Chapter 4

DETERMINANTS

DEFINITION 4.0.1 If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we define the *determinant* of A, (also denoted by $\det A$,) to be the scalar

$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$

The notation $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is also used for the determinant of A.

If A is a real matrix, there is a geometrical interpretation of $\det A$. If $P = (x_1,\, y_1)$ and $Q = (x_2,\, y_2)$ are points in the plane, forming a triangle with the origin $O = (0,\, 0)$, then apart from sign, $\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ is the area of the triangle OPQ. For, using polar coordinates, let $x_1 = r_1 \cos\theta_1$ and $y_1 = r_1 \sin\theta_1$, where $r_1 = OP$ and θ_1 is the angle made by the ray $\overrightarrow{OP}$ with the positive x–axis. Then triangle OPQ has area $\frac{1}{2}OP \cdot OQ \sin\alpha$, where $\alpha = \angle POQ$. If triangle OPQ has anti–clockwise orientation, then the ray $\overrightarrow{OQ}$ makes angle $\theta_2 = \theta_1 + \alpha$ with the positive x–axis. (See Figure 4.1.)

Also $x_2 = r_2 \cos\theta_2$ and $y_2 = r_2 \sin\theta_2$. Hence

$$\begin{aligned}
\text{Area}\, OPQ &= \frac{1}{2}OP \cdot OQ \sin\alpha \\
&= \frac{1}{2}OP \cdot OQ \sin(\theta_2 - \theta_1) \\
&= \frac{1}{2}OP \cdot OQ (\sin\theta_2 \cos\theta_1 - \cos\theta_2 \sin\theta_1) \\
&= \frac{1}{2}(OQ \sin\theta_2 \cdot OP \cos\theta_1 - OQ \cos\theta_2 \cdot OP \sin\theta_1)
\end{aligned}$$

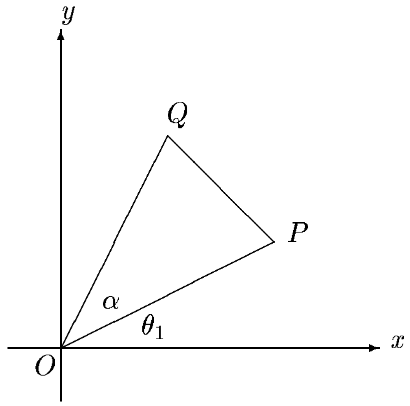

Figure 4.1: Area of triangle OPQ.

$$
\begin{aligned}
&= \frac{1}{2}(y_2x_1 - x_2y_1) \\
&= \frac{1}{2}\left|\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \end{array}\right|.
\end{aligned}
$$

Similarly, if triangle OPQ has clockwise orientation, then its area equals $-\frac{1}{2}\left|\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \end{array}\right|$.

For a general triangle $P_1P_2P_3$, with $P_i = (x_i, y_i)$, $i = 1, 2, 3$, we can take P_1 as the origin. Then the above formula gives

$$
\frac{1}{2}\left|\begin{array}{cc} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{array}\right| \quad \text{or} \quad -\frac{1}{2}\left|\begin{array}{cc} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{array}\right|,
$$

according as vertices $P_1P_2P_3$ are anti–clockwise or clockwise oriented.

We now give a recursive definition of the determinant of an $n \times n$ matrix $A = [a_{ij}]$, $n \geq 3$.

DEFINITION 4.0.2 (Minor) Let $M_{ij}(A)$ (or simply M_{ij} if there is no ambiguity) denote the determinant of the $(n-1) \times (n-1)$ submatrix of A formed by deleting the i–th row and j–th column of A. ($M_{ij}(A)$ is called the (i, j) *minor* of A.)

Assume that the determinant function has been defined for matrices of size $(n-1) \times (n-1)$. Then $\det A$ is defined by the so–called *first–row Laplace* *expansion*:

$$\begin{aligned}\det A &= a_{11}M_{11}(A) - a_{12}M_{12}(A) + \ldots + (-1)^{1+n}M_{1n}(A)\\ &= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}M_{1j}(A).\end{aligned}$$

For example, if $A = [a_{ij}]$ is a 3×3 matrix, the Laplace expansion gives

$$\begin{aligned}&\det A = a_{11}M_{11}(A) - a_{12}M_{12}(A) + a_{13}M_{13}(A)\\ =\;& a_{11}\begin{vmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}\\ =\;& a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})\\ =\;& a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.\end{aligned}$$

(The recursive definition also works for 2×2 determinants, if we define the determinant of a 1×1 matrix $[t]$ to be the scalar t:

$$\det A = a_{11}M_{11}(A) - a_{12}M_{12}(A) = a_{11}a_{22} - a_{12}a_{21}.)$$

EXAMPLE 4.0.1 If $P_1P_2P_3$ is a triangle with $P_i = (x_i, y_i)$, $i = 1, 2, 3$, then the area of triangle $P_1P_2P_3$ is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix} \quad \text{or} \quad -\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix},$$

according as the orientation of $P_1P_2P_3$ is anti–clockwise or clockwise.

For from the definition of 3×3 determinants, we have

$$\begin{aligned}\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix} &= \frac{1}{2}\left(x_1\begin{vmatrix} y_2 & 1\\ y_3 & 1\end{vmatrix} - y_1\begin{vmatrix} x_2 & 1\\ x_3 & 1\end{vmatrix} + \begin{vmatrix} x_2 & y_2\\ x_3 & y_3\end{vmatrix}\right)\\ &= \frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1\\ x_3 - x_1 & y_3 - y_1\end{vmatrix}.\end{aligned}$$

One property of determinants that follows immediately from the definition is the following:

THEOREM 4.0.1 If a row of a matrix is zero, then the value of the determinant is zero.

(The corresponding result for columns also holds, but here a simple proof by induction is needed.)

One of the simplest determinants to evaluate is that of a lower triangular matrix.

THEOREM 4.0.2 Let $A = [a_{ij}]$, where $a_{ij} = 0$ if $i < j$. Then

$$\det \mathrm{A} = a_{11}a_{22}\dots a_{nn}. \tag{4.1}$$

An important special case is when A is a diagonal matrix.

If $A = $diag $(a_1, \dots, a_n)$ then $\det A = a_1 \dots a_n$. In particular, for a scalar matrix tI_n, we have det $(tI_n) = t^n$.

Proof. Use induction on the size n of the matrix.

The result is true for $n = 2$. Now let $n > 2$ and assume the result true for matrices of size $n - 1$. If A is $n \times n$, then expanding $\det A$ along row 1 gives

$$\begin{aligned} \det A &= a_{11} \begin{vmatrix} a_{22} & 0 & \dots & 0 \\ a_{32} & a_{33} & \dots & 0 \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \\ &= a_{11}(a_{22}\dots a_{nn}) \end{aligned}$$

by the induction hypothesis.

If A is upper triangular, equation 4.1 remains true and the proof is again an exercise in induction, with the slight difference that the column version of theorem 4.0.1 is needed.

REMARK 4.0.1 It can be shown that the expanded form of the determinant of an $n \times n$ matrix A consists of $n!$ signed products $\pm a_{1i_1}a_{2i_2}\dots a_{ni_n}$, where $(i_1, i_2, \dots, i_n)$ is a permutation of $(1, 2, \dots, n)$, the sign being 1 or -1, according as the number of *inversions* of $(i_1, i_2, \dots, i_n)$ is even or odd. An inversion occurs when $i_r > i_s$ but $r < s$. (The proof is not easy and is omitted.)

The definition of the determinant of an $n \times n$ matrix was given in terms of the first–row expansion. The next theorem says that we can expand the determinant along any row or column. (The proof is not easy and is omitted.)

THEOREM 4.0.3

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}(A)$$

for $i = 1, \ldots, n$ (the so–called i–th row expansion) and

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}(A)$$

for $j = 1, \ldots, n$ (the so–called j–th column expansion).

REMARK 4.0.2 The expression $(-1)^{i+j}$ obeys the chess–board pattern of signs:

$$\begin{bmatrix} + & - & + & \ldots \\ - & + & - & \ldots \\ + & - & + & \ldots \\ \vdots & & & \end{bmatrix}.$$

The following theorems can be proved by straightforward inductions on the size of the matrix:

THEOREM 4.0.4 A matrix and its transpose have equal determinants; that is

$$\det A^t = \det A.$$

THEOREM 4.0.5 If two rows of a matrix are equal, the determinant is zero. Similarly for columns.

THEOREM 4.0.6 If two rows of a matrix are interchanged, the determinant changes sign.

EXAMPLE 4.0.2 If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are distinct points, then the line through P_1 and P_2 has equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

For, expanding the determinant along row 1, the equation becomes

$$ax + by + c = 0,$$

where

$$a = \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} = y_1 - y_2 \text{ and } b = -\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} = x_2 - x_1.$$

This represents a line, as not both a and b can be zero. Also this line passes through P_i, $i = 1, 2$. For the determinant has its first and i–th rows equal if $x = x_i$ and $y = y_i$ and is consequently zero.

There is a corresponding formula in three–dimensional geometry. If P_1, P_2, P_3 are non–collinear points in three–dimensional space, with $P_i = (x_i, y_i, z_i)$, $i = 1, 2, 3$, then the equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

represents the plane through P_1, P_2, P_3. For, expanding the determinant along row 1, the equation becomes $ax + by + cz + d = 0$, where

$$a = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \quad b = -\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}, \quad c = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

As we shall see in chapter 6, this represents a plane if at least one of a, b, c is non–zero. However, apart from sign and a factor $\frac{1}{2}$, the determinant expressions for a, b, c give the values of the areas of projections of triangle $P_1P_2P_3$ on the (y, z), (x, z) and (x, y) planes, respectively. Geometrically, it is then clear that at least one of a, b, c is non–zero. It is also possible to give an algebraic proof of this fact.

Finally, the plane passes through P_i, $i = 1, 2, 3$ as the determinant has its first and i–th rows equal if $x = x_i$, $y = y_i$, $z = z_i$ and is consequently zero. We now work towards proving that a matrix is non–singular if its determinant is non–zero.

DEFINITION 4.0.3 (Cofactor) The (i, j) cofactor of A, denoted by $C_{ij}(A)$ (or C_{ij} if there is no ambiguity) is defined by

$$C_{ij}(A) = (-1)^{i+j} M_{ij}(A).$$

REMARK 4.0.3 It is important to notice that $C_{ij}(A)$, like $M_{ij}(A)$, does not depend on a_{ij}. Use will be made of this observation presently.

In terms of the cofactor notation, Theorem 3.0.2 takes the form

THEOREM 4.0.7

$$\det A = \sum_{j=1}^{n} a_{ij}C_{ij}(A)$$

for $i = 1, \ldots, n$ and

$$\det A = \sum_{i=1}^{n} a_{ij}C_{ij}(A)$$

for $j = 1, \ldots, n$.

Another result involving cofactors is

THEOREM 4.0.8 Let A be an $n \times n$ matrix. Then

$$(a) \quad \sum_{j=1}^{n} a_{ij}C_{kj}(A) = 0 \quad \text{if } i \neq k.$$

Also

$$(b) \quad \sum_{i=1}^{n} a_{ij}C_{ik}(A) = 0 \quad \text{if } j \neq k.$$

Proof.

If A is $n \times n$ and $i \neq k$, let B be the matrix obtained from A by replacing row k by row i. Then $\det B = 0$ as B has two identical rows.

Now expand $\det B$ along row k. We get

$$\begin{aligned} 0 = \det B &= \sum_{j=1}^{n} b_{kj}C_{kj}(B) \\ &= \sum_{j=1}^{n} a_{ij}C_{kj}(A), \end{aligned}$$

in view of Remark 4.0.3.

DEFINITION 4.0.4 (Adjoint) If $A = [a_{ij}]$ is an $n \times n$ matrix, the *adjoint* of A, denoted by $\operatorname{adj} A$, is the transpose of the matrix of cofactors. Hence

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

Theorems 4.0.7 and 4.0.8 may be combined to give

THEOREM 4.0.9 Let A be an $n \times n$ matrix. Then

$$A(\operatorname{adj} A) = (\det A)I_n = (\operatorname{adj} A)A.$$

Proof.

$$\begin{aligned} (A \operatorname{adj} A)_{ik} &= \sum_{j=1}^{n} a_{ij}(\operatorname{adj} A)_{jk} \\ &= \sum_{j=1}^{n} a_{ij}C_{kj}(A) \\ &= \delta_{ik}\det A \\ &= ((\det A)I_n)_{ik}. \end{aligned}$$

Hence $A(\operatorname{adj} A) = (\det A)I_n$. The other equation is proved similarly.

COROLLARY 4.0.1 (Formula for the inverse) If $\det A \neq 0$, then A is non–singular and

$$A^{-1} = \frac{1}{\det A}\operatorname{adj} A.$$

EXAMPLE 4.0.3 The matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 8 & 9 \end{bmatrix}$$

is non–singular. For

$$\begin{aligned} \det A &= \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 8 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 8 & 8 \end{vmatrix} \\ &= -3 + 24 - 24 \\ &= -3 \neq 0. \end{aligned}$$

Also

$$
\begin{aligned}
A^{-1} &= \frac{1}{-3}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \\
&= -\frac{1}{3}\begin{bmatrix} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \\ -\begin{vmatrix} 4 & 6 \\ 8 & 9 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 8 & 9 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \\ \begin{vmatrix} 4 & 5 \\ 8 & 8 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 8 & 8 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \end{bmatrix} \\
&= -\frac{1}{3}\begin{bmatrix} -3 & 6 & -3 \\ 12 & -15 & 6 \\ -8 & 8 & -3 \end{bmatrix}.
\end{aligned}
$$

The following theorem is useful for simplifying and numerically evaluating a determinant. Proofs are obtained by expanding along the corresponding row or column.

THEOREM 4.0.10 The determinant is a linear function of each row and column.
For example

$$
(a) \quad \begin{vmatrix} a_{11}+a'_{11} & a_{12}+a'_{12} & a_{13}+a'_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a'_{11} & a'_{12} & a'_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
$$

$$
(b) \quad \begin{vmatrix} ta_{11} & ta_{12} & ta_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = t\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.
$$

COROLLARY 4.0.2 If a multiple of a row is added to *another* row, the value of the determinant is unchanged. Similarly for columns.

Proof. We illustrate with a 3×3 example, but the proof is really quite general.

$$
\begin{vmatrix} a_{11}+ta_{21} & a_{12}+ta_{22} & a_{13}+ta_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} ta_{21} & ta_{22} & ta_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + t \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + t \times 0$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

To evaluate a determinant numerically, it is advisable to reduce the matrix to row–echelon form, recording any sign changes caused by row interchanges, together with any factors taken out of a row, as in the following examples.

EXAMPLE 4.0.4 Evaluate the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 8 & 9 \end{vmatrix}.$$

Solution. Using row operations $R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 - 8R_1$ and then expanding along the first column, gives

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -8 & -15 \end{vmatrix} = \begin{vmatrix} -3 & -6 \\ -8 & -15 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 1 & 2 \\ -8 & -15 \end{vmatrix} = -3 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -3.$$

EXAMPLE 4.0.5 Evaluate the determinant

$$\begin{vmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 4 & 5 \\ 7 & 6 & 1 & 2 \\ 1 & 1 & 3 & 4 \end{vmatrix}.$$

Solution.

$$\begin{vmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 4 & 5 \\ 7 & 6 & 1 & 2 \\ 1 & 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -2 & 2 \\ 0 & -1 & -13 & -5 \\ 0 & 0 & 1 & 3 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -13 & -5 \\ 0 & 0 & 1 & 3 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -12 & -6 \\ 0 & 0 & 1 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -12 & -6 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 30 \end{vmatrix} = 60.$$

EXAMPLE 4.0.6 (Vandermonde determinant) Prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$$

Solution. Subtracting column 1 from columns 2 and 3 , then expanding along row 1, gives

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \\ &= \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix} \\ &= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} = (b-a)(c-a)(c-b). \end{aligned}$$

REMARK 4.0.4 From theorems 4.0.6, 4.0.10 and corollary 4.0.2, we deduce

(a) $\det(E_{ij}A) = -\det A$,

(b) $\det(E_i(t)A) = t\det A$, if $t \neq 0$,

(c) $\det(E_{ij}(t)A) = \det A$.

It follows that if A is row–equivalent to B, then $\det B = c \det A$, where $c \neq 0$. Hence $\det B \neq 0 \Leftrightarrow \det A \neq 0$ and $\det B = 0 \Leftrightarrow \det A = 0$. Consequently from theorem 2.5.8 and remark 2.5.7, we have the following important result:

THEOREM 4.0.11 Let A be an $n \times n$ matrix. Then

(i) A is non–singular if and only if $\det A \neq 0$;

(ii) A is singular if and only if $\det A = 0$;

(iii) the homogeneous system $AX = 0$ has a non–trivial solution if and only if $\det A = 0$.

EXAMPLE 4.0.7 Find the rational numbers a for which the following homogeneous system has a non–trivial solution and solve the system for these values of a:

$$\begin{aligned} x - 2y + 3z &= 0 \\ ax + 3y + 2z &= 0 \\ 6x + y + az &= 0. \end{aligned}$$

Solution. The coefficient determinant of the system is

$$\begin{aligned} \Delta = \begin{vmatrix} 1 & -2 & 3 \\ a & 3 & 2 \\ 6 & 1 & a \end{vmatrix} &= \begin{vmatrix} 1 & -2 & 3 \\ 0 & 3+2a & 2-3a \\ 0 & 13 & a-18 \end{vmatrix} \\ &= \begin{vmatrix} 3+2a & 2-3a \\ 13 & a-18 \end{vmatrix} \\ &= (3+2a)(a-18) - 13(2-3a) \\ &= 2a^2 + 6a - 80 = 2(a+8)(a-5). \end{aligned}$$

So $\Delta = 0 \Leftrightarrow a = -8$ or $a = 5$ and these values of a are the only values for which the given homogeneous system has a non–trivial solution.

If $a = -8$, the coefficient matrix has reduced row–echelon form equal to

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

and so the complete solution is $x = z$, $y = 2z$, with z arbitrary. If $a = 5$, the coefficient matrix has reduced row–echelon form equal to

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and so the complete solution is $x = -z$, $y = z$, with z arbitrary.

EXAMPLE 4.0.8 Find the values of t for which the following system is consistent and solve the system in each case:

$$\begin{aligned} x + y &= 1 \\ tx + y &= t \\ (1+t)x + 2y &= 3. \end{aligned}$$

Solution. Suppose that the given system has a solution (x_0, y_0). Then the following homogeneous system

$$\begin{aligned} x + y + z &= 0 \\ tx + y + tz &= 0 \\ (1+t)x + 2y + 3z &= 0 \end{aligned}$$

will have a non–trivial solution

$$x = x_0, \quad y = y_0, \quad z = -1.$$

Hence the coefficient determinant Δ is zero. However

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ t & 1 & t \\ 1+t & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ t & 1-t & 0 \\ 1+t & 1-t & 2-t \end{vmatrix} = \begin{vmatrix} 1-t & 0 \\ 1-t & 2-t \end{vmatrix} = (1-t)(2-t).$$

Hence $t = 1$ or $t = 2$. If $t = 1$, the given system becomes

$$\begin{aligned} x + y &= 1 \\ x + y &= 1 \\ 2x + 2y &= 3 \end{aligned}$$

which is clearly inconsistent. If $t = 2$, the given system becomes

$$\begin{aligned} x + y &= 1 \\ 2x + y &= 2 \\ 3x + 2y &= 3 \end{aligned}$$

which has the unique solution $x = 1$, $y = 0$.

To finish this section, we present an old (1750) method of solving a system of n equations in n unknowns called *Cramer's rule* . The method is not used in practice. However it has a theoretical use as it reveals explicitly how the solution depends on the coefficients of the augmented matrix.

THEOREM 4.0.12 (Cramer's rule) The system of n linear equations in n unknowns $x_1, \ldots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

has a unique solution if $\Delta = \det[a_{ij}] \neq 0$, namely

$$x_1 = \frac{\Delta_1}{\Delta},\ x_2 = \frac{\Delta_2}{\Delta}, \ldots, x_n = \frac{\Delta_n}{\Delta},$$

where Δ_i is the determinant of the matrix formed by replacing the i–th column of the coefficient matrix A by the entries $b_1, b_2, \ldots, b_n$.

Proof. Suppose the coefficient determinant $\Delta \neq 0$. Then by corollary 4.0.1, A^{-1} exists and is given by $A^{-1} = \frac{1}{\Delta}\operatorname{adj} A$ and the system has the unique solution

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A^{-1}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} &= \frac{1}{\Delta}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\ &= \frac{1}{\Delta}\begin{bmatrix} b_1C_{11} + b_2C_{21} + \ldots + b_nC_{n1} \\ b_2C_{12} + b_2C_{22} + \ldots + b_nC_{n2} \\ \vdots \\ b_nC_{1n} + b_2C_{2n} + \ldots + b_nC_{nn} \end{bmatrix}. \end{aligned}$$

However the i–th component of the last vector is the expansion of Δ_i along column i. Hence

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} = \begin{bmatrix} \Delta_1/\Delta \\ \Delta_2/\Delta \\ \vdots \\ \Delta_n/\Delta \end{bmatrix}.$$

4.1 PROBLEMS

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1. If the points $P_i = (x_i, y_i)$, $i = 1, 2, 3, 4$ form a quadrilateral with vertices in anti–clockwise orientation, prove that the area of the quadrilateral equals

$$\frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_4 \\ y_3 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & x_1 \\ y_4 & y_1 \end{vmatrix}\right).$$

(This formula generalizes to a simple polygon and is known as the *Surveyor's formula.*)

2. Prove that the following identity holds by expressing the left–hand side as the sum of 8 determinants:

$$\begin{vmatrix} a+x & b+y & c+z \\ x+u & y+v & z+w \\ u+a & v+b & w+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}.$$

3. Prove that

$$\begin{vmatrix} n^2 & (n+1)^2 & (n+2)^2 \\ (n+1)^2 & (n+2)^2 & (n+3)^2 \\ (n+2)^2 & (n+3)^2 & (n+4)^2 \end{vmatrix} = -8.$$

4. Evaluate the following determinants:

$$\text{(a)}\ \begin{vmatrix} 246 & 427 & 327 \\ 1014 & 543 & 443 \\ -342 & 721 & 621 \end{vmatrix} \qquad \text{(b)}\ \begin{vmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{vmatrix}.$$

[Answers: (a) -29400000; (b) 900.]

5. Compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \\ 5 & 2 & -3 \end{bmatrix}$$

by first computing the adjoint matrix.

[Answer: $A^{-1} = \frac{-1}{13}\begin{bmatrix} -11 & -4 & 2 \\ 29 & 7 & -10 \\ 1 & -2 & 1 \end{bmatrix}$.]

6. Prove that the following identities hold:

$$\text{(i)} \quad \begin{vmatrix} 2a & 2b & b-c \\ 2b & 2a & a+c \\ a+b & a+b & b \end{vmatrix} = -2(a-b)^2(a+b),$$

$$\text{(ii)} \quad \begin{vmatrix} b+c & b & c \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = 2a(b^2+c^2).$$

7. Let $P_i = (x_i, y_i)$, $i = 1, 2, 3$. If x_1, x_2, x_3 are distinct, prove that there is precisely one curve of the form $y = ax^2 + bx + c$ passing through P_1, P_2 and P_3.

8. Let
$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{bmatrix}.$$
Find the values of k for which $\det A = 0$ and hence, or otherwise, determine the value of k for which the following system has more than one solution:
$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + kz &= 3 \\ x + ky + 3z &= 2. \end{aligned}$$
Solve the system for this value of k and determine the solution for which $x^2 + y^2 + z^2$ has least value.

[Answer: $k = 2$; $x = 10/21$, $y = 13/21$, $z = 2/21$.]

9. By considering the coefficient determinant, find all rational numbers a and b for which the following system has (i) no solutions, (ii) exactly one solution, (iii) infinitely many solutions:
$$\begin{aligned} x - 2y + bz &= 3 \\ ax + \quad 2z &= 2 \\ 5x + 2y \quad &= 1. \end{aligned}$$
Solve the system in case (iii).

[Answer: (i) $ab = 12$ and $a \neq 3$, no solution; $ab \neq 12$, unique solution; $a = 3$, $b = 4$, infinitely many solutions; $x = -\frac{2}{3}z + \frac{2}{3}$, $y = \frac{5}{3}z - \frac{7}{6}$, with z arbitrary.]

10. Express the determinant of the matrix

$$B = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 2t+6 \\ 2 & 2 & 6-t & t \end{bmatrix}$$

as as polynomial in t and hence determine the rational values of t for which B^{-1} exists.

[Answer: $\det B = (t-2)(2t-1)$; $t \neq 2$ and $t \neq \frac{1}{2}$.]

11. If A is a 3×3 matrix over a field and $\det A \neq 0$, prove that

$$\begin{aligned} \text{(i)} \quad \det(\operatorname{adj} A) &= (\det A)^2, \\ \text{(ii)} \quad (\operatorname{adj} A)^{-1} &= \frac{1}{\det A} A = \operatorname{adj}(A^{-1}). \end{aligned}$$

12. Suppose that A is a real 3×3 matrix such that $A^t A = I_3$.

(i) Prove that $A^t(A - I_3) = -(A - I_3)^t$.

(ii) Prove that $\det A = \pm 1$.

(iii) Use (i) to prove that if $\det A = 1$, then $\det(A - I_3) = 0$.

13. If A is a square matrix such that one column is a linear combination of the remaining columns, prove that $\det A = 0$. Prove that the converse also holds.

14. Use Cramer's rule to solve the system

$$\begin{aligned} -2x + 3y - z &= 1 \\ x + 2y - z &= 4 \\ -2x - y + z &= -3. \end{aligned}$$

[Answer: $x = 2$, $y = 3$, $z = 4$.]

15. Use remark 4.0.4 to deduce that

$$\det E_{ij} = -1, \quad \det E_i(t) = t, \quad \det E_{ij}(t) = 1$$

and use theorem 2.5.8 and induction, to prove that

$$\det(BA) = \det B \det A,$$

if B is non–singular. Also prove that the formula holds when B is singular.

16. Prove that

$$\begin{vmatrix} a+b+c & a+b & a & a \\ a+b & a+b+c & a & a \\ a & a & a+b+c & a+b \\ a & a & a+b & a+b+c \end{vmatrix} = c^2(2b+c)(4a+2b+c).$$

17. Prove that

$$\begin{vmatrix} 1+u_1 & u_1 & u_1 & u_1 \\ u_2 & 1+u_2 & u_2 & u_2 \\ u_3 & u_3 & 1+u_3 & u_3 \\ u_4 & u_4 & u_4 & 1+u_4 \end{vmatrix} = 1+u_1+u_2+u_3+u_4.$$

18. Let $A \in M_{n\times n}(F)$. If $A^t = -A$, prove that $\det A = 0$ if n is odd and $1+1 \neq 0$ in F.

19. Prove that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 \\ r & r & 1 & 1 \\ r & r & r & 1 \end{vmatrix} = (1-r)^3.$$

20. Express the determinant

$$\begin{vmatrix} 1 & a^2-bc & a^4 \\ 1 & b^2-ca & b^4 \\ 1 & c^2-ab & c^4 \end{vmatrix}$$

as the product of one quadratic and four linear factors.

[Answer: $(b-a)(c-a)(c-b)(a+b+c)(b^2+bc+c^2+ac+ab+a^2)$.]