow{PA}$ are perpendicular, Pythagoras' theorem gives

$$\begin{aligned}
CP^2 &= AC^2 - PA^2\\
&= AC^2 - \|t\,\overrightarrow{AB}\,\|^2\\
&= AC^2 - t^2 AB^2\\
&= AC^2 - \left(\frac{\overrightarrow{AC} \cdot \overrightarrow{AB}}{AB^2}\right)^2 AB^2\\
&= \frac{AC^2 AB^2 - (\overrightarrow{AC} \cdot \overrightarrow{AB})^2}{AB^2},
\end{aligned}$$

as required.

EXAMPLE 8.5.3 The closest point on the line through $A = (1, 2, 1)$ and $B = (2, -1, 3)$ to the origin is $P = (\frac{17}{14}, \frac{19}{14}, \frac{20}{14})$ and the corresponding shortest distance equals $\frac{5}{14}\sqrt{42}$.

Another application of theorem 8.5.2 is to the projection of a line segment on another line:

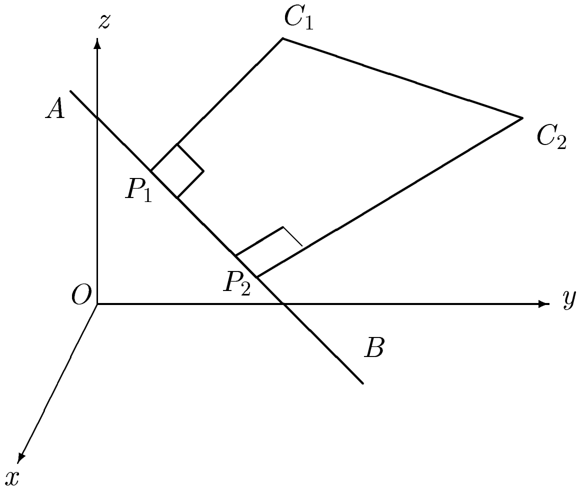

Figure 8.13: Projecting the segment C_1C_2 onto the line AB.

THEOREM 8.5.3 (The projection of a line segment onto a line) Let C_1, C_2 be points and P_1, P_2 be the feet of the perpendiculars from C_1 and C_2 to the line AB. Then

$$P_1P_2 = |\overrightarrow{C_1C_2} \cdot \hat{n}|,$$

where

$$\hat{n} = \frac{1}{AB}\overrightarrow{AB}.$$

Also

$$C_1C_2 \geq P_1P_2. \tag{8.14}$$

(See Figure 8.13.)

Proof. Using equations 8.12, we have

$$\mathbf{P_1} = \mathbf{A} + t_1\overrightarrow{AB}, \quad \mathbf{P_2} = \mathbf{A} + t_2\overrightarrow{AB},$$

where

$$t_1 = \frac{\overrightarrow{AC_1}\cdot\overrightarrow{AB}}{AB^2}, \quad t_2 = \frac{\overrightarrow{AC_2}\cdot\overrightarrow{AB}}{AB^2}.$$

Hence

$$\begin{aligned}\overrightarrow{P_1P_2} &= (\mathbf{A} + t_2\overrightarrow{AB}) - (\mathbf{A} + t_1\overrightarrow{AB}) \\ &= (t_2 - t_1)\overrightarrow{AB},\end{aligned}$$

so

$$\begin{aligned}
P_1P_2 &= ||\ \overrightarrow{P_1P_2}\ || = |t_2 - t_1|AB \\
&= \left| \frac{\overrightarrow{AC_2} \cdot \overrightarrow{AB}}{AB^2} - \frac{\overrightarrow{AC_1} \cdot \overrightarrow{AB}}{AB^2} \right| AB \\
&= \frac{\left| \overrightarrow{C_1C_2} \cdot \overrightarrow{AB} \right|}{AB^2} AB \\
&= \left| \overrightarrow{C_1C_2} \cdot \hat{n} \right|,
\end{aligned}$$

where $\hat{n}$ is the unit vector

$$\hat{n} = \frac{1}{AB}\ \overrightarrow{AB}\ .$$

Inequality 8.14 then follows from the Cauchy–Schwarz inequality 8.3.

DEFINITION 8.5.3 Two non–intersecting lines are called *skew* if they have non–parallel direction vectors.

Theorem 8.5.3 has an application to the problem of showing that two skew lines have a shortest distance between them. (The reader is referred to problem 16 at the end of the chapter.)

Before we turn to the study of planes, it is convenient to introduce the cross–product of two vectors.

8.6 The cross–product of two vectors

DEFINITION 8.6.1 Let $X = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $Y = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. Then $X \times Y$, the *cross–product* of X and Y, is defined by

$$X \times Y = a\mathbf{i} + b\mathbf{j} + c\mathbf{k},$$

where

$$a = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad b = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \quad c = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

The vector cross–product has the following properties which follow from properties of 2×2 and 3×3 determinants:

(i) $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j};$

(ii) $X \times X = 0$;

(iii) $Y \times X = -X \times Y$;

(iv) $X \times (Y + Z) = X \times Y + X \times Z$;

(v) $(tX) \times Y = t(X \times Y)$;

(vi) (Scalar triple product formula) if $Z = a_3\mathbf{i} + b_3\mathbf{j} + c_3\mathbf{k}$, then

$$X \cdot (Y \times Z) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (X \times Y) \cdot Z;$$

(vii) $X \cdot (X \times Y) = 0 = Y \cdot (X \times Y)$;

(viii) $||X \times Y|| = \sqrt{||X||^2||Y||^2 - (X \cdot Y)^2}$;

(ix) if X and Y are non–zero vectors and θ is the angle between X and Y, then

$$||X \times Y|| = ||X|| \cdot ||Y|| \sin \theta.$$

(See Figure 8.14.)

From theorem 8.4.3 and the definition of cross–product, it follows that non–zero vectors X and Y are parallel if and only if $X \times Y = 0$; hence by (vii), the cross–product of two non–parallel, non–zero vectors X and Y, is a non–zero vector perpendicular to both X and Y.

LEMMA 8.6.1 Let X and Y be non–zero, non–parallel vectors.

(i) Z is a linear combination of X and Y, if and only if Z is perpendicular to $X \times Y$;

(ii) Z is perpendicular to X and Y, if and only if Z is parallel to $X \times Y$.

Proof. Let X and Y be non–zero, non–parallel vectors. Then

$$X \times Y \neq 0.$$

Then if $X \times Y = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, we have

$$\det [X \times Y|X|Y]^t = \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (X \times Y) \cdot (X \times Y) > 0.$$

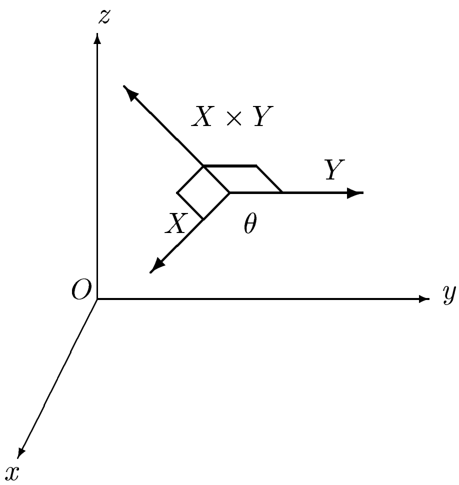

Figure 8.14: The vector cross–product.

Hence the matrix $[X \times Y|X|Y]$ is non–singular. Consequently the linear system

$$r(X \times Y) + sX + tY = Z \tag{8.15}$$

has a unique solution r, s, t.
(i) Suppose $Z = sX + tY$. Then

$$Z \cdot (X \times Y) = sX \cdot (X \times Y) + tY \cdot (X \times Y) = s0 + t0 = 0.$$

Conversely, suppose that

$$Z \cdot (X \times Y) = 0. \tag{8.16}$$

Now from equation 8.15, r, s, t exist satisfying

$$Z = r(X \times Y) + sX + tY.$$

Then equation 8.16 gives

$$\begin{aligned} 0 &= (r(X \times Y) + sX + tY) \cdot (X \times Y) \\ &= r||X \times Y||^2 + sX \cdot (X \times Y) + tY \cdot (Y \times X) \\ &= r||X \times Y||^2. \end{aligned}$$

Hence $r = 0$ and $Z = sX + tY$, as required.
(ii) Suppose $Z = \lambda(X \times Y)$. Then clearly Z is perpendicular to X and Y.

Conversely suppose that Z is perpendicular to X and Y.
Now from equation 8.15, r, s, t exist satisfying

$$Z = r(X \times Y) + sX + tY.$$

Then

$$\begin{aligned} sX \cdot X + tX \cdot Y &= X \cdot Z = 0 \\ sY \cdot X + tY \cdot Y &= Y \cdot Z = 0, \end{aligned}$$

from which it follows that

$$(sX + tY) \cdot (sX + tY) = 0.$$

Hence $sX + tY = 0$ and so $s = 0$, $t = 0$. Consequently $Z = r(X \times Y)$, as required.

The cross–product gives a compact formula for the distance from a point to a line, as well as the area of a triangle.

THEOREM 8.6.1 (Area of a triangle)
If A, B, C are distinct non–collinear points, then

(i) the distance d from C to the line AB is given by

$$d = \frac{|| \overrightarrow{AB} \times \overrightarrow{AC} ||}{AB}, \tag{8.17}$$

(ii) the area of the triangle ABC equals

$$\frac{|| \overrightarrow{AB} \times \overrightarrow{AC} ||}{2} = \frac{||\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}||}{2}. \tag{8.18}$$

Proof. The area Δ of triangle ABC is given by

$$\Delta = \frac{AB \cdot CP}{2},$$

where P is the foot of the perpendicular from C to the line AB. Now by formula 8.13, we have

$$\begin{aligned} CP &= \frac{\sqrt{AC^2 \cdot AB^2 - (\overrightarrow{AC} \cdot \overrightarrow{AB})^2}}{AB} \\ &= \frac{|| \overrightarrow{AB} \times \overrightarrow{AC} ||}{AB}, \end{aligned}$$

which, by property (viii) of the cross–product, gives formula 8.17. The second formula of equation 8.18 follows from the equations

$$\begin{aligned}
\overrightarrow{AB} \times \overrightarrow{AC} &= (\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A}) \\
&= \{(\mathbf{B} - \mathbf{A}) \times \mathbf{C}\} - \{(\mathbf{C} - \mathbf{A}) \times \mathbf{A}\} \\
&= \{(\mathbf{B} \times \mathbf{C} - \mathbf{A} \times \mathbf{C})\} - \{(\mathbf{B} \times \mathbf{A} - \mathbf{A} \times \mathbf{A})\} \\
&= \mathbf{B} \times \mathbf{C} - \mathbf{A} \times \mathbf{C} - \mathbf{B} \times \mathbf{A} \\
&= \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B},
\end{aligned}$$

as required.

8.7 Planes

DEFINITION 8.7.1 A *plane* is a set of points P satisfying an equation of the form

$$\mathbf{P} = \mathbf{P}_0 + sX + tY, \; s, t \in \mathbb{R}, \tag{8.19}$$

where X and Y are non–zero, non–parallel vectors.

For example, the xy–plane consists of the points $P = (x, y, 0)$ and corresponds to the plane equation

$$\mathbf{P} = x\mathbf{i} + y\mathbf{j} = \mathbf{O} + x\mathbf{i} + y\mathbf{j}.$$

In terms of coordinates, equation 8.19 takes the form

$$\begin{aligned}
x &= x_0 + sa_1 + ta_2 \\
y &= y_0 + sb_1 + tb_2 \\
z &= z_0 + sc_1 + tc_2,
\end{aligned}$$

where $P_0 = (x_0, y_0, z_0)$ and (a_1, b_1, c_1) and $(a_2, b_2, c_12$ are non–zero and non–proportional.

THEOREM 8.7.1 Let A, B, C be three non–collinear points. Then there is one and only one plane through these points, namely the plane given by the equation

$$\mathbf{P} = \mathbf{A} + s\overrightarrow{AB} + t\overrightarrow{AC}, \tag{8.20}$$

or equivalently

$$\overrightarrow{AP} = s\overrightarrow{AB} + t\overrightarrow{AC}\,. \tag{8.21}$$

(See Figure 8.15.)

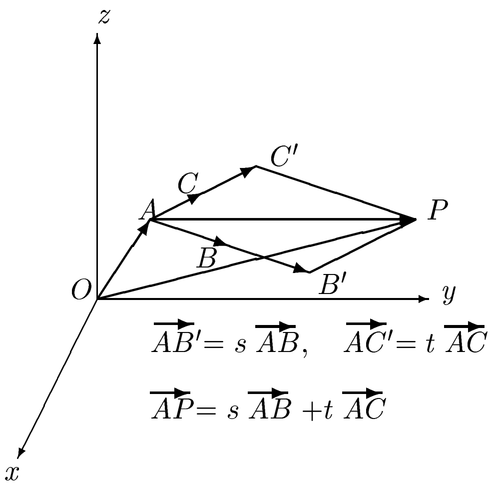

Figure 8.15: Vector equation for the plane ABC.

Proof. First note that equation 8.20 is indeed the equation of a plane through A, B and C, as $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are non–zero and non–parallel and $(s, t) = (0, 0)$, $(1, 0)$ and $(0, 1)$ give $P = A$, B and C, respectively. Call this plane $\mathcal{P}$.

Conversely, suppose $\mathbf{P} = \mathbf{P}_0 + sX + tY$ is the equation of a plane $\mathcal{Q}$ passing through A, B, C. Then $\mathbf{A} = \mathbf{P}_0 + s_0X + t_0Y$, so the equation for $\mathcal{Q}$ may be written

$$\mathbf{P} = \mathbf{A} + (s - s_0)X + (t - t_0)Y = \mathbf{A} + s'X + t'Y;$$

so in effect we can take $P_0 = A$ in the equation of $\mathcal{Q}$. Then the fact that B and C lie on $\mathcal{Q}$ gives equations

$$\mathbf{B} = \mathbf{A} + s_1X + t_1Y, \quad \mathbf{C} = \mathbf{A} + s_2X + t_2Y,$$

or

$$\overrightarrow{AB} = s_1X + t_1Y, \quad \overrightarrow{AC} = s_2X + t_2Y. \tag{8.22}$$

Then equations 8.22 and equation 8.20 show that

$$\mathcal{P} \subseteq \mathcal{Q}.$$

Conversely, it is straightforward to show that because $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not parallel, we have

$$\begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix} \neq 0.$$

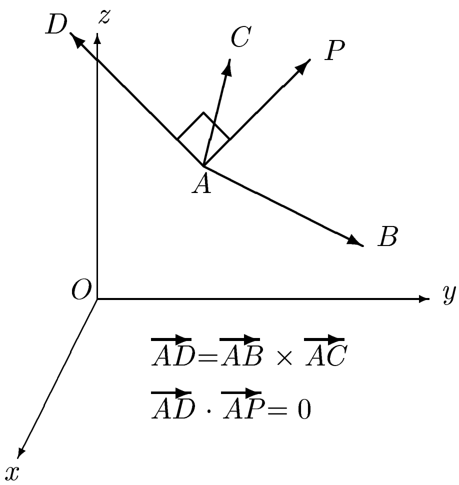

Figure 8.16: Normal equation of the plane ABC.

Hence equations 8.22 can be solved for X and Y as linear combinations of $\overrightarrow{AB}$ and $\overrightarrow{AC}$, allowing us to deduce that

$$\mathcal{Q} \subseteq \mathcal{P}.$$

Hence

$$\mathcal{Q} = \mathcal{P}.$$

THEOREM 8.7.2 (Normal equation for a plane) Let

$$A = (x_1, y_1, z_1),\ B = (x_2, y_2, z_2),\ C = (x_3, y_3, z_3)$$

be three non–collinear points. Then the plane through A, B, C is given by

$$\overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0, \qquad (8.23)$$

or equivalently,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0, \qquad (8.24)$$

where $P = (x, y, z)$. (See Figure 8.16.)

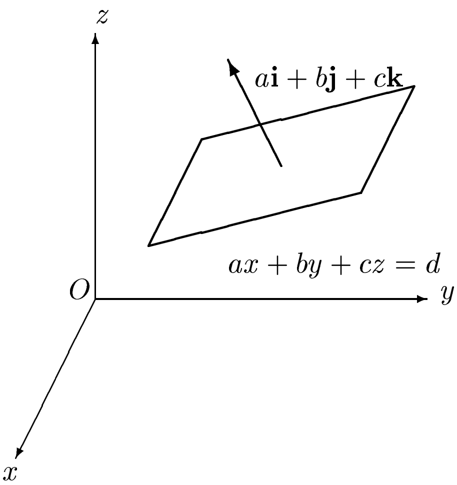

Figure 8.17: The plane $ax + by + cz = d$.

REMARK 8.7.1 Equation 8.24 can be written in more symmetrical form as

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0. \tag{8.25}$$

Proof. Let $\mathcal{P}$ be the plane through A, B, C. Then by equation 8.21, we have $P \in \mathcal{P}$ if and only if $\overrightarrow{AP}$ is a linear combination of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and so by lemma 8.6.1(i), using the fact that $\overrightarrow{AB} \times \overrightarrow{AC} \neq 0$ here, if and only if $\overrightarrow{AP}$ is perpendicular to $\overrightarrow{AB} \times \overrightarrow{AC}$. This gives equation 8.23.

Equation 8.24 is the scalar triple product version of equation 8.23, taking into account the equations

$$\begin{aligned} \overrightarrow{AP} &= (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}, \\ \overrightarrow{AB} &= (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}, \\ \overrightarrow{AC} &= (x_3 - x_1)\mathbf{i} + (y_3 - y_1)\mathbf{j} + (z_3 - z_1)\mathbf{k}. \end{aligned}$$

REMARK 8.7.2 Equation 8.24 gives rise to a linear equation in x, y and z:

$$ax + by + cz = d,$$

where $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \neq 0$. For

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} =$$

$$\begin{vmatrix} x & y & z \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} \tag{8.26}$$

and expanding the first determinant on the right–hand side of equation 8.26 along row 1 gives an expression

$$ax + by + cz$$

where

$$a = \begin{vmatrix} y_2 - y_1 & z_2 - z_1 \\ y_3 - y_1 & z_3 - z_1 \end{vmatrix}, \ b = -\begin{vmatrix} x_2 - x_1 & z_2 - z_1 \\ x_3 - x_1 & z_3 - z_1 \end{vmatrix}, \ c = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}.$$

But a, b, c are the components of $\overrightarrow{AB} \times \overrightarrow{AC}$, which in turn is non–zero, as A, B, C are non–collinear here.

Conversely if $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \neq 0$, the equation

$$ax + by + cz = d$$

does indeed represent a plane. For if say $a \neq 0$, the equation can be solved for x in terms of y and z:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{d}{a} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{c}{a} \\ 0 \\ 1 \end{bmatrix},$$

which gives the plane

$$\mathbf{P} = \mathbf{P}_0 + yX + zY,$$

where $P_0 = (-\frac{d}{a}, 0, 0)$ and $X = -\frac{b}{a}\mathbf{i} + \mathbf{j}$ and $Y = -\frac{c}{a}\mathbf{i} + \mathbf{k}$ are evidently non–parallel vectors.

REMARK 8.7.3 The plane equation $ax + by + cz = d$ is called the *normal form*, as it is easy to prove that if P_1 and P_2 are two points in the plane, then $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is perpendicular to $\overrightarrow{P_1P_2}$. Any non–zero vector with this property is called a *normal* to the plane. (See Figure 8.17.)

By lemma 8.6.1(ii), it follows that every vector X normal to a plane through three non–collinear points A, B, C is parallel to $\overrightarrow{AB} \times \overrightarrow{AC}$, since X is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

EXAMPLE 8.7.1 Show that the planes

$$x + y - 2z = 1 \quad \text{and} \quad x + 3y - z = 4$$

intersect in a line and find the distance from the point $C = (1,\, 0,\, 1)$ to this line.

Solution. Solving the two equations simultaneously gives

$$x = -\frac{1}{2} + \frac{5}{2}z, \quad y = \frac{3}{2} - \frac{1}{2}z, \tag{8.27}$$

where z is arbitrary. Hence

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = -\frac{1}{2}\mathbf{i} - \frac{3}{2}\mathbf{j} + z(\frac{5}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}),$$

which is the equation of a line $\mathcal{L}$ through $A = (-\frac{1}{2},\, -\frac{3}{2},\, 0)$ and having direction vector $\frac{5}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$.

We can now proceed in one of three ways to find the closest point on $\mathcal{L}$ to A.

One way is to use equation 8.17 with B defined by

$$\overrightarrow{AB} = \frac{5}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}.$$

Another method minimizes the distance CP, where P ranges over $\mathcal{L}$.

A third way is to find an equation for the plane through C, having $\frac{5}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$ as a normal. Such a plane has equation

$$5x - y + 2z = d,$$

where d is found by substituting the coordinates of C in the last equation.

$$d = 5 \times 1 - 0 + 2 \times 1 = 7.$$

We now find the point P where the plane intersects the line $\mathcal{L}$. Then $\overrightarrow{CP}$ will be perpendicular to $\mathcal{L}$ and CP will be the required shortest distance from C to $\mathcal{L}$. We find using equations 8.27 that

$$5(-\frac{1}{2} + \frac{5}{2}z) - (\frac{3}{2} - \frac{1}{2}z) + 2z = 7,$$

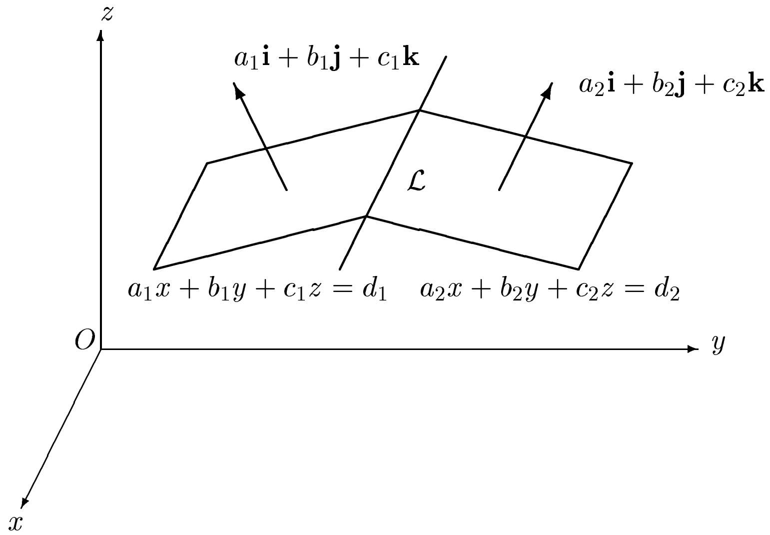

Figure 8.18: Line of intersection of two planes.

so $z = \frac{11}{15}$. Hence $P = (\frac{4}{3}, \frac{17}{15}, \frac{11}{15})$.

It is clear that through a given line and a point not on that line, there passes exactly one plane. If the line is given as the intersection of two planes, each in normal form, there is a simple way of finding an equation for this plane. More explicitly we have the following result:

THEOREM 8.7.3 Suppose the planes

$$a_1x + b_1y + c_1z = d_1 \tag{8.28}$$
$$a_2x + b_2y + c_2z = d_2 \tag{8.29}$$

have non–parallel normals. Then the planes intersect in a line $\mathcal{L}$.

Moreover the equation

$$\lambda(a_1x + b_1y + c_1z - d_1) + \mu(a_2x + b_2y + c_2z - d_2) = 0, \tag{8.30}$$

where λ and μ are not both zero, gives all planes through $\mathcal{L}$.

(See Figure 8.18.)

Proof. Assume that the normals $a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ are non–parallel. Then by theorem 8.4.3, not all of

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \tag{8.31}$$

are zero. If say $\Delta_1 \neq 0$, we can solve equations 8.28 and 8.29 for x and y in terms of z, as we did in the previous example, to show that the intersection forms a line $\mathcal{L}$.

We next have to check that if λ and μ are not both zero, then equation 8.30 represents a plane. (Whatever set of points equation 8.30 represents, this set certainly contains $\mathcal{L}$.)

$$(\lambda a_1 + \mu a_2)x + (\lambda b_1 + \mu b_2)y + (\lambda c_1 + \mu c_2)z - (\lambda d_1 + \mu d_2) = 0.$$

Then we clearly cannot have all the coefficients

$$\lambda a_1 + \mu a_2, \quad \lambda b_1 + \mu b_2, \quad \lambda c_1 + \mu c_2$$

zero, as otherwise the vectors $a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ would be parallel.

Finally, if $\mathcal{P}$ is a plane containing $\mathcal{L}$, let $P_0 = (x_0, y_0, z_0)$ be a point not on $\mathcal{L}$. Then if we define λ and μ by

$$\lambda = -(a_2x_0 + b_2y_0 + c_2z_0 - d_2), \quad \mu = a_1x_0 + b_1y_0 + c_1z_0 - d_1,$$

then at least one of λ and μ is non–zero. Then the coordinates of P_0 satisfy equation 8.30, which therefore represents a plane passing through $\mathcal{L}$ and P_0 and hence identical with $\mathcal{P}$.

EXAMPLE 8.7.2 Find an equation for the plane through $P_0 = (1, 0, 1)$ and passing through the line of intersection of the planes

$$x + y - 2z = 1 \quad \text{and} \quad x + 3y - z = 4.$$

Solution. The required plane has the form

$$\lambda(x + y - 2z - 1) + \mu(x + 3y - z - 4) = 0,$$

where not both of λ and μ are zero. Substituting the coordinates of P_0 into this equation gives

$$-2\lambda + \mu(-4) = 0, \quad \lambda = -2\mu.$$

So the required equation is

$$-2\mu(x + y - 2z - 1) + \mu(x + 3y - z - 4) = 0,$$

or

$$-x + y + 3z - 2 = 0.$$

Our final result is a formula for the distance from a point to a plane.

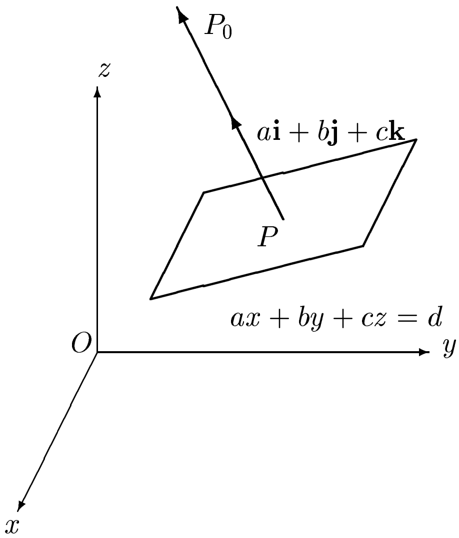

Figure 8.19: Distance from a point P_0 to the plane $ax + by + cz = d$.

THEOREM 8.7.4 (Distance from a point to a plane)
Let $P_0 = (x_0, y_0, z_0)$ and $\mathcal{P}$ be the plane

$$ax + by + cz = d. \tag{8.32}$$

Then there is a unique point P on $\mathcal{P}$ such that $\overrightarrow{P_0P}$ is normal to $\mathcal{P}$. Morever

$$P_0P = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

(See Figure 8.19.)
Proof. The line through P_0 normal to $\mathcal{P}$ is given by

$$\mathbf{P} = \mathbf{P}_0 + t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}),$$

or in terms of coordinates

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

Substituting these formulae in equation 8.32 gives

$$\begin{aligned} a(x_0 + at) + b(y_0 + bt) + c(z_0 + ct) &= d \\ t(a^2 + b^2 + c^2) &= -(ax_0 + by_0 + cz_0 - d), \end{aligned}$$

so

$$t = -\left(\frac{ax_0 + by_0 + cz_0 - d}{a^2 + b^2 + c^2}\right).$$

Then

$$\begin{aligned} P_0P = \|\ \overrightarrow{P_0P}\ \| &= \|t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\| \\ &= |t|\sqrt{a^2+b^2+c^2} \\ &= \frac{|ax_0+by_0+cz_0-d|}{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2} \\ &= \frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}. \end{aligned}$$

Other interesting geometrical facts about lines and planes are left to the problems at the end of this chapter.

8.8 PROBLEMS

.

1. Find the point where the line through $A = (3,\ -2,\ 7)$ and $B = (13,\ 3,\ -8)$ meets the xz–plane.

   [Ans: $(7,\ 0,\ 1)$.]

2. Let A, B, C be non–collinear points. If E is the mid–point of the segment BC and F is the point on the segment EA satisfying $\frac{AF}{EF} = 2$, prove that
$$\mathbf{F} = \frac{1}{3}(\mathbf{A} + \mathbf{B} + \mathbf{C}).$$
   (F is called the *centroid* of triangle ABC.)

3. Prove that the points $(2,\ 1,\ 4),\ (1,\ -1,\ 2),\ (3,\ 3,\ 6)$ are collinear.

4. If $A = (2,\ 3,\ -1)$ and $B = (3,\ 7,\ 4)$, find the points P on the line AB satisfying $PA/PB = 2/5$.

   [Ans: $\left(\frac{16}{7},\ \frac{29}{7},\ \frac{3}{7}\right)$ and $\left(\frac{4}{3},\ \frac{1}{3},\ -\frac{13}{3}\right)$.]

5. Let $\mathcal{M}$ be the line through $A = (1,\ 2,\ 3)$ parallel to the line joining $B = (-2,\ 2,\ 0)$ and $C = (4,\ -1,\ 7)$. Also $\mathcal{N}$ is the line joining $E = (1,\ -1,\ 8)$ and $F = (10,\ -1,\ 11)$. Prove that $\mathcal{M}$ and $\mathcal{N}$ intersect and find the point of intersection.

   [Ans: $(7,\ -1,\ 10)$.]

6. Prove that the triangle formed by the points $(-3,\,5,\,6)$, $(-2,\,7,\,9)$ and $(2,\,1,\,7)$ is a 30^o, 60^o, 90^o triangle.

7. Find the point on the line AB closest to the origin, where $A = (-2,\,1,\,3)$ and $B = (1,\,2,\,4)$. Also find this shortest distance.

   [Ans: $\left(-\frac{16}{11},\,\frac{13}{11},\,\frac{35}{11}\right)$ and $\sqrt{\frac{150}{11}}$.]

8. A line $\mathcal{N}$ is determined by the two planes

$$x + y - 2z = 1, \qquad \text{and} \qquad x + 3y - z = 4.$$

   Find the point P on $\mathcal{N}$ closest to the point $C = (1,\,0,\,1)$ and find the distance PC.

   [Ans: $\left(\frac{4}{3},\,\frac{17}{15},\,\frac{11}{15}\right)$ and $\frac{\sqrt{330}}{15}$.]

9. Find a linear equation describing the plane perpendicular to the line of intersection of the planes $x + y - 2z = 4$ and $3x - 2y + z = 1$ and which passes through $(6,\,0,\,2)$.

   [Ans: $3x + 7y + 5z = 28$.]

10. Find the length of the projection of the segment AB on the line $\mathcal{L}$, where $A = (1,\,2,\,3)$, $B = (5,\,-2,\,6)$ and $\mathcal{L}$ is the line CD, where $C = (7,\,1,\,9)$ and $D = (-1,\,5,\,8)$.

    [Ans: $\frac{17}{3}$.]

11. Find a linear equation for the plane through $A = (3,\,-1,\,2)$, perpendicular to the line $\mathcal{L}$ joining $B = (2,\,1,\,4)$ and $C = (-3,\,-1,\,7)$. Also find the point of intersection of $\mathcal{L}$ and the plane and hence determine the distance from A to $\mathcal{L}$. [Ans: $5x+2y-3z = 7$, $\left(\frac{111}{38},\,\frac{52}{38},\,\frac{131}{38}\right)$, $\sqrt{\frac{293}{38}}$.]

12. If P is a point inside the triangle ABC, prove that

$$\mathbf{P} = r\mathbf{A} + s\mathbf{B} + t\mathbf{C},$$

    where $r + s + t = 1$ and $r > 0$, $s > 0$, $t > 0$.

13. If B is the point where the perpendicular from $A = (6,\,-1,\,11)$ meets the plane $3x + 4y + 5z = 10$, find B and the distance AB.

    [Ans: $B = \left(\frac{123}{50},\,\frac{-286}{50},\,\frac{255}{50}\right)$ and $AB = \frac{59}{\sqrt{50}}$.]

14. Prove that the triangle with vertices $(-3,\,0,\,2)$, $(6,\,1,\,4)$, $(-5,\,1,\,0)$ has area $\frac{1}{2}\sqrt{333}$.

15. Find an equation for the plane through $(2,\,1,\,4)$, $(1,\,-1,\,2)$, $(4,\,-1,\,1)$.
[Ans: $2x - 7y + 6z = 21$.]

16. Lines $\mathcal{L}$ and $\mathcal{M}$ are non–parallel in 3–dimensional space and are given by equations
$$\mathbf{P} = \mathbf{A} + sX, \quad \mathbf{Q} = \mathbf{B} + tY.$$
   (i) Prove that there is precisely one pair of points P and Q such that $\overrightarrow{PQ}$ is perpendicular to X and Y.
   (ii) Explain why PQ is the shortest distance between lines $\mathcal{L}$ and $\mathcal{M}$. Also prove that
$$PQ = \frac{|\,(X \times Y)\cdot \overrightarrow{AB}|}{\|X \times Y\|}.$$

17. If $\mathcal{L}$ is the line through $A = (1,\,2,1)$ and $C = (3,\,-1,\,2)$, while $\mathcal{M}$ is the line through $B = (1,\,0,\,2)$ and $D = (2,\,1,\,3)$, prove that the shortest distance between $\mathcal{L}$ and $\mathcal{M}$ equals $\frac{13}{\sqrt{62}}$.

18. Prove that the volume of the tetrahedron formed by four non–coplanar points $A_i = (x_i,\,y_i,\,z_i)$, $1 \le i \le 4$, is equal to
$$\frac{1}{6}\,|\,(\overrightarrow{A_1A_2} \times \overrightarrow{A_1A_3})\cdot \overrightarrow{A_1A_4}|,$$
which in turn equals the absolute value of the determinant
$$\frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}.$$

19. The points $A = (1,\,1,\,5)$, $B = (2,\,2,\,1)$, $C = (1,\,-2,\,2)$ and $D = (-2,\,1,\,2)$ are the vertices of a tetrahedron. Find the equation of the line through A perpendicular to the face BCD and the distance of A from this face. Also find the shortest distance between the skew lines AD and BC.
[Ans: $\mathbf{P} = (1 + t)(\mathbf{i} + \mathbf{j} + 5\mathbf{k})$; $2\sqrt{3}$; 3.]