Difference Equations to **D**ifferential Equations

Section 7.3

Complex-Valued Functions: Motion in the Plane

In Section 7.2 we considered the problem of extending the elementary functions of calculus to complex-valued functions of a complex variable, while at the same time extending many of the concepts of the first six chapters to these new functions. In this section we will consider complex-valued functions of a real variable, that is, functions of the form $f : \mathbb{R} \to \mathbb{C}$. Such functions are often used to describe the motion of an object in the plane; if we think of the real variable t as measuring time, then we may interpret $f(t)$ as the location of an object in the complex plane at time t.

Since limits are at the foundation of most concepts in calculus, we begin with a definition of limit in this setting.

Definition Suppose $f : \mathbb{R} \to \mathbb{C}$ and f is defined for all t in an interval about the point a. We say that the *limit* of $f(t)$ as t approaches a is L, denoted

$$\lim_{t \to a} f(t) = L,$$

if whenever $\{t_n\}$ is a sequence of real numbers with $t_n \neq a$ for all n and

$$\lim_{n \to \infty} t_n = a,$$

then

$$\lim_{n \to \infty} f(t_n) = L.$$

Suppose $f : \mathbb{R} \to \mathbb{C}$. If we let $x(t) = \Re(f(t))$ and $y(t) = \Im(f(t))$, then

$$f(t) = x(t) + iy(t).$$

Hence, from our work in Section 7.2,

$$\lim_{t \to a} f(t) = \lim_{t \to a} x(t) + i \lim_{t \to a} y(t). \tag{7.3.1}$$

This result also holds if we modify our definition of limit to include one-sided limits and limits to ∞ or $-\infty$.

Example Suppose a particle moves in the plane so that its position at time t is given by

$$f(t) = \cos(2\pi t) + i \sin(2\pi t) = e^{2\pi i t}.$$

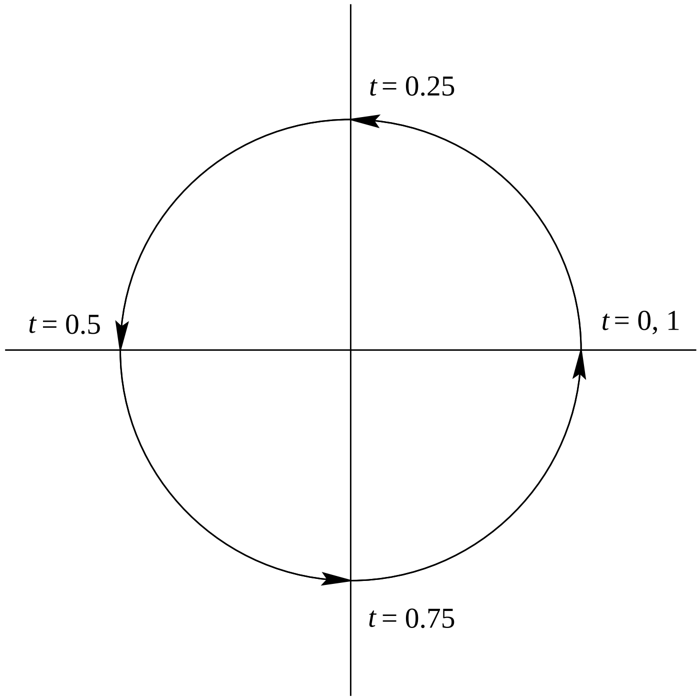

Figure 7.3.1 Motion of a particle on the unit circle centered at the origin

If we let C denote the unit circle centered at the origin, then $f(t)$ is a point in the complex plane on C, $2\pi t$ units from $(1, 0)$ in the counterclockwise direction along the circumference of C. For example, at time $t = 0$ the particle is at $f(0) = 1$, at time $t = \frac{1}{4}$ the particle is at $f\left(\frac{1}{4}\right) = i$, at time $t = \frac{1}{2}$ the particle is at $f\left(\frac{1}{2}\right) = -1$, at $t = \frac{3}{4}$ the particle is at $f\left(\frac{3}{4}\right) = -i$, and at time $t = 1$ the particle is at $f(1) = 1$. Note that f has period 1, so the particle traverses C once in the counterclockwise direction as t goes from 0 to 1, after which the particle will repeat this motion over every interval of time of length 1. See Figure 7.3.1. As an example of a limit, we note that

$$\lim_{t\to-\frac{1}{4}} f(t) = \lim_{t\to-\frac{1}{4}} \cos(2t) + i \lim_{t\to-\frac{1}{4}} \sin(2t) = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = -i.$$

Note that the path of the particle shown in Figure 7.3.1 is not the graph of the function f, but rather a plot of $f(t)$ for values of t from 0 to 1. In general, if $f(t) = x(t) + iy(t)$ represents the position of a particle moving in the complex plane at time t, we can obtain a good representation of the path of the particle over an interval of time $[a, b]$ by plotting the points $(x(t), y(t))$ for a large number of points in $[a, b]$ and connecting these points with straight lines, similar to the procedure we used for plotting the graph of a function in Section 2.1.

Example Suppose a particle moves in the plane so that its position at time t is given by

$$z(t) = \tanh(t) + i\text{sech}(t).$$

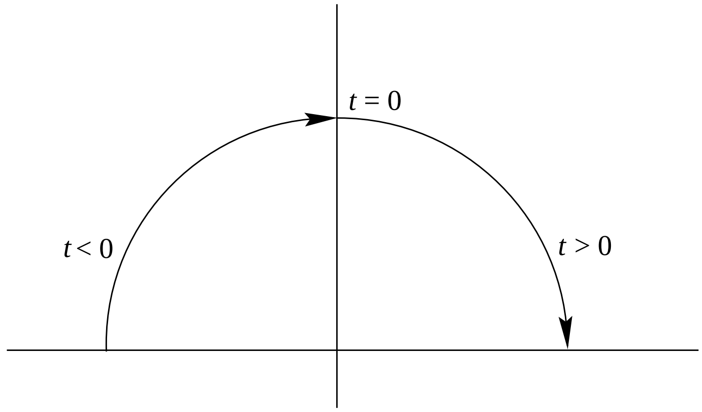

Figure 7.3.2 Motion on the upper half of the unit circle

Then $\Re(z(t)) = \tanh(t)$, $\Im(z(t)) = \operatorname{sech}(t)$, and

$$|z(t)| = \sqrt{\tanh^2(t) + \operatorname{sech}^2(t)} = 1.$$

Thus the particle is moving along the unit circle C as in the previous example. However, since $\operatorname{sech}(t) > 0$ for all t, the particle is always on the upper half of C. Moreover, $z(0) = i$,

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \tanh(t) + i \lim_{t\to\infty} \operatorname{sech}(t) = 1$$

and

$$\lim_{t\to-\infty} z(t) = \lim_{t\to-\infty} \tanh(t) + i \lim_{t\to-\infty} \operatorname{sech}(t) = -1.$$

Combining these results with the fact that $\Re(z(t)) = \tanh(t)$ is an increasing function, we see that as time flows from $-\infty$ to ∞, the particle moves from left to right on the upper half of C, coming from the point $(-1, 0)$ as t increases from $-\infty$ and approaching the point $(1, 0)$ as t increases toward ∞. See Figure 7.3.2.

We may now define continuity and differentiability in analogy with our previous definitions.

Definition We say a function $f : \mathbb{R} \to \mathbb{C}$ is *continuous* at a if $\lim_{t\to a} f(t) = f(a)$.

If $x(t) = \Re(f(t))$ and $y(t) = \Im(f(t))$, then

$$\lim_{t\to a} f(t) = f(a)$$

if and only if

$$\lim_{t\to a} x(t) = x(a)$$

and

$$\lim_{t\to a} y(t) = y(a).$$

Hence f is continuous at a if and only if both x and y are continuous at a. For example, the functions in the previous examples are both continuous for every t in $(-\infty, \infty)$ since the functions $\cos(t)$, $\sin(t)$, $\tanh(t)$, and $\text{sech}(t)$ are continuous for all t in $(-\infty, \infty)$.

Definition If $f : \mathbb{R} \to \mathbb{C}$, then the derivative of f at a, denoted $f'(a)$, is given by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}, \tag{7.3.2}$$

provided the limit exists.

Now if $f(t) = x(t) + iy(t)$, where x and y are both differentiable, then

$$\begin{aligned} f'(t) &= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \\ &= \lim_{h\to 0} \frac{x(t+h) + iy(t+h) - (x(t) + iy(t))}{h} \\ &= \lim_{h\to 0} \frac{x(t+h) - x(t)}{h} + i \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} \\ &= x'(t) + iy'(t). \end{aligned}$$

Hence differentiating a function $f : \mathbb{R} \to \mathbb{C}$ reduces to differentiating the real and complex parts of f.

Proposition If $x : \mathbb{R} \to \mathbb{R}$ and $y : \mathbb{R} \to \mathbb{R}$ are differentiable and $f(t) = x(t) + iy(t)$, then

$$f'(t) = x'(t) + iy'(t). \tag{7.3.3}$$

Example If $f(t) = \cos(t) + i\sin(t)$, then $f'(t) = -\sin(t) + i\cos(t)$.

Of course, it is also possible to express a function $f : \mathbb{R} \to C$ in polar form. If we let $r(t) = |f(t)|$ and $\theta(t) = \arg(f(t))$, then r and θ are real-valued functions and

$$f(t) = r(t)e^{i\theta(t)}. \tag{7.3.4}$$

If r and θ are differentiable, then

$$\begin{aligned} f'(t) &= \frac{d}{dt} r(t)e^{i\theta(t)} \\ &= \frac{d}{dt}(r(t)\cos(\theta(t)) + ir(t)\sin(\theta(t))) \\ &= -r(t)\sin(\theta(t))\theta'(t) + r'(t)\cos(\theta(t)) + ir(t)\cos(\theta(t))\theta'(t) + ir'(t)\sin(\theta(t)) \\ &= r(t)\theta'(t)(-\sin(\theta(t)) + i\cos(\theta(t))) + r'(t)(\cos(\theta(t)) + i\sin(\theta(t))) \\ &= ir(t)\theta'(t)\left(-\frac{\sin(\theta(t))}{i} + \cos(\theta(t))\right) + r'(t)e^{i\theta(t)} \\ &= ir(t)\theta'(t)(i\sin(\theta(t)) + \cos(\theta(t))) + r'(t)e^{i\theta(t)} \\ &= ir(t)\theta'(t)e^{i\theta(t)} + r'(t)e^{i\theta(t)}. \end{aligned}$$

Note that this result is exactly what we would obtain if we treated i as a real constant and differentiated (7.3.4) using the product and chain rules. Hence instead of remembering the formula, we need only remember that we should differentiate a function given in polar form using the product rule and treating i as we would any constant. In particular, taking $r(t) = 1$ for all t, we have

$$\frac{d}{dt}e^{i\theta(t)} = i\theta'(t)e^{i\theta(t)}. \tag{7.3.5}$$

Example If $f(t) = 4te^{it^2}$, then

$$f'(t) = 4t(2it)e^{it^2} + 4e^{it^2} = (4 + 8it^2)e^{it^2}.$$

To understand the derivative geometrically, consider the setting where $z(t) = x(t) + iy(t)$ represents the position at time t of a particle moving in the plane. Then $x'(t)$ represents the velocity of the particle in the x direction and $y'(t)$ represents the velocity of the particle in the y direction. In other words, if all forces acting on the particle were to cease at time t_0, then, according to Newton's first law, during the next unit of time the particle would move in a straight line, $x'(t_0)$ units in the x direction and $y'(t_0)$ units in the y direction. That is, in one unit of time the particle would travel along a straight line from $(x(t_0), y(t_0))$ to $(x(t_0) + x'(t_0), y(t_0) + y'(t_0))$. Hence the particle would move, in a straight line, from $z(t_0)$ to $z(t_0) + z'(t_0)$. Moreover, since the distance traveled in this unit of time is $|z'(t_0)|$, the speed of the particle at time t_0 is given by $|z'(t_0)|$. In short, $\arg(z'(t_0))$ tells us the direction in which the particle is moving at time t_0 and $|z'(t_0)|$ tells us the speed at which the particle is traveling at that instant. These considerations make the next definition reasonable.

Definition If $z(t)$ gives the position at time t of a particle moving in the plane, then we call $z'(t)$ the *velocity* of the particle and we call $|z'(t)|$ the *speed* of the particle.

Notice that this definition is directly analogous to our treatment of motion along a straight line in earlier chapters. In that case, if $f(t)$ represented the position of a particle moving along a straight line, then we called $f'(t)$ the velocity of the particle and $|f'(t)|$ the speed of the particle at time t.

Also notice that if for a given time t_0 we draw an arrow from $z(t_0)$ to $z(t_0) + z'(t_0)$, then this arrow points in the direction of motion of the particle at time t_0. Moreover, if $x'(t_0) \neq 0$, then this arrow has slope

$$\frac{y'(t_0)}{x'(t_0)}.$$

Now, from the chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

from which we obtain

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{y'(t)}{x'(t)}.$$

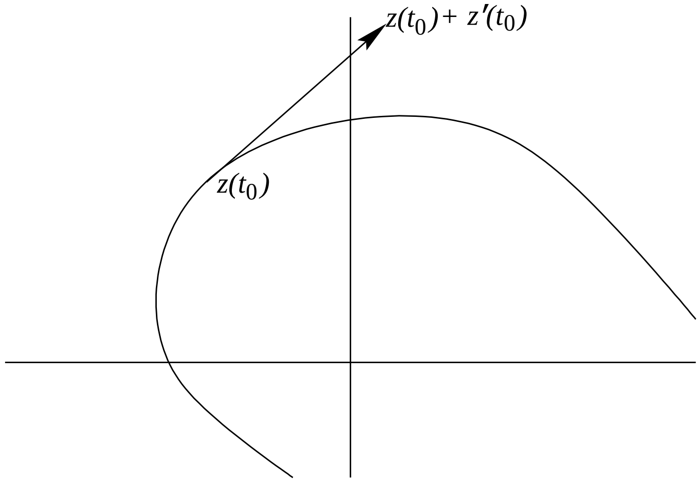

Figure 7.3.3 Motion in the complex plane

Hence the arrow from $z(t_0)$ to $z(t_0) + z'(t_0)$ points along the line tangent to the curve of motion at $z(t_0)$. Moreover, the length of this arrow is the speed of the particle at the instant t_0

Example If the position of a particle moving in the plane is given by $z(t) = 2e^{i\frac{t}{2}}$ at time t, then the particle is traveling counterclockwise on the circle C of radius 2 centered at the origin. The velocity of the particle is given by

$$z'(t) = ie^{i\frac{t}{2}} = e^{i\frac{\pi}{2}}e^{i\frac{t}{2}} = e^{i\left(\frac{t}{2}+\frac{\pi}{2}\right)}$$

and $|z'(t)| = 1$. Hence at any time t, the particle is moving at unit speed with velocity pointing in the direction of $z(t)$ rotated counterclockwise through an angle of $\frac{\pi}{2}$. See Figure 7.3.4.

Also in analogy with the one-dimensional case, if $z(t)$ is the position of a particle moving in the plane at time t, then the *acceleration* of the particle is given by $z''(t)$, the derivative of the velocity. Newton's second law of motion applies in this setting, telling us that if the particle has mass m, then the force acting on the particle at time t is

$$F(t) = mz''(t).$$

Hence the magnitude of the force acting on the particle is

$$|F(t)| = m|z''(t)|$$

and the force acts in the direction of an arrow pointing from $z(t)$ to $z(t) + z''(t)$.

Example In our previous example, position was given by

$$z(t) = 2e^{i\frac{t}{2}}$$

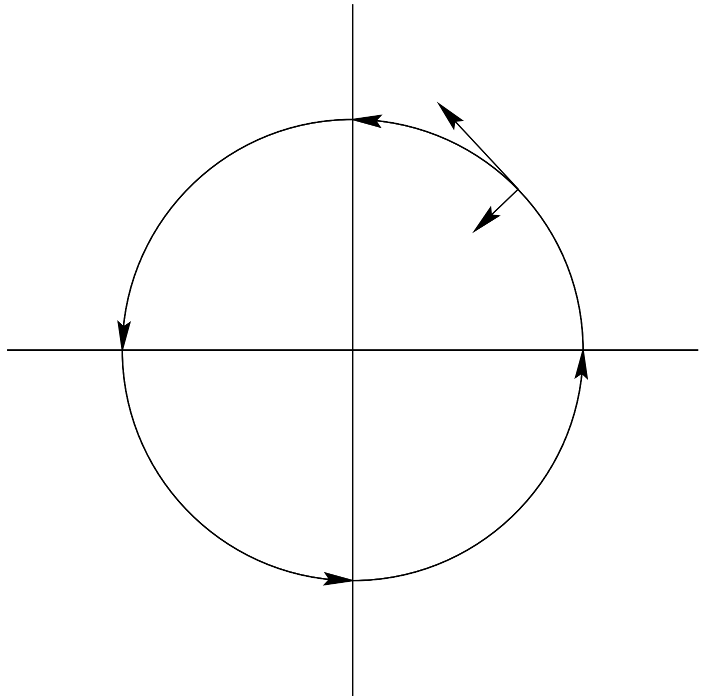

Figure 7.3.4 Arrows indicating velocity and acceleration at time $t = \frac{\pi}{2}$

and velocity by

$$z'(t) = ie^{i\frac{t}{2}}.$$

Thus the acceleration of the particle is

$$z''(t) = -\frac{1}{2}e^{i\frac{t}{2}}.$$

Note that

$$z''(t) = -\frac{1}{4}z(t),$$

showing that the force acting on the particle is directed toward the center of the circle C. See Figure 7.3.4.

Since we may compute the derivative of a complex-valued function by differentiating its complex and real parts separately, it is reasonable to define the definite integral of such a function in terms of the integrals of its real and complex parts.

Definition Suppose $f : \mathbb{R} \to \mathbb{C}$ with $f(t) = x(t) + iy(t)$. If x and y are both integrable on the interval $[a, b]$, then we define the *definite integral* of f over the interval $[a, b]$ by

$$\int_a^b f(t)dt = \int_a^b x(t)dt + i\int_a^b y(t)dt. \tag{7.3.6}$$

Example If

$$f(t) = \sin(t) + i\cos\left(\frac{t}{2}\right),$$

then

$$\begin{aligned}\int_0^\pi f(t)dt &= \int_0^\pi \sin(t)dt + i\int_0^\pi \cos\left(\frac{t}{2}\right)dt \\ &= -\cos(t)\Big|_0^\pi + 2i\sin\left(\frac{t}{2}\right)\Big|_0^\pi \\ &= (1+1) + i(2-0) \\ &= 2+2i.\end{aligned}$$

If $F:\mathbb{R}\to\mathbb{C}$ and $f:\mathbb{R}\to C$ are continuous on $[a,b]$, with $F(t) = X(t)+iY(t)$, $f(t) = x(t)+iy(t)$, and $f(t) = F'(t)$ for all t in (a,b), then

$$\begin{aligned}\int_a^b f(t)dt &= \int_a^b x(t)dt + i\int_a^b y(t)dt \\ &= X(t)\Big|_a^b + iY(t)\Big|_a^b \\ &= (X(b)-X(a)) + i(Y(a)-Y(b)) \\ &= (X(b)+iY(b)) - (X(a)+iY(a)) \\ &= F(b)-F(a).\end{aligned}$$

Hence we have a version of the Fundamental Theorem of Integral Calculus which may be applied directly to complex-valued functions of a real variable.

Proposition If $F:\mathbb{R}\to\mathbb{C}$ and $f:\mathbb{R}\to C$ are continuous on $[a,b]$ with $f(t) = F'(t)$ for all t in $[a,b]$, then

$$\int_a^b f(t)dt = F(b)-F(a).$$

Example If $f(t) = e^{3it}$, then

$$\int_0^\pi f(t)dt = \int_0^\pi e^{3it}dt = \frac{1}{3i}e^{3it}\Big|_0^\pi = \frac{1}{3i}(e^{3i\pi}-e^0) = \frac{1}{3i}(-1-1) = -\frac{2}{3i} = \frac{2}{3}i.$$

Example Integration by parts, which we derived in Section 4.5 from the product rule, is applicable in our current situation. For example, to evaluate

$$\int_0^\pi 3te^{it}dt,$$

we let

$$u = 3t \qquad dv = e^{it}dt$$

$$du = 3dt \quad v = \frac{1}{i}e^{it} = -ie^{it}.$$

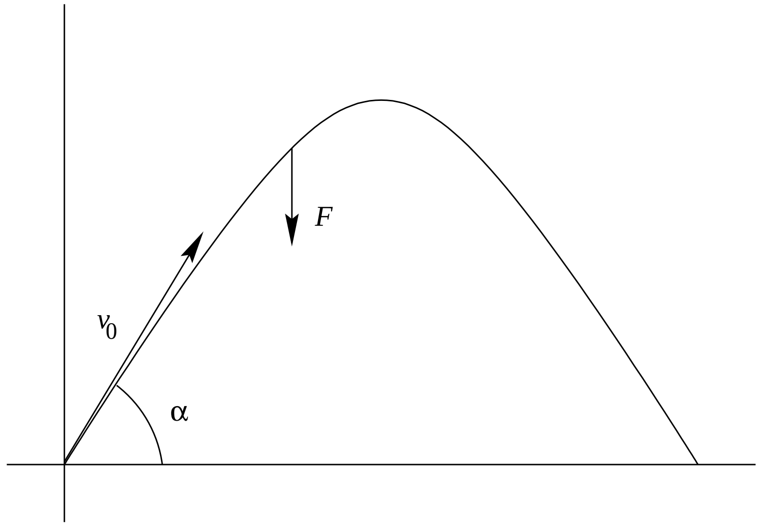

Figure 7.3.5 Motion of a projectile

Then

$$\begin{aligned}\int_0^{\pi} 3te^{it}dt &= -3tie^{it}\Big|_0^{\pi} + \int_0^{\pi} 3ie^{it}dt \\ &= -3\pi ie^{i\pi} + 0 + 3e^{it}\Big|_0^{\pi} \\ &= -3\pi i(-1) + 3e^{i\pi} - 3e^0 \\ &= 3\pi i - 3 - 3 \\ &= -6 + 3\pi i.\end{aligned}$$

In our final example for this section, we consider the problem of finding the motion of a projectile moving close to the surface of the earth. This problem will not only tie together many of the concepts of this section, but it will also provide a preview of Section 7.4 and our discussion of differential equations in Chapter 8.

Example Suppose a projectile of mass m is fired from the surface of the earth at an angle α, where $0 < \alpha < \frac{\pi}{2}$. We will consider the motion of the projectile as a path in the complex plane with its position at time t given by $z(t)$. Further, we assume that its initial position is $z(0) = 0$ and its initial velocity is $z'(0) = v_0$. Ignoring the effects of air resistance, the only force acting on the projectile during its flight is the force of gravity, acting vertically downward. Hence at any time t the force is given by $F = -mgi$, where g is the acceleration due to gravity (32 feet/second^2 or 9.8 meters/second^2). See Figure 7.3.5. Thus Newton's second law of motion gives us

$$-mgi = mz''(t),$$

that is,

$$-gi = z''(t),$$

at any time t. If we let $v(t)$ be the velocity of the projectile at time t, then

$$v'(t) = z''(t) = -gi.$$

Hence, by the previous proposition,

$$v(t) - v(0) = \int_0^t v'(s)ds = -\int_0^t gids = -gis\Big|_0^t = -gti.$$

Thus

$$v(t) = -gti + v_0.$$

Integrating again, we have, since $z'(t) = v(t)$,

$$z(t) - z(0) = \int_0^t (-gsi + v_0)ds = \left(-\frac{1}{2}gis^2 + v_0 s\right)\Big|_0^t = v_0 t - \frac{1}{2}gt^2 i.$$

Now if $s_0 = |v_0|$, that is, if s_0 is the initial speed of the projectile, then

$$v_0 = s_0 e^{i\alpha} = s_0\cos(\alpha) + s_0\sin(\alpha)i.$$

Hence

$$z(t) = (s_0\cos(\alpha) + s_0\sin(\alpha)i)t - \frac{1}{2}gt^2 i = s_0\cos(\alpha)t + \left(s_0\sin(\alpha)t - \frac{1}{2}gt^2\right)i.$$

Thus

$$\Re(z(t)) = s_0\cos(\alpha)t$$

and

$$\Im(z(t)) = s_0\sin(\alpha)t - \frac{1}{2}gt^2.$$

That is, if we write $z(t) = x(t) + iy(t)$, where $x : \mathbb{R} \to \mathbb{R}$ and $y : \mathbb{R} \to \mathbb{R}$, then

$$x(t) = s_0\cos(\alpha)t \tag{7.3.7}$$

and

$$y(t) = s_0\sin(\alpha)t - \frac{1}{2}gt^2. \tag{7.3.8}$$

Note that $x(t)$ gives the horizontal distance traveled at time t and $y(t)$ gives the height of the projectile above the ground at time t. For example, if the projectile is fired at an angle of $\alpha = \frac{\pi}{6}$ with an initial speed of $s_0 = 50$ feet per second, then its position at time t is specified by

$$x(t) = 25\sqrt{3}t \text{ feet}$$

and

$$y(t) = 25t - 16t^2 \text{ feet}.$$

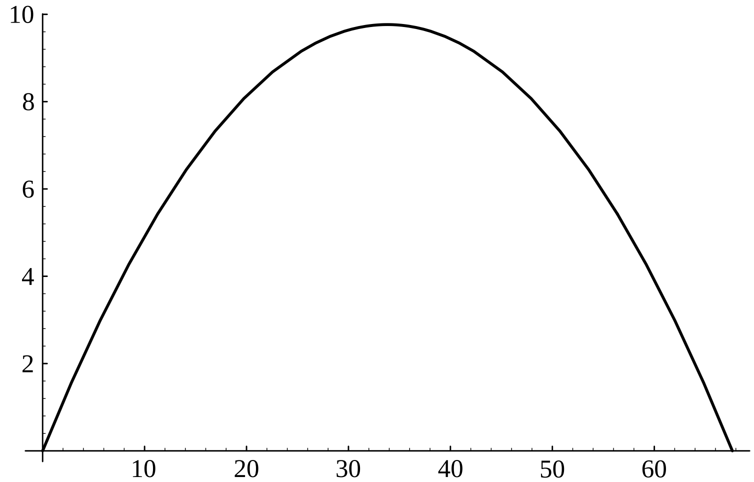

Figure 7.3.6 Motion of a projectile

A plot of this motion is shown in Figure 7.3.6.

In the next section we will consider a more complicated motion problem, namely, the two-body problem, the problem of determining the orbit of a planet about its sun.

Problems

1. For each of the following, suppose the given function specifies the position of a particle moving in the complex plane. Plot the path of the motion over the given time interval, indicating the direction of motion with arrows on the curve.

   (a) $f(t) = \cos(2t) + i\sin(2t)$, $0 \leq t \leq \pi$

   (b) $z(t) = 4\cos(t) + i\sin(t)$, $0 \leq t \leq 2\pi$

   (c) $g(t) = \sin\left(\frac{t}{2}\right) + i\cos\left(\frac{t}{2}\right)$, $0 \leq t \leq 3\pi$

   (d) $z(t) = \operatorname{sech}(2t) + i\tanh(2t)$, $-\infty < t < \infty$

   (e) $f(t) = 2t + it^2$, $-4 \leq t \leq 4$

   (f) $g(t) = t^2 + it^4$, $-2 \leq t \leq 2$

   (g) $z(t) = 3e^{it}$, $-\pi \leq t \leq \pi$

   (h) $h(t) = 3te^{it}$, $0 \leq t \leq 6\pi$

   (i) $z(t) = \frac{3}{t}e^{2it}$, $1 \leq t \leq 20$

2. Differentiate each of the functions in the previous problem.

3. For each of the following, suppose the given function specifies the position of a particle moving in the complex plane. Find the velocity, speed, and the acceleration for each at the specified time.

   (a) $z(t) = \cos(2t) + i\sin(2t)$, $t = \frac{\pi}{6}$

(b) $f(t) = 3\sin(t) + i\cos(2t)$, $t = \pi$

(c) $z(t) = \tanh(t) + i\text{sech}(t)$, $t = 3$

(d) $h(t) = 4t^2 + i(4t - 1)$, $t = 1$

(e) $z(t) = 5e^{it}$, $t = \dfrac{\pi}{2}$

(f) $f(t) = 4t^2 e^{2it}$, $t = \dfrac{5\pi}{3}$

4. Evaluate the following integrals.

(a) $\displaystyle\int_0^4 (2t + it)dt$

(b) $\displaystyle\int_0^{\pi} (\sin(t) + i\cos(3t))dt$

(c) $\displaystyle\int_0^{\frac{\pi}{2}} (-3\sin(2t) + it^3)dt$

(d) $\displaystyle\int_0^{\frac{\pi}{3}} 5e^{it}dt$

(e) $\displaystyle\int_0^{\pi} 2te^{3it}dt$

(f) $\displaystyle\int_0^{\pi} t^2e^{it}dt$

5. Suppose $z(t)$ specifies the position at time t of a particle moving in the complex plane. If we know $z(0) = 1 + i$ and $z'(t) = \cos(t) + i\sin(t)$, find $z(t)$ and plot the path of the object for $0 \le t \le 2\pi$.

6. In the last example of this section we saw that if a projectile is fired from the surface of the earth at an angle α, $0 < \alpha < \frac{\pi}{2}$, with an initial speed of s_0 feet per second, then the x and y coordinates of its position after t seconds are given by

$$x = s_0\cos(\alpha)t$$

and

$$y = s_0\sin(\alpha)t - 16t^2.$$

(a) Find the time t at which the projectile strikes the ground.

(b) The *range* R of the projectile is the value of x when the projectile strikes the ground. Use your result from (a) to find R.

(c) Show that R is maximized when $\alpha = \frac{\pi}{4}$.

(d) Solve the first equation for t in terms of x and substitute this result into the second equation to show that path of the projectile is a parabola.

7. A projectile is fired from the surface of the earth at an angle α, $0 < \alpha < \frac{\pi}{2}$, with an initial speed of 150 feet per second.

(a) Using the results of the previous problem, find the maximum range for the projectile.

(b) What is the range of the projectile if $\alpha = \frac{\pi}{6}$? If $\alpha = \frac{\pi}{3}$? In each case, when does the projectile strike the ground?

(c) Plot the path of motion for $\alpha = \frac{\pi}{6}$, $\alpha = \frac{\pi}{4}$, and $\alpha = \frac{\pi}{3}$.

8. Suppose a particle moves in the complex plane so that its position at time t is given by $z(t) = x(t) + iy(t)$, where

$$x(t) = \int_0^t \cos\left(\frac{\pi s^2}{2}\right) ds$$

and

$$y(t) = \int_0^t \sin\left(\frac{\pi s^2}{2}\right) ds.$$

   (a) Plot the path of motion for $-5 \leq t \leq 5$, indicating the direction of motion with arrows on the curve.

   (b) Find the velocity and acceleration of the particle.

9. If $f : \mathbb{R} \to \mathbb{R}$ is such that

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty,$$

then the function

$$\varphi(\lambda) = \int_{-\infty}^{\infty} f(t) e^{i\lambda t} dt$$

is called the *Fourier transform* of f.

   (a) Show that

$$\varphi(\lambda) = \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} \int_{-\infty}^{\infty} t^n f(t) dt.$$

   (b) Show that

$$\varphi'(\lambda) = \sum_{n=0}^{\infty} \frac{i^{n+1}\lambda^n}{n!} \int_{-\infty}^{\infty} t^{n+1} f(t) dt.$$

   (c) Show that

$$\frac{\varphi'(0)}{i} = \int_{-\infty}^{\infty} t f(t) dt.$$

   (d) Show that

$$\frac{\varphi^{(n)}(0)}{i^n} = \int_{-\infty}^{\infty} t^n f(t) dt$$

   for $n = 0, 1, 2, \ldots$.

10. Let

$$f(t) = \begin{cases} e^{-t}, & \text{for } t \geq 0, \\ 0, & \text{for } t < 0. \end{cases}$$

   (a) With reference to the previous problem, show that the Fourier transform of f is

$$\varphi(\lambda) = \frac{1}{1 - i\lambda}.$$

(b) Use the results from (a) and Problem 9 to evaluate

$$\int_0^\infty t^n e^{-t} dt$$

for $n = 0, 1, 2, 3, 4$.

(c) For $s > 0$, the function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

is called the *gamma function*. Show that

$$\Gamma(n+1) = n!$$

for $n = 0, 1, 2, \ldots$.

11. With reference to Problem 9, find the Fourier transform of

$$f(t) = e^{-\frac{t^2}{2}}$$

and use it to evaluate

$$\int_{-\infty}^\infty t^n e^{-\frac{t^2}{2}} dt$$

for $n = 0, 1, 2, 3, 4$.