Chapter 1

LINEAR EQUATIONS

1.1 Introduction to linear equations

A *linear equation* in n unknowns $x_1, x_2, \cdots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n, b$ are given real numbers.

For example, with x and y instead of x_1 and x_2, the linear equation $2x + 3y = 6$ describes the line passing through the points (3, 0) and (0, 2).

Similarly, with x, y and z instead of x_1, x_2 and x_3, the linear equation $2x + 3y + 4z = 12$ describes the plane passing through the points (6, 0, 0), (0, 4, 0), (0, 0, 3).

A *system* of m linear equations in n unknowns $x_1, x_2, \cdots, x_n$ is a family of linear equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

We wish to determine if such a system has a solution, that is to find out if there exist numbers $x_1, x_2, \cdots, x_n$ which satisfy each of the equations simultaneously. We say that the system is *consistent* if it has a solution. Otherwise the system is called *inconsistent.*

Note that the above system can be written concisely as

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1, 2, \cdots, m.$$

The matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called the *coefficient matrix* of the system, while the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

is called the *augmented matrix* of the system.

Geometrically, solving a system of linear equations in two (or three) unknowns is equivalent to determining whether or not a family of lines (or planes) has a common point of intersection.

EXAMPLE 1.1.1 Solve the equation

$$2x + 3y = 6.$$

Solution. The equation $2x + 3y = 6$ is equivalent to $2x = 6 - 3y$ or $x = 3 - \frac{3}{2}y$, where y is arbitrary. So there are infinitely many solutions.

EXAMPLE 1.1.2 Solve the system

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 0. \end{aligned}$$

Solution. We subtract the second equation from the first, to get $2y = 1$ and $y = \frac{1}{2}$. Then $x = y - z = \frac{1}{2} - z$, where z is arbitrary. Again there are infinitely many solutions.

EXAMPLE 1.1.3 Find a polynomial of the form $y = a_0 + a_1x + a_2x^2 + a_3x^3$ which passes through the points $(-3, -2)$, $(-1, 2)$, $(1, 5)$, $(2, 1)$.

Solution. When x has the values -3, -1, 1, 2, then y takes corresponding values -2, 2, 5, 1 and we get four equations in the unknowns a_0, a_1, a_2, a_3:

$$\begin{aligned} a_0 - 3a_1 + 9a_2 - 27a_3 &= -2 \\ a_0 - a_1 + a_2 - a_3 &= 2 \\ a_0 + a_1 + a_2 + a_3 &= 5 \\ a_0 + 2a_1 + 4a_2 + 8a_3 &= 1. \end{aligned}$$

This system has the unique solution $a_0 = 93/20$, $a_1 = 221/120$, $a_2 = -23/20$,
$a_3 = -41/120$. So the required polynomial is

$$y = \frac{93}{20} + \frac{221}{120}x - \frac{23}{20}x^2 - \frac{41}{120}x^3.$$

In [26, pages 33–35] there are examples of systems of linear equations which arise from simple electrical networks using Kirchhoff's laws for electrical circuits.

Solving a system consisting of a single linear equation is easy. However if we are dealing with two or more equations, it is desirable to have a systematic method of determining if the system is consistent and to find all solutions.

Instead of restricting ourselves to linear equations with rational or real coefficients, our theory goes over to the more general case where the coefficients belong to an arbitrary *field*. A *field* F is a set F which possesses operations of *addition* and *multiplication* which satisfy the familiar rules of rational arithmetic. There are ten basic properties that a field must have:

THE FIELD AXIOMS.

1. $(a + b) + c = a + (b + c)$ for all a, b, c in F;

2. $(ab)c = a(bc)$ for all a, b, c in F;

3. $a + b = b + a$ for all a, b in F;

4. $ab = ba$ for all a, b in F;

5. there exists an element 0 in F such that $0 + a = a$ for all a in F;

6. there exists an element 1 in F such that $1a = a$ for all a in F;

7. to every a in F, there corresponds an *additive inverse* $-a$ in F, satisfying
$$a + (-a) = 0;$$

8. to every non–zero a in F, there corresponds a *multiplicative inverse* a^{-1} in F, satisfying
$$aa^{-1} = 1;$$

9. $a(b + c) = ab + ac$ for all a, b, c in F;

10. $0 \neq 1$.

With standard definitions such as $a - b = a + (-b)$ and $\dfrac{a}{b} = ab^{-1}$ for $b \neq 0$, we have the following familiar rules:

$$\begin{aligned}
-(a+b) &= (-a) + (-b), \quad (ab)^{-1} = a^{-1}b^{-1};\\
-(-a) &= a, \quad (a^{-1})^{-1} = a;\\
-(a-b) &= b - a, \quad (\frac{a}{b})^{-1} = \frac{b}{a};\\
\frac{a}{b} + \frac{c}{d} &= \frac{ad+bc}{bd};\\
\frac{a}{b}\frac{c}{d} &= \frac{ac}{bd};\\
\frac{ab}{ac} &= \frac{b}{c}, \quad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b};\\
-(ab) &= (-a)b = a(-b);\\
-\left(\frac{a}{b}\right) &= \frac{-a}{b} = \frac{a}{-b};\\
0a &= 0;\\
(-a)^{-1} &= -(a^{-1}).
\end{aligned}$$

Fields which have only finitely many elements are of great interest in many parts of mathematics and its applications, for example to coding theory. It is easy to construct fields containing exactly p elements, where p is a prime number. First we must explain the idea of *modular addition* and *modular multiplication*. If a is an integer, we define $a \pmod{p}$ to be the *least remainder on dividing a by p*: That is, if $a = bp + r$, where b and r are integers and $0 \leq r < p$, then $a \pmod{p} = r$.

For example, $-1 \pmod{2} = 1$, $3 \pmod{3} = 0$, $5 \pmod{3} = 2$.

Then addition and multiplication mod p are defined by

$$\begin{aligned} a \oplus b &= (a+b) \pmod{p} \\ a \otimes b &= (ab) \pmod{p}. \end{aligned}$$

For example, with $p = 7$, we have $3 \oplus 4 = 7 \pmod{7} = 0$ and $3 \otimes 5 = 15 \pmod{7} = 1$. Here are the complete addition and multiplication tables mod 7:

$\oplus$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

$\otimes$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

If we now let $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$, then it can be proved that $\mathbb{Z}_p$ forms a field under the operations of modular addition and multiplication mod p. For example, the additive inverse of 3 in $\mathbb{Z}_7$ is 4, so we write $-3 = 4$ when calculating in $\mathbb{Z}_7$. Also the multiplicative inverse of 3 in $\mathbb{Z}_7$ is 5 , so we write $3^{-1} = 5$ when calculating in $\mathbb{Z}_7$.

In practice, we write $a \oplus b$ and $a \otimes b$ as $a+b$ and ab or $a \times b$ when dealing with linear equations over $\mathbb{Z}_p$.

The simplest field is $\mathbb{Z}_2$, which consists of two elements 0, 1 with addition satisfying $1+1=0$. So in $\mathbb{Z}_2$, $-1 = 1$ and the arithmetic involved in solving equations over $\mathbb{Z}_2$ is very simple.

EXAMPLE 1.1.4 Solve the following system over $\mathbb{Z}_2$:

$$\begin{aligned} x+y+z &= 0 \\ x+z &= 1. \end{aligned}$$

Solution. We add the first equation to the second to get $y = 1$. Then $x = 1 - z = 1 + z$, with z arbitrary. Hence the solutions are $(x,\, y,\, z) = (1,\, 1,\, 0)$ and $(0,\, 1,\, 1)$.

We use $\mathbb{Q}$ and $\mathbb{R}$ to denote the fields of rational and real numbers, respectively. Unless otherwise stated, the field used will be $\mathbb{Q}$.

1.2 Solving linear equations

We show how to solve any system of linear equations over an arbitrary field, using the *GAUSS–JORDAN* algorithm. We first need to define some terms.

DEFINITION 1.2.1 (Row–echelon form) A matrix is in *row–echelon form* if

(i) all zero rows (if any) are at the bottom of the matrix and

(ii) if two successive rows are non–zero, the second row starts with more zeros than the first (moving from left to right).

For example, the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row–echelon form, whereas the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is not in row–echelon form.

The *zero* matrix of any size is always in row–echelon form.

DEFINITION 1.2.2 (Reduced row–echelon form) A matrix is in *reduced row–echelon form* if

1. it is in row–echelon form,
2. the leading (leftmost non–zero) entry in each non–zero row is 1,
3. all other elements of the column in which the leading entry 1 occurs are zeros.

For example the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are in reduced row–echelon form, whereas the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are not in reduced row–echelon form, but are in row–echelon form.

The *zero* matrix of any size is always in reduced row–echelon form.

Notation. If a matrix is in reduced row–echelon form, it is useful to denote the column numbers in which the leading entries 1 occur, by $c_1, c_2, \ldots, c_r$, with the remaining column numbers being denoted by $c_{r+1}, \ldots, c_n$, where r is the number of non–zero rows. For example, in the 4×6 matrix above, we have $r = 3$, $c_1 = 2$, $c_2 = 4$, $c_3 = 5$, $c_4 = 1$, $c_5 = 3$, $c_6 = 6$.

The following operations are the ones used on systems of linear equations and do not change the solutions.

DEFINITION 1.2.3 (Elementary row operations) There are three types of *elementary row operations* that can be performed on matrices:

1. Interchanging two rows:

   $R_i \leftrightarrow R_j$ interchanges rows i and j.

2. Multiplying a row by a non–zero scalar:

   $R_i \rightarrow tR_i$ multiplies row i by the non–zero scalar t.

3. Adding a multiple of one row to another row:

   $R_j \rightarrow R_j + tR_i$ adds t times row i to row j.

DEFINITION 1.2.4 *[Row equivalence]*Matrix A is *row–equivalent* to matrix B if B is obtained from A by a sequence of elementary row operations.

EXAMPLE 1.2.1 Working from left to right,

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \quad R_2 \rightarrow R_2 + 2R_3 \quad \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 5 \\ 1 & -1 & 2 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 2 \\ 4 & -1 & 5 \end{bmatrix} \quad R_1 \rightarrow 2R_1 \quad \begin{bmatrix} 2 & 4 & 0 \\ 1 & -1 & 2 \\ 4 & -1 & 5 \end{bmatrix} = B.$$

Thus A is row–equivalent to B. Clearly B is also row–equivalent to A, by performing the inverse row–operations $R_1 \to \frac{1}{2}R_1$, $R_2 \leftrightarrow R_3$, $R_2 \to R_2 - 2R_3$ on B.

It is not difficult to prove that if A and B are row–equivalent augmented matrices of two systems of linear equations, then the two systems have the same solution sets – a solution of the one system is a solution of the other. For example the systems whose augmented matrices are A and B in the above example are respectively

$$\left\{\begin{array}{rcl} x+2y &=& 0\\ 2x+y &=& 1\\ x-y &=& 2 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcl} 2x+4y &=& 0\\ x-y &=& 2\\ 4x-y &=& 5 \end{array}\right.$$

and these systems have precisely the same solutions.

1.3 The Gauss–Jordan algorithm

We now describe the *GAUSS–JORDAN ALGORITHM*. This is a process which starts with a given matrix A and produces a matrix B in reduced row–echelon form, which is row–equivalent to A. If A is the augmented matrix of a system of linear equations, then B will be a much simpler matrix than A from which the consistency or inconsistency of the corresponding system is immediately apparent and in fact the complete solution of the system can be read off.

<u>STEP 1</u>.

Find the first non–zero column moving from left to right, (column c_1) and select a non–zero entry from this column. By interchanging rows, if necessary, ensure that the first entry in this column is non–zero. Multiply row 1 by the multiplicative inverse of a_{1c_1} thereby converting a_{1c_1} to 1. For each non–zero element a_{ic_1}, $i > 1$, (if any) in column c_1, add $-a_{ic_1}$ times row 1 to row i, thereby ensuring that all elements in column c_1, apart from the first, are zero.

<u>STEP 2</u>. If the matrix obtained at Step 1 has its 2nd, $\ldots, m$th rows all zero, the matrix is in reduced row–echelon form. Otherwise suppose that the first column which has a non–zero element in the rows below the first is column c_2. Then $c_1 < c_2$. By interchanging rows below the first, if necessary, ensure that a_{2c_2} is non–zero. Then convert a_{2c_2} to 1 and by adding suitable multiples of row 2 to the remaing rows, where necessary, ensure that all remaining elements in column c_2 are zero.

The process is repeated and will eventually stop after r steps, either because we run out of rows, or because we run out of non–zero columns. In general, the final matrix will be in reduced row–echelon form and will have r non–zero rows, with leading entries 1 in columns $c_1, \ldots, c_r$, respectively.

EXAMPLE 1.3.1

$$\left[\begin{array}{rrrr} 0 & 0 & 4 & 0 \\ 2 & 2 & -2 & 5 \\ 5 & 5 & -1 & 5 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{rrrr} 2 & 2 & -2 & 5 \\ 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \end{array}\right]$$

$$R_1 \rightarrow \tfrac{1}{2}R_1 \quad \left[\begin{array}{rrrr} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \end{array}\right] \quad R_3 \rightarrow R_3 - 5R_1 \quad \left[\begin{array}{rrrr} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & -\frac{15}{2} \end{array}\right]$$

$$R_2 \rightarrow \tfrac{1}{4}R_2 \quad \left[\begin{array}{rrrr} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -\frac{15}{2} \end{array}\right] \quad \left\{\begin{array}{l} R_1 \rightarrow R_1 + R_2 \\ R_3 \rightarrow R_3 - 4R_2 \end{array}\right. \quad \left[\begin{array}{rrrr} 1 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{15}{2} \end{array}\right]$$

$$R_3 \rightarrow \tfrac{-2}{15}R_3 \quad \left[\begin{array}{rrrr} 1 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad R_1 \rightarrow R_1 - \tfrac{5}{2}R_3 \quad \left[\begin{array}{rrrr} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last matrix is in reduced row–echelon form.

REMARK 1.3.1 It is possible to show that a given matrix over an arbitrary field is row–equivalent to *precisely one* matrix which is in reduced row–echelon form.

A flow–chart for the Gauss–Jordan algorithm, based on [1, page 83] is presented in figure 1.1 below.

1.4 Systematic solution of linear systems.

Suppose a system of m linear equations in n unknowns $x_1, \cdots, x_n$ has augmented matrix A and that A is row–equivalent to a matrix B which is in reduced row–echelon form, via the Gauss–Jordan algorithm. Then A and B are $m \times (n+1)$. Suppose that B has r non–zero rows and that the leading entry 1 in row i occurs in column number c_i, for $1 \le i \le r$. Then

$$1 \le c_1 < c_2 < \cdots, < c_r \le n+1.$$

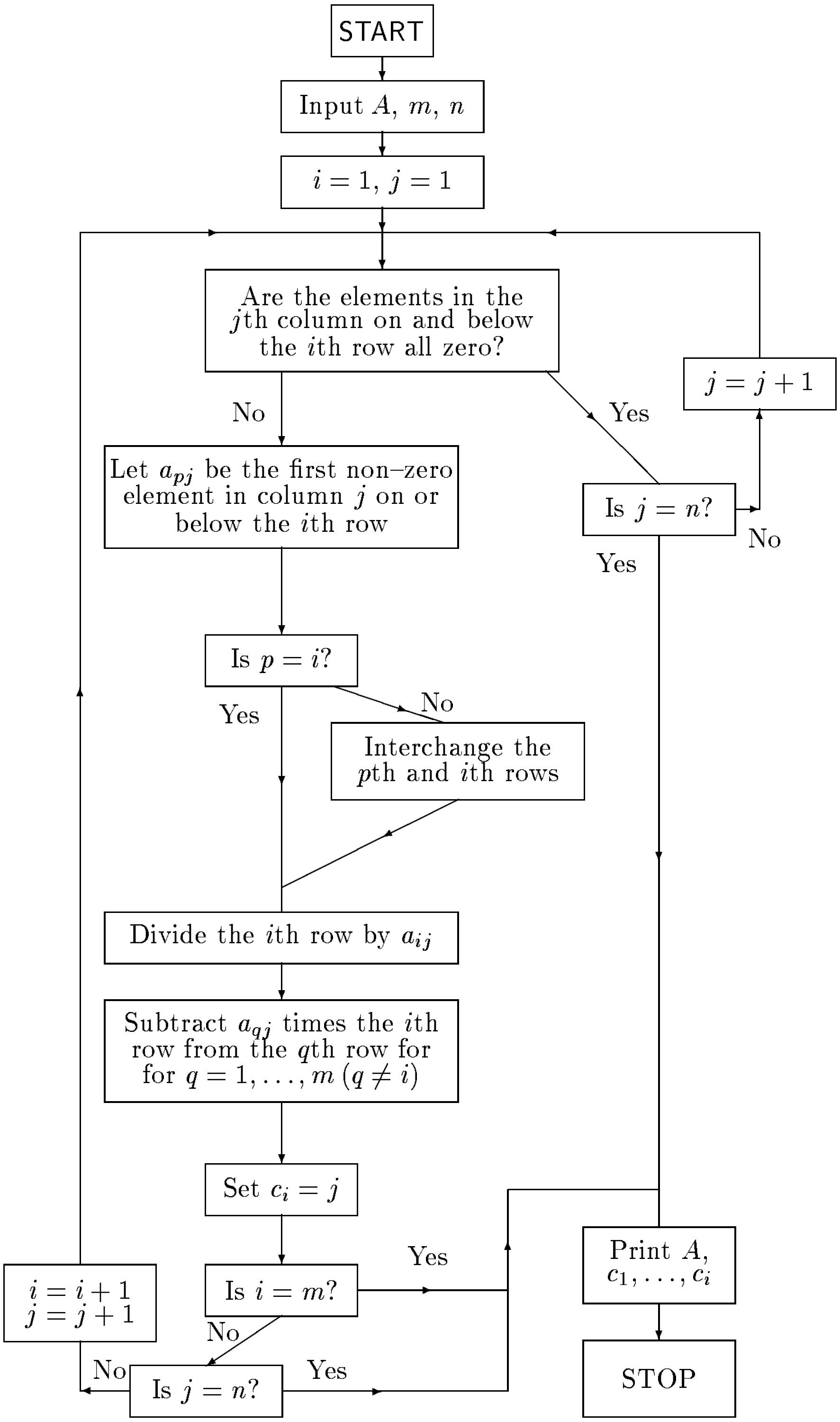

Figure 1.1: Gauss–Jordan algorithm.

Also assume that the remaining column numbers are $c_{r+1}, \cdots, c_{n+1}$, where

$$1 \le c_{r+1} < c_{r+2} < \cdots < c_n \le n+1.$$

Case 1: $c_r = n+1$. The system is inconsistent. For the last non–zero row of B is $[0, 0, \cdots, 1]$ and the corresponding equation is

$$0x_1 + 0x_2 + \cdots + 0x_n = 1,$$

which has no solutions. Consequently the original system has no solutions.

Case 2: $c_r \le n$. The system of equations corresponding to the non–zero rows of B is consistent. First notice that $r \le n$ here.

If $r = n$, then $c_1 = 1$, $c_2 = 2$, $\cdots$, $c_n = n$ and

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 & d_1 \\ 0 & 1 & \cdots & 0 & d_2 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & d_n \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

There is a unique solution $x_1 = d_1$, $x_2 = d_2$, $\cdots$, $x_n = d_n$.

If $r < n$, there will be more than one solution (infinitely many if the field is infinite). For all solutions are obtained by taking the unknowns $x_{c_1}, \cdots, x_{c_r}$ as *dependent* unknowns and using the r equations corresponding to the non–zero rows of B to express these unknowns in terms of the remaining *independent* unknowns $x_{c_{r+1}}, \ldots, x_{c_n}$, which can take on arbitrary values:

$$\begin{aligned} x_{c_1} &= b_{1\,n+1} - b_{1c_{r+1}}x_{c_{r+1}} - \cdots - b_{1c_n}x_{c_n} \\ &\vdots \\ x_{c_r} &= b_{r\,n+1} - b_{rc_{r+1}}x_{c_{r+1}} - \cdots - b_{rc_n}x_{c_n}. \end{aligned}$$

In particular, taking $x_{c_{r+1}} = 0, \ldots, x_{c_{n-1}} = 0$ and $x_{c_n} = 0, 1$ respectively, produces at least two solutions.

EXAMPLE 1.4.1 Solve the system

$$\begin{aligned} x + y &= 0 \\ x - y &= 1 \\ 4x + 2y &= 1. \end{aligned}$$

Solution. The augmented matrix of the system is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

which is row equivalent to

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

We read off the unique solution $x = \frac{1}{2}$, $y = -\frac{1}{2}$.
(Here $n = 2$, $r = 2$, $c_1 = 1$, $c_2 = 2$. Also $c_r = c_2 = 2 < 3 = n + 1$ and $r = n$.)

EXAMPLE 1.4.2 Solve the system

$$\begin{aligned} 2x_1 + 2x_2 - 2x_3 &= 5 \\ 7x_1 + 7x_2 + x_3 &= 10 \\ 5x_1 + 5x_2 - x_3 &= 5. \end{aligned}$$

Solution. The augmented matrix is

$$A = \begin{bmatrix} 2 & 2 & -2 & 5 \\ 7 & 7 & 1 & 10 \\ 5 & 5 & -1 & 5 \end{bmatrix}$$

which is row equivalent to

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We read off inconsistency for the original system.
(Here $n = 3$, $r = 3$, $c_1 = 1$, $c_2 = 3$. Also $c_r = c_3 = 4 = n + 1$.)

EXAMPLE 1.4.3 Solve the system

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_1 + x_2 - x_3 &= 2. \end{aligned}$$

Solution. The augmented matrix is

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

which is row equivalent to

$$B = \begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & -1 & \frac{1}{2} \end{bmatrix}.$$

The complete solution is $x_1 = \frac{3}{2}$, $x_2 = \frac{1}{2} + x_3$, with x_3 arbitrary. (Here $n = 3$, $r = 2$, $c_1 = 1$, $c_2 = 2$. Also $c_r = c_2 = 2 < 4 = n + 1$ and $r < n$.)

EXAMPLE 1.4.4 Solve the system

$$\begin{aligned} 6x_3 + 2x_4 - 4x_5 - 8x_6 &= 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 &= 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 &= 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 &= 1. \end{aligned}$$

Solution. The augmented matrix is

$$A = \begin{bmatrix} 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{bmatrix}$$

which is row equivalent to

$$B = \begin{bmatrix} 1 & -\frac{3}{2} & 0 & \frac{11}{6} & -\frac{19}{6} & 0 & \frac{1}{24} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & 0 & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The complete solution is

$$\begin{aligned} x_1 &= \tfrac{1}{24} + \tfrac{3}{2}x_2 - \tfrac{11}{6}x_4 + \tfrac{19}{6}x_5, \\ x_3 &= \tfrac{5}{3} - \tfrac{1}{3}x_4 + \tfrac{2}{3}x_5, \\ x_6 &= \tfrac{1}{4}, \end{aligned}$$

with x_2, x_4, x_5 arbitrary.
(Here $n = 6$, $r = 3$, $c_1 = 1$, $c_2 = 3$, $c_3 = 6$; $c_r = c_3 = 6 < 7 = n + 1$; $r < n$.)

EXAMPLE 1.4.5 Find the rational number t for which the following system is consistent and solve the system for this value of t.

$$\begin{aligned} x+y &= 2 \\ x-y &= 0 \\ 3x-y &= t. \end{aligned}$$

Solution. The augmented matrix of the system is

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 3 & -1 & t \end{bmatrix}$$

which is row–equivalent to the simpler matrix

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & t-2 \end{bmatrix}.$$

Hence if $t \neq 2$ the system is inconsistent. If $t = 2$ the system is consistent and

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We read off the solution $x = 1$, $y = 1$.

EXAMPLE 1.4.6 For which rationals a and b does the following system have (i) no solution, (ii) a unique solution, (iii) infinitely many solutions?

$$\begin{aligned} x-2y+3z &= 4 \\ 2x-3y+az &= 5 \\ 3x-4y+5z &= b. \end{aligned}$$

Solution. The augmented matrix of the system is

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 2 & -3 & a & 5 \\ 3 & -4 & 5 & b \end{bmatrix}$$

$$\left\{\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}\right. \quad \left[\begin{array}{rrrr} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 2 & -4 & b-12 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2 \quad \left[\begin{array}{rrrr} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 0 & -2a+8 & b-6 \end{array}\right] = B.$$

Case 1. $a \neq 4$. Then $-2a+8 \neq 0$ and we see that B can be reduced to a matrix of the form

$$\left[\begin{array}{cccc} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & \frac{b-6}{-2a+8} \end{array}\right]$$

and we have the unique solution $x = u$, $y = v$, $z = (b-6)/(-2a+8)$.

Case 2. $a = 4$. Then

$$B = \left[\begin{array}{rrrr} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & b-6 \end{array}\right].$$

If $b \neq 6$ we get no solution, whereas if $b = 6$ then

$$B = \left[\begin{array}{rrrr} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 + 2R_2 \quad \left[\begin{array}{rrrr} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$ We read off the complete solution $x = -2 + z$, $y = -3 + 2z$, with z arbitrary.

EXAMPLE 1.4.7 Find the reduced row–echelon form of the following matrix over $\mathbb{Z}_3$:

$$\left[\begin{array}{cccc} 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 0 \end{array}\right].$$

Hence solve the system

$$\begin{aligned} 2x + y + 2z &= 1 \\ 2x + 2y + z &= 1. \end{aligned}$$

over $\mathbb{Z}_3$.

Solution.

$$\left[\begin{array}{cccc} 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 0 \end{array}\right] \quad R_2 \to R_2 - R_1 \quad \left[\begin{array}{cccc} 2 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right] = \left[\begin{array}{cccc} 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$R_1 \to 2R_1 \quad \left[\begin{array}{cccc} 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right] \quad R_1 \to R_1 + R_2 \quad \left[\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right].$$

The last matrix is in reduced row–echelon form.

To solve the system of equations whose augmented matrix is the given matrix over $\mathbb{Z}_3$, we see from the reduced row–echelon form that $x = 1$ and $y = 2 - 2z = 2 + z$, where $z = 0, 1, 2$. Hence there are three solutions to the given system of linear equations: $(x, y, z) = (1, 2, 0)$, $(1, 0, 1)$ and $(1, 1, 2)$.

1.5 Homogeneous systems

A system of homogeneous linear equations is a system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

Such a system is always consistent as $x_1 = 0, \cdots, x_n = 0$ is a solution. This solution is called the *trivial* solution. Any other solution is called a *non–trivial* solution.

For example the homogeneous system

$$\begin{aligned} x - y &= 0 \\ x + y &= 0 \end{aligned}$$

has only the trivial solution, whereas the homogeneous system

$$\begin{aligned} x - y + z &= 0 \\ x + y + z &= 0 \end{aligned}$$

has the complete solution $x = -z$, $y = 0$, z arbitrary. In particular, taking $z = 1$ gives the non–trivial solution $x = -1$, $y = 0$, $z = 1$.

There is simple but fundamental theorem concerning homogeneous systems.

THEOREM 1.5.1 *A homogeneous system of m linear equations in n unknowns always has a non–trivial solution if $m < n$.*

Proof. Suppose that $m < n$ and that the coefficient matrix of the system is row–equivalent to B, a matrix in reduced row–echelon form. Let r be the number of non–zero rows in B. Then $r \leq m < n$ and hence $n - r > 0$ and so the number $n - r$ of arbitrary unknowns is in fact positive. Taking one of these unknowns to be 1 gives a non–trivial solution.

REMARK 1.5.1 Let two systems of homogeneous equations in n unknowns have coefficient matrices A and B, respectively. If each row of B is a linear combination of the rows of A (i.e. a sum of multiples of the rows of A) and each row of A is a linear combination of the rows of B, then it is easy to prove that the two systems have identical solutions. The converse is true, but is not easy to prove. Similarly if A and B have the same reduced row–echelon form, apart from possibly zero rows, then the two systems have identical solutions and conversely.

There is a similar situation in the case of two systems of linear equations (not necessarily homogeneous), with the proviso that in the statement of the converse, the extra condition that both the systems are consistent, is needed.

1.6 PROBLEMS

1. Which of the following matrices of rationals is in reduced row–echelon form?

$$\text{(a)} \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}$$

$$\text{(d)} \begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(e)} \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(f)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{(g)} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ [Answers: (a), (e), (g)]

2. Find reduced row–echelon forms which are row–equivalent to the following matrices:

$$\text{(a)} \begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{(d)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix}.$$

[Answers:

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.]

3. Solve the following systems of linear equations by reducing the augmented matrix to reduced row–echelon form:

$$(a) \quad \begin{aligned} x + y + z &= 2 \\ 2x + 3y - z &= 8 \\ x - y - z &= -8 \end{aligned} \qquad (b) \quad \begin{aligned} x_1 + x_2 - x_3 + 2x_4 &= 10 \\ 3x_1 - x_2 + 7x_3 + 4x_4 &= 1 \\ -5x_1 + 3x_2 - 15x_3 - 6x_4 &= 9 \end{aligned}$$

$$(c) \quad \begin{aligned} 3x - y + 7z &= 0 \\ 2x - y + 4z &= \tfrac{1}{2} \\ x - y + z &= 1 \\ 6x - 4y + 10z &= 3 \end{aligned} \qquad (d) \quad \begin{aligned} 2x_2 + 3x_3 - 4x_4 &= 1 \\ 2x_3 + 3x_4 &= 4 \\ 2x_1 + 2x_2 - 5x_3 + 2x_4 &= 4 \\ 2x_1 - 6x_3 + 9x_4 &= 7 \end{aligned}$$

[Answers: (a) $x = -3$, $y = \frac{19}{4}$, $z = \frac{1}{4}$; (b) inconsistent;

(c) $x = -\frac{1}{2} - 3z$, $y = -\frac{3}{2} - 2z$, with z arbitrary;

(d) $x_1 = \frac{19}{2} - 9x_4$, $x_2 = -\frac{5}{2} + \frac{17}{4}x_4$, $x_3 = 2 - \frac{3}{2}x_4$, with x_4 arbitrary.]

4. Show that the following system is consistent if and only if $c = 2a - 3b$ and solve the system in this case.

$$\begin{aligned} 2x - y + 3z &= a \\ 3x + y - 5z &= b \\ -5x - 5y + 21z &= c. \end{aligned}$$

[Answer: $x = \frac{a+b}{5} + \frac{2}{5}z$, $y = \frac{-3a+2b}{5} + \frac{19}{5}z$, with z arbitrary.]

5. Find the value of t for which the following system is consistent and solve the system for this value of t.

$$\begin{aligned} x + y &= 1 \\ tx + y &= t \\ (1 + t)x + 2y &= 3. \end{aligned}$$

[Answer: $t = 2$; $x = 1$, $y = 0$.]

6. Solve the homogeneous system

$$\begin{aligned}
-3x_1 + x_2 + x_3 + x_4 &= 0\\
x_1 - 3x_2 + x_3 + x_4 &= 0\\
x_1 + x_2 - 3x_3 + x_4 &= 0\\
x_1 + x_2 + x_3 - 3x_4 &= 0.
\end{aligned}$$

[Answer: $x_1 = x_2 = x_3 = x_4$, with x_4 arbitrary.]

7. For which rational numbers λ does the homogeneous system

$$\begin{aligned}
x + (\lambda - 3)y &= 0\\
(\lambda - 3)x + y &= 0
\end{aligned}$$

have a non–trivial solution?

[Answer: $\lambda = 2,\ 4$.]

8. Solve the homogeneous system

$$\begin{aligned}
3x_1 + x_2 + x_3 + x_4 &= 0\\
5x_1 - x_2 + x_3 - x_4 &= 0.
\end{aligned}$$

[Answer: $x_1 = -\frac{1}{4}x_3$, $x_2 = -\frac{1}{4}x_3 - x_4$, with x_3 and x_4 arbitrary.]

9. Let A be the coefficient matrix of the following homogeneous system of n equations in n unknowns:

$$\begin{aligned}
(1-n)x_1 + x_2 + \cdots + x_n &= 0\\
x_1 + (1-n)x_2 + \cdots + x_n &= 0\\
\cdots &= 0\\
x_1 + x_2 + \cdots + (1-n)x_n &= 0.
\end{aligned}$$

Find the reduced row–echelon form of A and hence, or otherwise, prove that the solution of the above system is $x_1 = x_2 = \cdots = x_n$, with x_n arbitrary.

10. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix over a field F. Prove that A is row–equivalent to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if $ad - bc \neq 0$, but is row–equivalent to a matrix whose second row is zero, if $ad - bc = 0$.

11. For which rational numbers a does the following system have (i) no solutions (ii) exactly one solution (iii) infinitely many solutions?

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2. \end{aligned}$$

[Answer: $a = -4$, no solution; $a = 4$, infinitely many solutions; $a \neq \pm 4$, exactly one solution.]

12. Solve the following system of homogeneous equations over $\mathbb{Z}_2$:

$$\begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 + x_4 + x_5 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 0 \\ x_3 + x_4 &= 0. \end{aligned}$$

[Answer: $x_1 = x_2 = x_4 + x_5$, $x_3 = x_4$, with x_4 and x_5 arbitrary elements of $\mathbb{Z}_2$.]

13. Solve the following systems of linear equations over $\mathbb{Z}_5$:

$$(a) \quad \begin{aligned} 2x + y + 3z &= 4 \\ 4x + y + 4z &= 1 \\ 3x + y + 2z &= 0 \end{aligned} \qquad (b) \quad \begin{aligned} 2x + y + 3z &= 4 \\ 4x + y + 4z &= 1 \\ x + y &= 3. \end{aligned}$$

[Answer: (a) $x = 1$, $y = 2$, $z = 0$; (b) $x = 1 + 2z$, $y = 2 + 3z$, with z an arbitrary element of $\mathbb{Z}_5$.]

14. If $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ are solutions of a system of linear equations, prove that

$$((1-t)\alpha_1 + t\beta_1, \ldots, (1-t)\alpha_n + t\beta_n)$$

is also a solution.

15. If $(\alpha_1, \ldots, \alpha_n)$ is a solution of a system of linear equations, prove that the complete solution is given by $x_1 = \alpha_1 + y_1, \ldots, x_n = \alpha_n + y_n$, where $(y_1, \ldots, y_n)$ is the general solution of the associated homogeneous system.

16. Find the values of a and b for which the following system is consistent. Also find the complete solution when $a = b = 2$.

$$\begin{aligned} x + y - z + w &= 1 \\ ax + y + z + w &= b \\ 3x + 2y + \quad aw &= 1 + a. \end{aligned}$$

[Answer: $a \neq 2$ or $a = 2 = b$; $x = 1 - 2z$, $y = 3z - w$, with z, w arbitrary.]

17. Let $F = \{0, 1, a, b\}$ be a field consisting of 4 elements.

(a) Determine the addition and multiplication tables of F. (Hint: Prove that the elements $1+0$, $1+1$, $1+a$, $1+b$ are distinct and deduce that $1 + 1 + 1 + 1 = 0$; then deduce that $1 + 1 = 0$.)

(b) A matrix A, whose elements belong to F, is defined by

$$A = \begin{bmatrix} 1 & a & b & a \\ a & b & b & 1 \\ 1 & 1 & 1 & a \end{bmatrix},$$

prove that the reduced row–echelon form of A is given by the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$