Chapter 5

COMPLEX NUMBERS

5.1 Constructing the complex numbers

One way of introducing the field $\mathbb{C}$ of complex numbers is via the arithmetic of 2×2 matrices.

DEFINITION 5.1.1 A complex number is a matrix of the form

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix},$$

where x and y are real numbers.

Complex numbers of the form $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$ are scalar matrices and are called *real* complex numbers and are denoted by the symbol $\{x\}$.

The real complex numbers $\{x\}$ and $\{y\}$ are respectively called the *real part* and *imaginary part* of the complex number $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$.

The complex number $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is denoted by the symbol i.

We have the identities

$$\begin{aligned} \begin{bmatrix} x & -y \\ y & x \end{bmatrix} &= \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} 0 & -y \\ y & 0 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} \\ &= \{x\} + i\{y\}, \end{aligned}$$

$$i^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \{-1\}.$$

Complex numbers of the form $i\{y\}$, where y is a non–zero real number, are called *imaginary numbers*.

If two complex numbers are equal, we can equate their real and imaginary parts:

$$\{x_1\} + i\{y_1\} = \{x_2\} + i\{y_2\} \Rightarrow x_1 = x_2 \text{ and } y_1 = y_2,$$

if x_1, x_2, y_1, y_2 are real numbers. Noting that $\{0\} + i\{0\} = \{0\}$, gives the useful special case is

$$\{x\} + i\{y\} = \{0\} \Rightarrow x = 0 \text{ and } y = 0,$$

if x and y are real numbers.

The sum and product of two real complex numbers are also real complex numbers:

$$\{x\} + \{y\} = \{x + y\}, \quad \{x\}\{y\} = \{xy\}.$$

Also, as real complex numbers are scalar matrices, their arithmetic is very simple. They form a field under the operations of matrix addition and multiplication. The additive identity is $\{0\}$, the additive inverse of $\{x\}$ is $\{-x\}$, the multiplicative identity is $\{1\}$ and the multiplicative inverse of $\{x\}$ is $\{x^{-1}\}$. Consequently

$$\{x\} - \{y\} = \{x\} + (-\{y\}) = \{x\} + \{-y\} = \{x - y\},$$

$$\frac{\{x\}}{\{y\}} = \{x\}\{y\}^{-1} = \{x\}\{y^{-1}\} = \{xy^{-1}\} = \left\{\frac{x}{y}\right\}.$$

It is customary to blur the distinction between the real complex number $\{x\}$ and the real number x and write $\{x\}$ as x. Thus we write the complex number $\{x\} + i\{y\}$ simply as $x + iy$.

More generally, the sum of two complex numbers is a complex number:

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2); \tag{5.1}$$

and (using the fact that scalar matrices commute with all matrices under matrix multiplication and $\{-1\}A = -A$ if A is a matrix), the product of two complex numbers is a complex number:

$$\begin{aligned}
&(x_1 + iy_1)(x_2 + iy_2) = x_1(x_2 + iy_2) + (iy_1)(x_2 + iy_2) \\
&= x_1x_2 + x_1(iy_2) + (iy_1)x_2 + (iy_1)(iy_2) \\
&= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 \\
&= (x_1x_2 + \{-1\}y_1y_2) + i(x_1y_2 + y_1x_2) \\
&= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2), \qquad (5.2)
\end{aligned}$$

The set $\mathbb{C}$ of complex numbers forms a field under the operations of matrix addition and multiplication. The additive identity is 0, the additive inverse of $x + iy$ is the complex number $(-x) + i(-y)$, the multiplicative identity is 1 and the multiplicative inverse of the non–zero complex number $x + iy$ is the complex number $u + iv$, where

$$u = \frac{x}{x^2 + y^2} \text{ and } v = \frac{-y}{x^2 + y^2}.$$

(If $x + iy \neq 0$, then $x \neq 0$ or $y \neq 0$, so $x^2 + y^2 \neq 0$.)

From equations 5.1 and 5.2, we observe that addition and multiplication of complex numbers is performed just as for real numbers, replacing i^2 by -1, whenever it occurs.

A useful identity satisfied by complex numbers is

$$r^2 + s^2 = (r + is)(r - is).$$

This leads to a method of expressing the ratio of two complex numbers in the form $x + iy$, where x and y are real complex numbers.

$$\begin{aligned}
\frac{x_1 + iy_1}{x_2 + iy_2} &= \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} \\
&= \frac{(x_1x_2 + y_1y_2) + i(-x_1y_2 + y_1x_2)}{x_2^2 + y_2^2}.
\end{aligned}$$

The process is known as *rationalization of the denominator.*

5.2 Calculating with complex numbers

We can now do all the standard linear algebra calculations over the field of complex numbers – find the reduced row–echelon form of an matrix whose elements are complex numbers, solve systems of linear equations, find inverses and calculate determinants.

For example,

$$\begin{aligned}
\begin{vmatrix} 1+i & 2-i \\ 7 & 8-2i \end{vmatrix} &= (1+i)(8-2i) - 7(2-i) \\
&= (8-2i) + i(8-2i) - 14 + 7i \\
&= -4 + 13i \neq 0.
\end{aligned}$$

Then by Cramer's rule, the linear system

$$\begin{aligned} (1+i)z + (2-i)w &= 2+7i \\ 7z + (8-2i)w &= 4-9i \end{aligned}$$

has the unique solution

$$\begin{aligned} z &= \frac{\begin{vmatrix} 2+7i & 2-i \\ 4-9i & 8-2i \end{vmatrix}}{-4+13i} \\ &= \frac{(2+7i)(8-2i)-(4-9i)(2-i)}{-4+13i} \\ &= \frac{2(8-2i)+(7i)(8-2i)-\{(4(2-i)-9i(2-i)\}}{-4+13i} \\ &= \frac{16-4i+56i-14i^2-\{8-4i-18i+9i^2\}}{-4+13i} \\ &= \frac{31+74i}{-4+13i} \\ &= \frac{(31+74i)(-4-13i)}{(-4)^2+13^2} \\ &= \frac{838-699i}{(-4)^2+13^2} \\ &= \frac{838}{185} - \frac{699}{185}i \end{aligned}$$

and similarly $w = \frac{-698}{185} + \frac{229}{185}i$.

An important property enjoyed by complex numbers is that every complex number has a square root:

THEOREM 5.2.1
If w is a non–zero complex number, then the equation $z^2 = w$ has precisely two solutions $z \in \mathbb{C}$.

Proof. Case 1. Suppose $b = 0$. Then if $a > 0$, $z = \sqrt{a}$ is a solution, while if $a < 0$, $i\sqrt{-a}$ is a solution.

Case 2. Suppose $b \neq 0$. Let $z = x + iy$, $w = a + ib$, x, y, a, $b \in \mathbb{R}$. Then the equation $z^2 = w$ becomes

$$(x+iy)^2 = x^2 - y^2 + 2xyi = a + ib,$$

so equating real and imaginary parts gives

$$x^2 - y^2 = a \quad \text{and} \quad 2xy = b.$$

Hence $x \neq 0$ and $y = b/(2x)$. Consequently

$$x^2 - \left(\frac{b}{2x}\right)^2 = a,$$

so $4x^4 - 4ax^2 - b^2 = 0$ and $4(x^2)^2 - 4a(x^2) - b^2 = 0$. Hence

$$x^2 = \frac{4a \pm \sqrt{16a^2 + 16b^2}}{8} = \frac{a \pm \sqrt{a^2 + b^2}}{2}.$$

However $x^2 > 0$, so we must take the $+$ sign, as $a - \sqrt{a^2 + b^2} < 0$. Hence

$$x^2 = \frac{a + \sqrt{a^2 + b^2}}{2}, \quad x = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}.$$

Then y is determined by $y = b/(2x)$.

EXAMPLE 5.2.1 Solve the equation $z^2 = 1 + i$.

Solution. Put $z = x + iy$. Then the equation becomes

$$(x + iy)^2 = x^2 - y^2 + 2xyi = 1 + i,$$

so equating real and imaginary parts gives

$$x^2 - y^2 = 1 \text{ and } 2xy = 1.$$

Hence $x \neq 0$ and $y = b/(2x)$. Consequently

$$x^2 - \left(\frac{1}{2x}\right)^2 = 1,$$

so $4x^4 - 4x^2 - 1 = 0$. Hence

$$x^2 = \frac{4 \pm \sqrt{16 + 16}}{8} = \frac{1 \pm \sqrt{2}}{2}.$$

Hence

$$x^2 = \frac{1 + \sqrt{2}}{2} \quad \text{and} \quad x = \pm\sqrt{\frac{1 + \sqrt{2}}{2}}.$$

Then

$$y = \frac{1}{2x} = \pm\frac{1}{\sqrt{2}\sqrt{1+\sqrt{2}}}.$$

Hence the solutions are

$$z = \pm\left(\sqrt{\frac{1+\sqrt{2}}{2}} + \frac{i}{\sqrt{2}\sqrt{1+\sqrt{2}}}\right).$$

EXAMPLE 5.2.2 Solve the equation $z^2 + (\sqrt{3}+i)z + 1 = 0$.

Solution. Because every complex number has a square root, the familiar formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the solution of the general quadratic equation $az^2 + bz + c = 0$ can be used, where now $a(\neq 0)$, b, $c \in \mathbb{C}$. Hence

$$\begin{aligned} z &= \frac{-(\sqrt{3}+i) \pm \sqrt{(\sqrt{3}+i)^2 - 4}}{2} \\ &= \frac{-(\sqrt{3}+i) \pm \sqrt{(3+2\sqrt{3}i-1) - 4}}{2} \\ &= \frac{-(\sqrt{3}+i) \pm \sqrt{-2+2\sqrt{3}i}}{2}. \end{aligned}$$

Now we have to solve $w^2 = -2 + 2\sqrt{3}i$. Put $w = x + iy$. Then $w^2 = x^2 - y^2 + 2xyi = -2 + 2\sqrt{3}i$ and equating real and imaginary parts gives $x^2 - y^2 = -2$ and $2xy = 2\sqrt{3}$. Hence $y = \sqrt{3}/x$ and so $x^2 - 3/x^2 = -2$. So $x^4 + 2x^2 - 3 = 0$ and $(x^2+3)(x^2-1) = 0$. Hence $x^2 - 1 = 0$ and $x = \pm 1$. Then $y = \pm\sqrt{3}$. Hence $(1+\sqrt{3}i)^2 = -2 + 2\sqrt{3}i$ and the formula for z now becomes

$$\begin{aligned} z &= \frac{-\sqrt{3} - i \pm (1+\sqrt{3}i)}{2} \\ &= \frac{1-\sqrt{3}+(1+\sqrt{3})i}{2} \quad \text{or} \quad \frac{-1-\sqrt{3}-(1+\sqrt{3})i}{2}. \end{aligned}$$

EXAMPLE 5.2.3 Find the cube roots of 1.

Solution. We have to solve the equation $z^3 = 1$, or $z^3 - 1 = 0$. Now $z^3 - 1 = (z-1)(z^2 + z + 1)$. So $z^3 - 1 = 0 \Rightarrow z - 1 = 0$ or $z^2 + z + 1 = 0$. But

$$z^2 + z + 1 = 0 \Rightarrow z = \frac{-1 \pm \sqrt{1^2 - 4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}.$$

So there are 3 cube roots of 1, namely i and $(-1 \pm \sqrt{3}i)/2$.

We state the next theorem without proof. It states that every non–constant polynomial with complex number coefficients has a root in the field of complex numbers.

THEOREM 5.2.2 (Gauss) If $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where $a_n \neq 0$ and $n \geq 1$, then $f(z) = 0$ for some $z \in \mathbb{C}$.

It follows that in view of the *factor* theorem, which states that if $a \in F$ is a root of a polynomial $f(z)$ with coefficients from a field F, then $z - a$ is a factor of $f(z)$, that is $f(z) = (z - a)g(z)$, where the coefficients of $g(z)$ also belong to F. By repeated application of this result, we can factorize any polynomial with complex coefficients into a product of linear factors with complex coefficients:

$$f(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n).$$

There are available a number of computational algorithms for finding good approximations to the roots of a polynomial with complex coefficients.

5.3 Geometric representation of $\mathbb{C}$

Complex numbers can be represented as points in the plane, using the correspondence $x + iy \leftrightarrow (x, y)$. The representation is known as the *Argand diagram* or *complex plane*. The real complex numbers lie on the x–axis, which is then called the *real axis*, while the imaginary numbers lie on the y–axis, which is known as the *imaginary axis*. The complex numbers with positive imaginary part lie in the *upper half plane*, while those with negative imaginary part lie in the *lower half plane*.

Because of the equation

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

complex numbers add vectorially, using the parallellogram law. Similarly, the complex number $z_1 - z_2$ can be represented by the vector from (x_2, y_2) to (x_1, y_1), where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. (See Figure 5.1.)

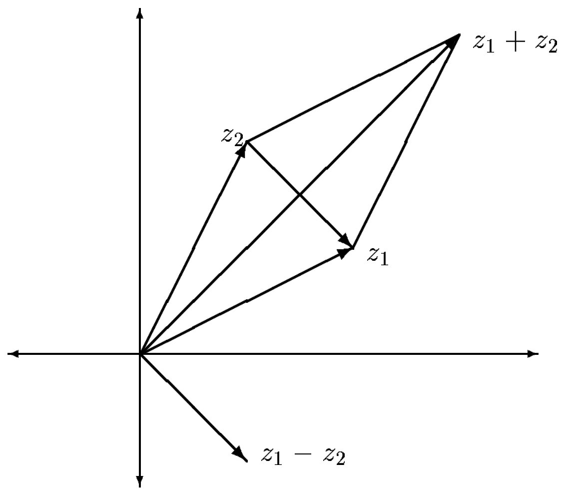

Figure 5.1: Complex addition and subraction.

The geometrical representation of complex numbers can be very useful when complex number methods are used to investigate properties of triangles and circles. It is very important in the branch of calculus known as Complex Function theory, where geometric methods play an important role.

We mention that the line through two distinct points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ has the form $z = (1-t)z_1 + tz_2$, $t \in \mathbb{R}$, where $z = x + iy$ is any point on the line and $z_i = x_i + iy_i$, $i = 1, 2$. For the line has parametric equations

$$x = (1-t)x_1 + tx_2, \quad y = (1-t)y_1 + ty_2$$

and these can be combined into a single equation $z = (1-t)z_1 + tz_2$.

Circles have various equation representations in terms of complex numbers, as will be seen later.

5.4 Complex conjugate

DEFINITION 5.4.1 (Complex conjugate) If $z = x + iy$, the *complex conjugate* of z is the complex number defined by $\overline{z} = x - iy$. Geometrically, the complex conjugate of z is obtained by reflecting z in the real axis (see Figure 5.2).

The following properties of the complex conjugate are easy to verify:

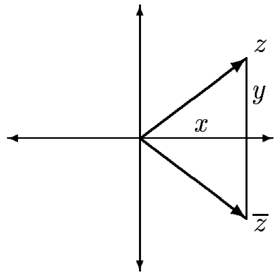

Figure 5.2: The complex conjugate of z: $\overline{z}$.

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$;
2. $\overline{-z} = -\,\overline{z}$.
3. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$;
4. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$;
5. $\overline{(1/z)} = 1/\overline{z}$;
6. $\overline{(z_1/z_2)} = \overline{z_1}/\overline{z_2}$;
7. z is real if and only if $\overline{z} = z$;
8. With the standard convention that the real and imaginary parts are denoted by $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$, we have
$$\mathrm{Re}\, z = \frac{z + \overline{z}}{2}, \quad \mathrm{Im}\, z = \frac{z - \overline{z}}{2i};$$
9. If $z = x + iy$, then $z\overline{z} = x^2 + y^2$.

THEOREM 5.4.1 If $f(z)$ is a polynomial with real coefficients, then its non–real roots occur in complex–conjugate pairs, i.e. if $f(z) = 0$, then $f(\overline{z}) = 0$.

Proof. Suppose $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$, where $a_n, \ldots, a_0$ are real. Then

$$\begin{aligned} 0 = \overline{0} = \overline{f(z)} &= \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} \\ &= \overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \cdots + \overline{a_1}\,\overline{z} + \overline{a_0} \\ &= a_n \overline{z}^n + a_{n-1} \overline{z}^{n-1} + \cdots + a_1 \overline{z} + a_0 \\ &= f(\overline{z}). \end{aligned}$$

EXAMPLE 5.4.1 Discuss the position of the roots of the equation

$$z^4 = -1$$

in the complex plane.

Solution. The equation $z^4 = -1$ has real coefficients and so its roots come in complex conjugate pairs. Also if z is a root, so is $-z$. Also there are clearly no real roots and no imaginary roots. So there must be one root w in the first quadrant, with all remaining roots being given by $\overline{w}$, $-w$ and $-\overline{w}$. In fact, as we shall soon see, the roots lie evenly spaced on the unit circle.

The following theorem is useful in deciding if a polynomial $f(z)$ has a multiple root a; that is if $(z-a)^m$ divides $f(z)$ for some $m \geq 2$. (The proof is left as an exercise.)

THEOREM 5.4.2 If $f(z) = (z-a)^m g(z)$, where $m \geq 2$ and $g(z)$ is a polynomial, then $f'(a) = 0$ and the polynomial and its derivative have a common root.

From theorem 5.4.1 we obtain a result which is very useful in the explicit integration of rational functions (i.e. ratios of polynomials) with real coefficients.

THEOREM 5.4.3 If $f(z)$ is a non–constant polynomial with real coefficients, then $f(z)$ can be factorized as a product of real linear factors and real quadratic factors.

Proof. In general $f(z)$ will have r real roots $z_1, \ldots, z_r$ and $2s$ non–real roots $z_{r+1}, \overline{z}_{r+1}, \ldots, z_{r+s}, \overline{z}_{r+s}$, occurring in complex–conjugate pairs by theorem 5.4.1. Then if a_n is the coefficient of highest degree in $f(z)$, we have the factorization

$$\begin{aligned} f(z) &= a_n(z-z_1)\cdots(z-z_r) \times \\ &\quad \times (z-z_{r+1})(z-\overline{z}_{r+1})\cdots(z-z_{r+s})(z-\overline{z}_{r+s}). \end{aligned}$$

We then use the following identity for $j = r+1, \ldots, r+s$ which in turn shows that paired terms give rise to real quadratic factors:

$$\begin{aligned} (z-z_j)(z-\overline{z}_j) &= z^2 - (z_j + \overline{z}_j)z + z_j\overline{z}_j \\ &= z^2 - 2\mathrm{Re}\, z_j + (x_j^2 + y_j^2), \end{aligned}$$

where $z_j = x_j + iy_j$.

A well–known example of such a factorization is the following:

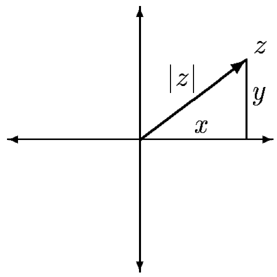

Figure 5.3: The modulus of z: $|z|$.

EXAMPLE 5.4.2 Find a factorization of z^4+1 into real linear and quadratic factors.

Solution. Clearly there are no real roots. Also we have the preliminary factorization $z^4+1=(z^2-i)(z^2+i)$. Now the roots of z^2-i are easily verified to be $\pm(1+i)/\sqrt{2}$, so the roots of z^2+i must be $\pm(1-i)/\sqrt{2}$. In other words the roots are $w=(1+i)/\sqrt{2}$ and $\overline{w}$, $-w$, $-\overline{w}$. Grouping conjugate–complex terms gives the factorization

$$\begin{aligned} z^4+1 &= (z-w)(z-\overline{w})(z+w)(z+\overline{w}) \\ &= (z^2-2z\mathrm{Re}\, w+w\overline{w})(z^2+2z\mathrm{Re}\, w+w\overline{w}) \\ &= (z^2-\sqrt{2}z+1)(z^2+\sqrt{2}z+1). \end{aligned}$$

5.5 Modulus of a complex number

DEFINITION 5.5.1 (Modulus) If $z=x+iy$, the *modulus* of z is the non–negative real number $|z|$ defined by $|z|=\sqrt{x^2+y^2}$. Geometrically, the modulus of z is the distance from z to 0 (see Figure 5.3).

More generally, $|z_1-z_2|$ is the distance between z_1 and z_2 in the complex plane. For

$$\begin{aligned} |z_1-z_2|=|(x_1+iy_1)-(x_2+iy_2)| &= |(x_1-x_2)+i(y_1-y_2)| \\ &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}. \end{aligned}$$

The following properties of the modulus are easy to verify, using the identity $|z|^2=z\overline{z}$:

(i) $|z_1z_2|=|z_1||z_2|$;

(ii) $|z^{-1}| = |z|^{-1}$;

(iii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$.

For example, to prove (i):

$$\begin{aligned} |z_1z_2|^2 &= (z_1z_2)\overline{z_1z_2} = (z_1z_2)\overline{z_1}\,\overline{z_2} \\ &= (z_1\overline{z_1})(z_2\overline{z_2}) = |z_1|^2|z_2|^2 = (|z_1||z_2|)^2. \end{aligned}$$

Hence $|z_1z_2| = |z_1||z_2|$.

EXAMPLE 5.5.1 Find $|z|$ when $z = \dfrac{(1+i)^4}{(1+6i)(2-7i)}$.

Solution.

$$\begin{aligned} |z| &= \frac{|1+i|^4}{|1+6i||2-7i|} \\ &= \frac{(\sqrt{1^2+1^2})^4}{\sqrt{1^2+6^2}\sqrt{2^2+(-7)^2}} \\ &= \frac{4}{\sqrt{37}\sqrt{53}}. \end{aligned}$$

THEOREM 5.5.1 (Ratio formulae) If z lies on the line through z_1 and z_2:

$$z = (1-t)z_1 + tz_2, \quad t \in \mathbb{R},$$

we have the useful *ratio formulae*:

$$\text{(i)} \quad \left|\frac{z-z_1}{z-z_2}\right| = \left|\frac{t}{1-t}\right| \quad \text{if } z \neq z_2,$$

$$\text{(ii)} \quad \left|\frac{z-z_1}{z_1-z_2}\right| = |t|.$$

Circle equations. The equation $|z - z_0| = r$, where $z_0 \in \mathbb{C}$ and $r > 0$, represents the circle centre z_0 and radius r. For example the equation $|z - (1+2i)| = 3$ represents the circle $(x-1)^2 + (y-2)^2 = 9$.

Another useful circle equation is the *circle of Apollonius* :

$$\left|\frac{z-a}{z-b}\right| = \lambda,$$

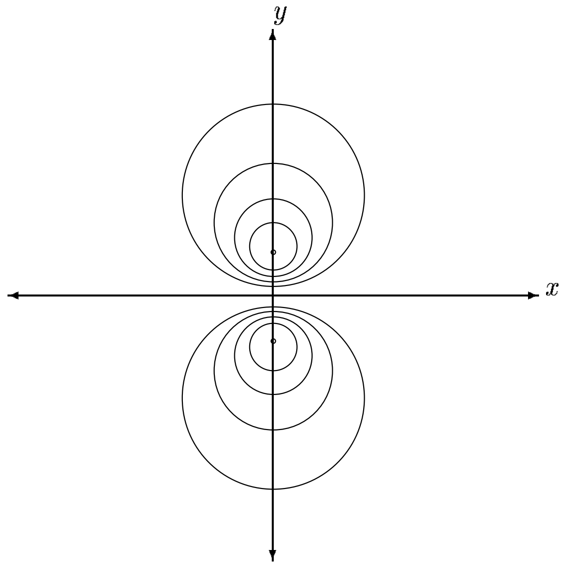

Figure 5.4: Apollonius circles: $\frac{|z+2i|}{|z-2i|} = \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}; \frac{4}{1}, \frac{8}{3}, \frac{2}{1}, \frac{8}{5}$.

where a and b are distinct complex numbers and λ is a positive real number, $\lambda \neq 1$. (If $\lambda = 1$, the above equation represents the perpendicular bisector of the segment joining a and b.)

An algebraic proof that the above equation represents a circle, runs as follows. We use the following identities:

$$\begin{array}{lrcl}
\text{(i)} & |z-a|^2 & = & |z|^2 - 2\mathrm{Re}\,(\overline{z}a) + |a|^2 \\
\text{(ii)} & \mathrm{Re}\,(z_1 \pm z_2) & = & \mathrm{Re}\, z_1 \pm \mathrm{Re}\, z_2 \\
\text{(iii)} & \mathrm{Re}\,(tz) & = & t\mathrm{Re}\, z \text{ if } t \in \mathbb{R}.
\end{array}$$

We have

$$\begin{aligned}
& \left|\frac{z-a}{z-b}\right| = \lambda \Leftrightarrow |z-a|^2 = \lambda^2|z-b|^2 \\
\Leftrightarrow \quad & |z|^2 - 2\mathrm{Re}\,\{\overline{z}a\} + |a|^2 = \lambda^2(|z|^2 - 2\mathrm{Re}\,\{\overline{z}b\} + |b|^2) \\
\Leftrightarrow \quad & (1-\lambda^2)|z|^2 - 2\mathrm{Re}\,\{\overline{z}(a-\lambda^2 b)\} = \lambda^2|b|^2 - |a|^2 \\
\Leftrightarrow \quad & |z|^2 - 2\mathrm{Re}\left\{\overline{z}\left(\frac{a-\lambda^2 b}{1-\lambda^2}\right)\right\} = \frac{\lambda^2|b|^2-|a|^2}{1-\lambda^2} \\
\Leftrightarrow \quad & |z|^2 - 2\mathrm{Re}\left\{\overline{z}\left(\frac{a-\lambda^2 b}{1-\lambda^2}\right)\right\} + \left|\frac{a-\lambda^2 b}{1-\lambda^2}\right|^2 = \frac{\lambda^2|b|^2-|a|^2}{1-\lambda^2} + \left|\frac{a-\lambda^2 b}{1-\lambda^2}\right|^2.
\end{aligned}$$

Now it is easily verified that

$$|a-\lambda^2 b|^2+(1-\lambda^2)(\lambda^2|b|^2-|a|^2)=\lambda^2|a-b|^2.$$

So we obtain

$$\begin{aligned}\left|\frac{z-a}{z-b}\right|=\lambda \quad &\Leftrightarrow \quad \left|z-\left(\frac{a-\lambda^2 b}{1-\lambda^2}\right)\right|^2=\frac{\lambda^2|a-b|^2}{|1-\lambda^2|^2}\\ &\Leftrightarrow \quad \left|z-\left(\frac{a-\lambda^2 b}{1-\lambda^2}\right)\right|=\frac{\lambda|a-b|}{|1-\lambda^2|}.\end{aligned}$$

The last equation represents a circle centre z_0, radius r, where

$$z_0=\frac{a-\lambda^2 b}{1-\lambda^2} \quad \text{and} \quad r=\frac{\lambda|a-b|}{|1-\lambda^2|}.$$

There are two special points on the circle of Apollonius, the points z_1 and z_2 defined by

$$\frac{z_1-a}{z_1-b}=\lambda \quad \text{and} \quad \frac{z_2-a}{z_2-b}=-\lambda,$$

or

$$z_1=\frac{a-\lambda b}{1-\lambda} \quad \text{and} \quad z_2=\frac{a+\lambda b}{1+\lambda}. \tag{5.3}$$

It is easy to verify that z_1 and z_2 are distinct points on the line through a and b and that $z_0=\frac{z_1+z_2}{2}$. Hence the circle of Apollonius is the circle based on the segment z_1, z_2 as diameter.

EXAMPLE 5.5.2 Find the centre and radius of the circle

$$|z-1-i|=2|z-5-2i|.$$

Solution. Method 1. Proceed algebraically and simplify the equation

$$|x+iy-1-i|=2|x+iy-5-2i|$$

or

$$|x-1+i(y-1)|=2|x-5+i(y-2)|.$$

Squaring both sides gives

$$(x-1)^2+(y-1)^2=4((x-5)^2+(y-2)^2),$$

which reduces to the circle equation

$$x^2+y^2-\frac{38}{3}x-\frac{14}{3}y+38=0.$$

Completing the square gives

$$(x - \frac{19}{3})^2 + (y - \frac{7}{3})^2 = \left(\frac{19}{3}\right)^2 + \left(\frac{7}{3}\right)^2 - 38 = \frac{68}{9},$$

so the centre is $(\frac{19}{3}, \frac{7}{3})$ and the radius is $\sqrt{\frac{68}{9}}$.

Method 2. Calculate the diametrical points z_1 and z_2 defined above by equations 5.3:

$$\begin{aligned} z_1 - 1 - i &= 2(z_1 - 5 - 2i) \\ z_2 - 1 - i &= -2(z_2 - 5 - 2i). \end{aligned}$$

We find $z_1 = 9 + 3i$ and $z_2 = (11 + 5i)/3$. Hence the centre z_0 is given by

$$z_0 = \frac{z_1 + z_2}{2} = \frac{19}{3} + \frac{7}{3}i$$

and the radius r is given by

$$r = |z_1 - z_0| = \left|\left(\frac{19}{3} + \frac{7}{3}i\right) - (9 + 3i)\right| = \left|-\frac{8}{3} - \frac{2}{3}i\right| = \frac{\sqrt{68}}{3}.$$

5.6 Argument of a complex number

Let $z = x + iy$ be a non–zero complex number, $r = |z| = \sqrt{x^2 + y^2}$. Then we have $x = r\cos\theta$, $y = r\sin\theta$, where θ is the angle made by z with the positive x–axis. So θ is unique up to addition of a multiple of 2π radians.

DEFINITION 5.6.1 (Argument) Any number θ satisfying the above pair of equations is called an *argument* of z and is denoted by $\arg z$. The particular argument of z lying in the range $-\pi < \theta \le \pi$ is called the *principal argument* of z and is denoted by $\operatorname{Arg} z$ (see Figure 5.5).

We have $z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$ and this representation of z is called the *polar representation* or *modulus–argument form* of z.

EXAMPLE 5.6.1 $\operatorname{Arg} 1 = 0$, $\operatorname{Arg}(-1) = \pi$, $\operatorname{Arg} i = \frac{\pi}{2}$, $\operatorname{Arg}(-i) = -\frac{\pi}{2}$.

We note that $y/x = \tan\theta$ if $x \ne 0$, so θ is determined by this equation up to a multiple of π. In fact

$$\operatorname{Arg} z = \tan^{-1}\frac{y}{x} + k\pi,$$

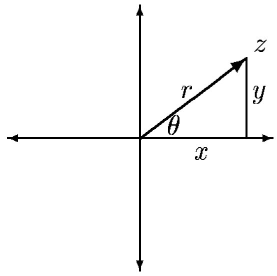

Figure 5.5: The argument of z: $\arg z = \theta$.

where $k = 0$ if $x > 0$; $k = 1$ if $x < 0$, $y > 0$; $k = -1$ if $x < 0$, $y < 0$.

To determine $\operatorname{Arg} z$ graphically, it is simplest to draw the triangle formed by the points 0, x, z on the complex plane, mark in the positive acute angle α between the rays $0, x$ and $0, z$ and determine $\operatorname{Arg} z$ geometrically, using the fact that $\alpha = \tan^{-1}(|y|/|x|)$, as in the following examples:

EXAMPLE 5.6.2 Determine the principal argument of z for the followig complex numbers:

$$z = 4 + 3i,\ -4 + 3i,\ -4 - 3i,\ 4 - 3i.$$

Solution. Referring to Figure 5.6, we see that $\operatorname{Arg} z$ has the values

$$\alpha,\ \pi - \alpha,\ -\pi + \alpha,\ -\alpha,$$

where $\alpha = \tan^{-1} \frac{3}{4}$.

An important property of the argument of a complex number states that the sum of the arguments of two non–zero complex numbers is an argument of their product:

THEOREM 5.6.1 If θ_1 and θ_2 are arguments of z_1 and z_2, then $\theta_1 + \theta_2$ is an argument of $z_1 z_2$.

Proof. Let z_1 and z_2 have polar representations $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)), \end{aligned}$$

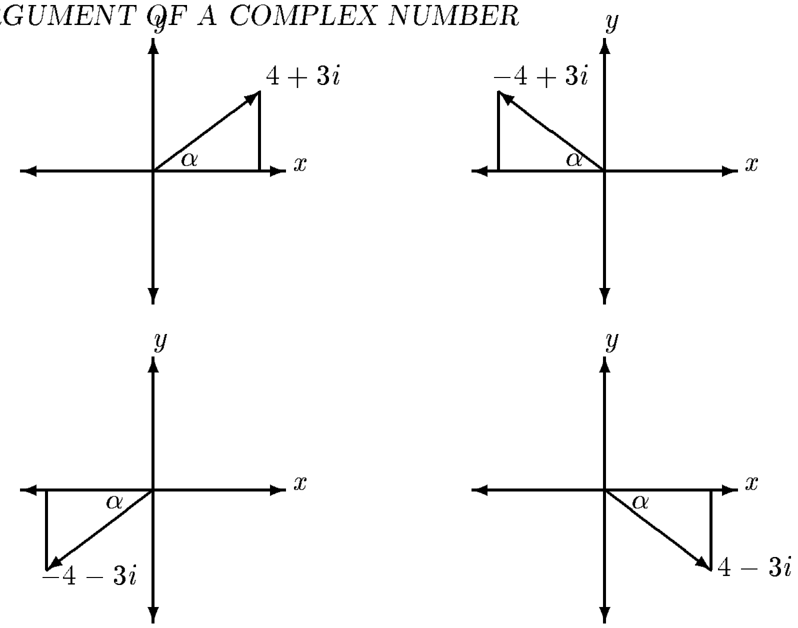

Figure 5.6: Argument examples.

which is the polar representation of z_1z_2, as $r_1r_2 = |z_1||z_2| = |z_1z_2|$. Hence $\theta_1 + \theta_2$ is an argument of z_1z_2.

An easy induction gives the following generalization to a product of n complex numbers:

COROLLARY 5.6.1 If $\theta_1, \ldots, \theta_n$ are arguments for $z_1, \ldots, z_n$ respectively, then $\theta_1 + \cdots + \theta_n$ is an argument for $z_1 \cdots z_n$.

Taking $\theta_1 = \cdots = \theta_n = \theta$ in the previous corollary gives

COROLLARY 5.6.2 If θ is an argument of z, then $n\theta$ is an argument for z^n.

THEOREM 5.6.2 If θ is an argument of the non–zero complex number z, then $-\theta$ is an argument of z^{-1}.

Proof. Let θ be an argument of z. Then $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$. Hence

$$\begin{aligned} z^{-1} &= r^{-1}(\cos\theta + i\sin\theta)^{-1} \\ &= r^{-1}(\cos\theta - i\sin\theta) \\ &= r^{-1}(\cos(-\theta) + i\sin(-\theta)). \end{aligned}$$

Now $r^{-1} = |z|^{-1} = |z^{-1}|$, so $-\theta$ is an argument of z^{-1}.

COROLLARY 5.6.3 If θ_1 and θ_2 are arguments of z_1 and z_2, then $\theta_1 - \theta_2$ is an argument of z_1/z_2.

In terms of principal arguments, we have the following equations:

$$\begin{array}{lrcl}
\text{(i)} & \operatorname{Arg}(z_1z_2) & = & \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2k_1\pi, \\
\text{(ii)} & \operatorname{Arg}(z^{-1}) & = & -\operatorname{Arg} z + 2k_2\pi, \\
\text{(iii)} & \operatorname{Arg}(z_1/z_2) & = & \operatorname{Arg} z_1 - \operatorname{Arg} z_2 + 2k_3\pi, \\
\text{(iv)} & \operatorname{Arg}(z_1\cdots z_n) & = & \operatorname{Arg} z_1 + \cdots + \operatorname{Arg} z_n + 2k_4\pi, \\
\text{(v)} & \operatorname{Arg}(z^n) & = & n\operatorname{Arg} z + 2k_5\pi,
\end{array}$$

where k_1, k_2, k_3, k_4, k_5 are integers.

In numerical examples, we can write (i), for example, as

$$\operatorname{Arg}(z_1z_2) \equiv \operatorname{Arg} z_1 + \operatorname{Arg} z_2.$$

EXAMPLE 5.6.3 Find the modulus and principal argument of

$$z = \left(\frac{\sqrt{3}+i}{1+i}\right)^{17}$$

and hence express z in modulus–argument form.

Solution. $|z| = \dfrac{|\sqrt{3}+i|^{17}}{|1+i|^{17}} = \dfrac{2^{17}}{(\sqrt{2})^{17}} = 2^{17/2}$.

$$\begin{aligned}
\operatorname{Arg} z &\equiv 17\operatorname{Arg}\left(\frac{\sqrt{3}+i}{1+i}\right) \\
&= 17(\operatorname{Arg}(\sqrt{3}+i) - \operatorname{Arg}(1+i)) \\
&= 17\left(\frac{\pi}{6} - \frac{\pi}{4}\right) = \frac{-17\pi}{12}.
\end{aligned}$$

Hence $\operatorname{Arg} z = \left(\frac{-17\pi}{12}\right) + 2k\pi$, where k is an integer. We see that $k = 1$ and hence $\operatorname{Arg} z = \frac{7\pi}{12}$. Consequently $z = 2^{17/2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$.

DEFINITION 5.6.2 If θ is a real number, then we define $e^{i\theta}$ by

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

More generally, if $z = x + iy$, then we define e^z by

$$e^z = e^x e^{iy}.$$

For example,

$$e^{\frac{i\pi}{2}} = i,\ e^{i\pi} = -1,\ e^{-\frac{i\pi}{2}} = -i.$$

The following properties of the complex exponential function are left as exercises:

THEOREM 5.6.3

$$\begin{array}{lrcl}
\text{(i)} & e^{z_1}e^{z_2} & = & e^{z_1+z_2}, \\
\text{(ii)} & e^{z_1}\cdots e^{z_n} & = & e^{z_1+\cdots+z_n}, \\
\text{(iii)} & e^{z} & \neq & 0, \\
\text{(iv)} & (e^{z})^{-1} & = & e^{-z}, \\
\text{(v)} & e^{z_1}/e^{z_2} & = & e^{z_1-z_2}, \\
\text{(vi)} & \overline{e^{z}} & = & e^{\overline{z}}.
\end{array}$$

THEOREM 5.6.4 The equation

$$e^z = 1$$

has the complete solution $z = 2k\pi i$, $k \in \mathbb{Z}$.

Proof. First we observe that

$$e^{2k\pi i} = \cos\,(2k\pi) + i \sin\,(2k\pi) = 1.$$

Conversely, suppose $e^z = 1$, $z = x + iy$. Then $e^x(\cos y + i \sin y) = 1$. Hence $e^x \cos y = 1$ and $e^x \sin y = 0$. Hence $\sin y = 0$ and so $y = n\pi$, $n \in \mathbb{Z}$. Then $e^x \cos\,(n\pi) = 1$, so $e^x(-1)^n = 1$, from which follows $(-1)^n = 1$ as $e^x > 0$. Hence $n = 2k$, $k \in \mathbb{Z}$ and $e^x = 1$. Hence $x = 0$ and $z = 2k\pi i$.

5.7 De Moivre's theorem

The next theorem has many uses and is a special case of theorem 5.6.3(ii). Alternatively it can be proved directly by induction on n.

THEOREM 5.7.1 (De Moivre) If n is a positive integer, then

$$(\cos\theta + i\sin\theta)^n = \cos\, n\theta + i \sin\, n\theta.$$

As a first application, we consider the equation $z^n = 1$.

THEOREM 5.7.2 The equation $z^n = 1$ has n distinct solutions, namely the complex numbers $\zeta_k = e^{\frac{2k\pi i}{n}}$, $k = 0, 1, \ldots, n-1$. These lie equally spaced on the unit circle $|z| = 1$ and are obtained by starting at 1, moving round the circle anti–clockwise, incrementing the argument in steps of $\frac{2\pi}{n}$. (See Figure 5.7)

We notice that the roots are the powers of the special root $\zeta = e^{\frac{2\pi i}{n}}$.

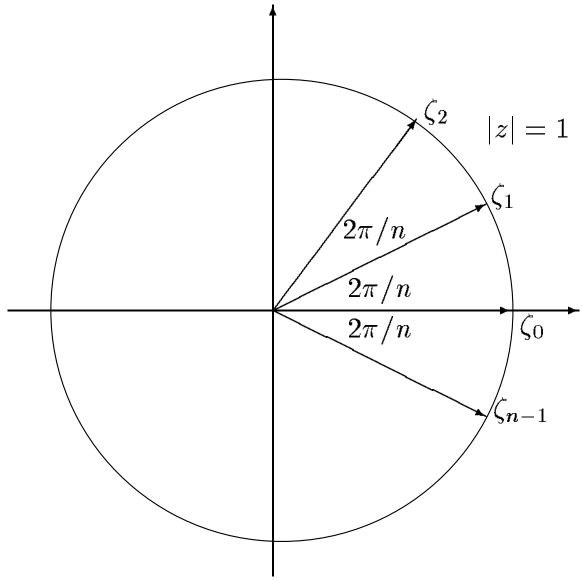

Figure 5.7: The nth roots of unity.

Proof. With ζ_k defined as above,

$$\zeta_k^n = \left(e^{\frac{2k\pi i}{n}}\right)^n = e^{\frac{2k\pi i}{n}n} = 1,$$

by De Moivre's theorem. However $|\zeta_k| = 1$ and $\arg \zeta_k = \frac{2k\pi}{n}$, so the complex numbers ζ_k, $k = 0, 1, \ldots, n-1$, lie equally spaced on the unit circle. Consequently these numbers must be precisely all the roots of $z^n - 1$. For the polynomial $z^n - 1$, being of degree n over a field, can have at most n distinct roots in that field.

The more general equation $z^n = a$, where $a \in, \mathbb{C}, a \neq 0$, can be reduced to the previous case:

Let α be argument of z, so that $a = |a|e^{i\alpha}$. Then if $w = |a|^{1/n}e^{\frac{i\alpha}{n}}$, we have

$$\begin{aligned} w^n &= \left(|a|^{1/n}e^{\frac{i\alpha}{n}}\right)^n \\ &= (|a|^{1/n})^n \left(e^{\frac{i\alpha}{n}}\right)^n \\ &= |a|e^{i\alpha} = a. \end{aligned}$$

So w is a particular solution. Substituting for a in the original equation, we get $z^n = w^n$, or $(z/w)^n = 1$. Hence the complete solution is $z/w =$

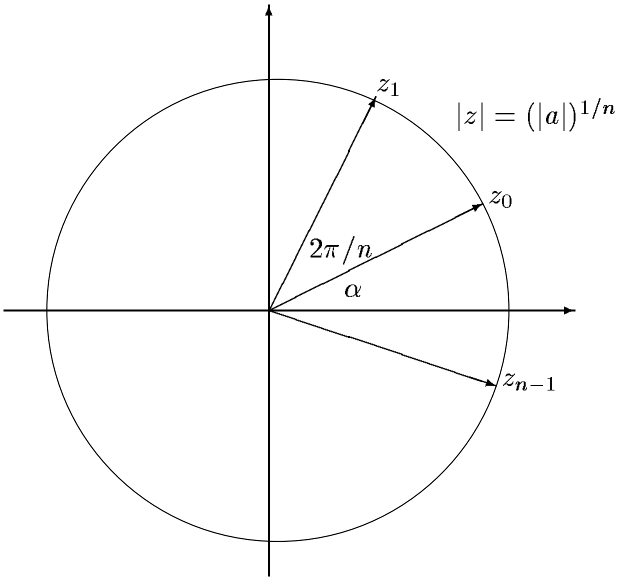

Figure 5.8: The roots of $z^n = a$.

$e^{\frac{2k\pi i}{n}}$, $k = 0, 1, \ldots, n-1$, or

$$z_k = |a|^{1/n} e^{\frac{i\alpha}{n}} e^{\frac{2k\pi i}{n}} = |a|^{1/n} e^{\frac{i(\alpha+2k\pi)}{n}}, \tag{5.4}$$

$k = 0, 1, \ldots, n-1$. So the roots are equally spaced on the circle

$$|z| = |a|^{1/n}$$

and are generated from the special solution having argument equal to $(\arg a)/n$, by incrementing the argument in steps of $2\pi/n$. (See Figure 5.8.)

EXAMPLE 5.7.1 Factorize the polynomial $z^5 - 1$ as a product of real linear and quadratic factors.

Solution. The roots are 1, $e^{\frac{2\pi i}{5}}$, $e^{\frac{-2\pi i}{5}}$, $e^{\frac{4\pi i}{5}}$, $e^{\frac{-4\pi i}{5}}$, using the fact that non–real roots come in conjugate–complex pairs. Hence

$$z^5 - 1 = (z-1)(z - e^{\frac{2\pi i}{5}})(z - e^{\frac{-2\pi i}{5}})(z - e^{\frac{4\pi i}{5}})(z - e^{\frac{-4\pi i}{5}}).$$

Now

$$\begin{aligned}(z - e^{\frac{2\pi i}{5}})(z - e^{\frac{-2\pi i}{5}}) &= z^2 - z(e^{\frac{2\pi i}{5}} + e^{\frac{-2\pi i}{5}}) + 1 \\ &= z^2 - 2z\cos\tfrac{2\pi}{5} + 1.\end{aligned}$$

Similarly

$$(z - e^{\frac{4\pi i}{5}})(z - e^{\frac{-4\pi i}{5}}) = z^2 - 2z\cos\tfrac{4\pi}{5} + 1.$$

This gives the desired factorization.

EXAMPLE 5.7.2 Solve $z^3 = i$.

Solution. $|i| = 1$ and $\mathrm{Arg}\, i = \frac{\pi}{2} = \alpha$. So by equation 5.4, the solutions are

$$z_k = |i|^{1/3} e^{\frac{i(\alpha+2k\pi)}{3}},\ k = 0,\ 1,\ 2.$$

First, $k = 0$ gives

$$z_0 = e^{\frac{i\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}.$$

Next, $k = 1$ gives

$$z_1 = e^{\frac{5\pi i}{6}} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = \frac{-\sqrt{3}}{2} + \frac{i}{2}.$$

Finally, $k = 2$ gives

$$z_1 = e^{\frac{9\pi i}{6}} = \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6} = -i.$$

We finish this chapter with two more examples of De Moivre's theorem.

EXAMPLE 5.7.3 If

$$\begin{aligned} C &= 1 + \cos\theta + \cdots + \cos\,(n-1)\theta, \\ S &= \sin\theta + \cdots + \sin\,(n-1)\theta, \end{aligned}$$

prove that

$$C = \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\cos\tfrac{(n-1)\theta}{2} \text{ and } S = \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\sin\tfrac{(n-1)\theta}{2},$$

if $\theta \neq 2k\pi$, $k \in \mathbb{Z}$.

Solution.

$$\begin{aligned}
C + iS &= 1 + (\cos\theta + i\sin\theta) + \cdots + (\cos\ (n-1)\theta + i\sin\ (n-1)\theta) \\
&= 1 + e^{i\theta} + \cdots + e^{i(n-1)\theta} \\
&= 1 + z + \cdots + z^{n-1},\ \text{where } z = e^{i\theta} \\
&= \frac{1-z^n}{1-z},\ \text{if } z \neq 1,\ \text{i.e. } \theta \neq 2k\pi, \\
&= \frac{1-e^{in\theta}}{1-e^{i\theta}} = \frac{e^{\frac{in\theta}{2}}(e^{\frac{-in\theta}{2}} - e^{\frac{in\theta}{2}})}{e^{\frac{i\theta}{2}}(e^{\frac{-i\theta}{2}} - e^{\frac{i\theta}{2}})} \\
&= e^{i(n-1)\frac{\theta}{2}} \frac{\sin\ \frac{n\theta}{2}}{\sin\ \frac{\theta}{2}} \\
&= (\cos\ (n-1)\tfrac{\theta}{2} + i\sin\ (n-1)\tfrac{\theta}{2}) \frac{\sin\ \frac{n\theta}{2}}{\sin\ \frac{\theta}{2}}.
\end{aligned}$$

The result follows by equating real and imaginary parts.

EXAMPLE 5.7.4 Express $\cos\ n\theta$ and $\sin\ n\theta$ in terms of $\cos\theta$ and $\sin\theta$, using the equation $\cos\ n\theta + \sin\ n\theta = (\cos\theta + i\sin\theta)^n$.

Solution. The binomial theorem gives

$$\begin{aligned}
(\cos\theta + i\sin\theta)^n = \cos^n\theta + \tbinom{n}{1}\cos^{n-1}\theta(i\sin\theta) + \tbinom{n}{2}\cos^{n-2}\theta(i\sin\theta)^2 + \cdots \\
+ (i\sin\theta)^n.
\end{aligned}$$

Equating real and imaginary parts gives

$$\begin{aligned}
\cos\ n\theta &= \cos^n\theta - \tbinom{n}{2}\cos^{n-2}\theta\sin^2\theta + \cdots \\
\sin\ n\theta &= \tbinom{n}{1}\cos^{n-1}\theta\sin\theta - \tbinom{n}{3}\cos^{n-3}\theta\sin^3\theta + \cdots.
\end{aligned}$$

5.8 PROBLEMS

1. Express the following complex numbers in the form $x + iy$, x, y real:

   (i) $(-3+i)(14-2i)$; (ii) $\dfrac{2+3i}{1-4i}$; (iii) $\dfrac{(1+2i)^2}{1-i}$.

   [Answers: (i) $-40+20i$; (ii) $-\frac{10}{17}+\frac{11}{17}i$; (iii) $-\frac{7}{2}+\frac{i}{2}$.]

2. Solve the following equations:

$$\text{(i)} \qquad iz + (2-10i)z = 3z + 2i,$$

$$\text{(ii)} \qquad \begin{aligned} (1+i)z + (2-i)w &= -3i \\ (1+2i)z + (3+i)w &= 2+2i. \end{aligned}$$

[Answers:(i) $z = -\frac{9}{41} - \frac{i}{41}$; (ii) $z = -1 + 5i$, $w = \frac{19}{5} - \frac{8i}{5}$.]

3. Express $1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{99}$ in the form $x + iy$, x, y real. [Answer: $(1 + 2^{50})i$.]

4. Solve the equations: (i) $z^2 = -8 - 6i$; (ii) $z^2 - (3+i)z + 4 + 3i = 0$.

   [Answers: (i) $z = \pm(1-3i)$; (ii) $z = 2 - i,\ 1 + 2i$.]

5. Find the modulus and principal argument of each of the following complex numbers:

   (i) $4 + i$; (ii) $-\frac{3}{2} - \frac{i}{2}$; (iii) $-1 + 2i$; (iv) $\frac{1}{2}(-1 + i\sqrt{3})$.

   [Answers: (i) $\sqrt{17}$, $\tan^{-1}\frac{1}{4}$; (ii) $\frac{\sqrt{10}}{2}$, $-\pi + \tan^{-1}\frac{1}{3}$; (iii) $\sqrt{5}$, $\pi - \tan^{-1} 2$.]

6. Express the following complex numbers in modulus-argument form:

   (i) $z = (1+i)(1+i\sqrt{3})(\sqrt{3} - i)$.

   (ii) $z = \dfrac{(1+i)^5(1-i\sqrt{3})^5}{(\sqrt{3}+i)^4}$.

   [Answers:
   (i) $z = 4\sqrt{2}(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$; (ii) $z = 2^{7/2}(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12})$.]

7. (i) If $z = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ and $w = 3(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$, find the polar form of
   (a) zw; (b) $\frac{z}{w}$; (c) $\frac{w}{z}$; (d) $\frac{z^5}{w^2}$.

   (ii) Express the following complex numbers in the form $x + iy$:
   (a) $(1+i)^{12}$; (b) $\left(\frac{1-i}{\sqrt{2}}\right)^{-6}$.

   [Answers: (i): (a) $6(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$; (b) $\frac{2}{3}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$;

   (c) $\frac{3}{2}(\cos -\frac{\pi}{12} + i\sin -\frac{\pi}{12})$; (d) $\frac{32}{9}(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12})$;

   (ii): (a) -64; (b) $-i$.]

8. Solve the equations:

$$\text{(i) } z^2 = 1 + i\sqrt{3}; \text{ (ii) } z^4 = i; \text{ (iii) } z^3 = -8i; \text{ (iv) } z^4 = 2 - 2i.$$

[Answers: (i) $z = \pm\frac{(\sqrt{3}+i)}{\sqrt{2}}$; (ii) $i^k(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}), k = 0, 1, 2, 3$; (iii) $z = 2i, -\sqrt{3} - i, \sqrt{3} - i$; (iv) $z = i^k 2^{\frac{3}{8}}(\cos\frac{\pi}{16} - i\sin\frac{\pi}{16}), k = 0, 1, 2, 3.$]

9. Find the reduced row–echelon form of the complex matrix

$$\begin{bmatrix} 2+i & -1+2i & 2 \\ 1+i & -1+i & 1 \\ 1+2i & -2+i & 1+i \end{bmatrix}.$$

[Answer: $\begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.]

10. (i) Prove that the line equation $lx + my = n$ is equivalent to

$$\overline{p}z + p\overline{z} = 2n,$$

where $p = l + im$.

(ii) Use (ii) to deduce that reflection in the straight line

$$\overline{p}z + p\overline{z} = n$$

is described by the equation

$$\overline{p}w + p\overline{z} = n.$$

[Hint: The complex number $l + im$ is perpendicular to the given line.]

(iii) Prove that the line $|z - a| = |z - b|$ may be written as $\overline{p}z + p\overline{z} = n$, where $p = b - a$ and $n = |b|^2 - |a|^2$. Deduce that if z lies on the Apollonius circle $\frac{|z-a|}{|z-b|} = \lambda$, then w, the reflection of z in the line $|z - a| = |z - b|$, lies on the Apollonius circle $\frac{|z-a|}{|z-b|} = \frac{1}{\lambda}$.

11. Let a and b be distinct complex numbers and $0 < \alpha < \pi$.

(i) Prove that each of the following sets in the complex plane represents a circular arc and sketch the circular arcs on the same diagram:

$$\operatorname{Arg}\frac{z-a}{z-b} = \alpha,\ -\alpha,\ \pi-\alpha,\ \alpha-\pi.$$

Also show that $\operatorname{Arg}\dfrac{z-a}{z-b} = \pi$ represents the line segment joining a and b, while $\operatorname{Arg}\dfrac{z-a}{z-b} = 0$ represents the remaining portion of the line through a and b.

(ii) Use (i) to prove that four distinct points z_1, z_2, z_3, z_4 are concyclic or collinear, if and only if the *cross–ratio*

$$\frac{z_4 - z_1}{z_4 - z_2}\Big/\frac{z_3 - z_1}{z_3 - z_2}$$

is real.

(iii) Use (ii) to derive *Ptolemy's* Theorem: Four distinct points A, B, C, D are concyclic or collinear, if and only if one of the following holds:

$$\begin{aligned} AB\cdot CD + BC\cdot AD &= AC\cdot BD \\ BD\cdot AC + AD\cdot BC &= AB\cdot CD \\ BD\cdot AC + AB\cdot CD &= AD\cdot BC. \end{aligned}$$