Section 7.1

The Algebra of Complex Numbers

At this point we have considered only real-valued functions of a real variable. That is, all of our work has centered on functions of the form $f : \mathbb{R} \to \mathbb{R}$, functions which take a real number to a real number. In this chapter we will discuss complex numbers and the calculus of associated functions. In particular, if we let $\mathbb{C}$ represent the set of all complex numbers, then we will be interested in functions of the form $f : \mathbb{R} \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$. We will begin the story in this section with a discussion of what complex numbers are and how we work with them.

Perhaps because of their name, it is sometimes thought that complex numbers are in some way more mysterious than real numbers, that a number such as $i = \sqrt{-1}$ is not as "real" as a number like 2 or -351.127 or even π. However, all of these numbers are equally meaningful, they are all useful mathematical abstractions. Although complex numbers are a relatively recent invention of mathematics, dating back just over 200 years in their current form, it is also the case that negative numbers, which were once called fictitious numbers to indicate that they were less "real" than positive numbers, have only been accepted for about the same period of time, and we have only started to understand the nature of real numbers during the past 150 years or so. In fact, if you think about their underlying meaning, π is a far more "complex" number than i.

Although complex numbers originate with attempts to solve certain algebraic equations, such as

$$x^2 + 1 = 0,$$

we will give a geometric definition which identifies complex numbers with points in the plane. This definition not only gives complex numbers a concrete geometrical meaning, but also provides us with a powerful algebraic tool for working with points in the plane.

Definition A *complex number* is an ordered pair of real numbers with addition defined by

$$(a, b) + (c, d) = (a + c, b + d) \tag{7.1.1}$$

and multiplication defined by

$$(a, b) \times (c, d) = (ac - bd, ad + bc), \tag{7.1.2}$$

where a, b, c, and d are any real numbers.

We will let i denote the complex number $(0, 1)$. Then, by our definition of multiplication,

$$i^2 = (0, 1) \times (0, 1) = (0 - 1, 0 + 0) = (-1, 0). \tag{7.1.3}$$

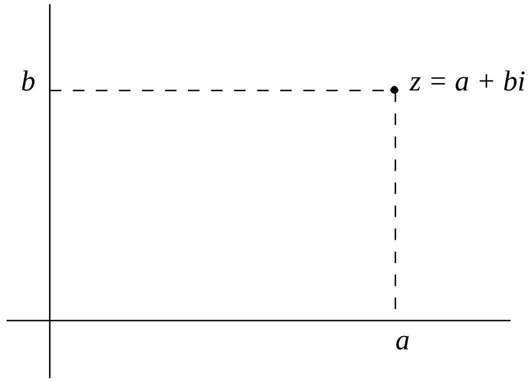

Figure 7.1.1 Geometric representation of a complex number

If we identify the real number a with the complex number $(a, 0)$, then we have

$$ai = (a, 0) \times (0, 1) = (0 - 0, a + 0) = (0, a).$$

Then for any two real numbers, we have

$$(a, b) = (a, 0) + (0, b) = a + bi. \tag{7.1.4}$$

That is, $a + bi$ is another way to write the complex number (a, b). In particular, with this convention, (7.1.3) becomes

$$i^2 = -1, \tag{7.1.5}$$

that is,

$$i = \sqrt{-1}. \tag{7.1.6}$$

Moreover, we may write (7.1.1) as

$$(a + bi) + (c + di) = (a + c) + (b + d)i \tag{7.1.7}$$

and (7.1.2) as

$$(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i. \tag{7.1.8}$$

In fact, we may view the latter as a consequence of the ordinary algebraic expansion of the product

$$(a + bi)(c + di)$$

combined with the equality $i^2 = -1$. That is,

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

It also follows from this formulation that if r is a real number, which we identify with $r + 0i$, and $z = a + bi$ is a complex number, then

$$rz = r(a + bi) = (r + 0i)(a + bi) = ra + rbi. \tag{7.1.9}$$

As indicated above, we let $\mathbb{C}$ denote the set of all complex numbers. Because of our identification of $\mathbb{C}$ with the plane, we usually refer to $\mathbb{C}$ as the *complex plane*. Since the description of complex numbers as points in the plane is often associated with the work of Carl Friedrich Gauss (1777-1855) (although appearing first in the work of Caspar Wessel (1745-1818)), $\mathbb{C}$ is also referred to as the *Gaussian plane*.

Example $(3 + 4i) + (5 - 6i) = 8 - 2i$.

Example $(2 + i)(3 - 2i) = 6 - 4i + 3i - 2i^2 = 8 - i$.

Example $-3(4 + 2i) = -12 - 6i$.

We have yet to define subtraction and division for complex numbers. If z and w are complex numbers, we may define

$$z - w = z + (-1)w. \tag{7.1.10}$$

It follows that if $z = a + bi$ and $w = c + di$, then

$$z - w = a + bi + (-c - di) = (a - c) + (b - d)i. \tag{7.1.11}$$

As a first step toward defining division, note that if $z = a + bi$ with either $a \neq 0$ or $b \neq 0$, then

$$(a + bi)\left(\frac{a - bi}{a^2 + b^2}\right) = \frac{a^2 - b^2 i^2}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1.$$

In other words,

$$\frac{a - bi}{a^2 + b^2}$$

is the multiplicative inverse, or reciprocal, of $z = a + bi$. Hence we will write

$$z^{-1} = \frac{1}{z} = \frac{a - bi}{a^2 + b^2}. \tag{7.1.12}$$

Given another complex number w, we may define w divided by z by

$$\frac{w}{z} = wz^{-1}. \tag{7.1.13}$$

Definition Given a complex number $z = a + bi$, the number $a - bi$ is called the *conjugate* of z and is denoted $\bar{z}$.

Note that if $z = a + bi$, then

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2. \tag{7.1.14}$$

Hence

$$\sqrt{z\bar{z}} = \sqrt{a^2 + b^2}, \tag{7.1.15}$$

which is the distance in the complex plane from z to the origin.

Definition Given a complex number z, the *magnitude* of z, denoted $|z|$, is defined by

$$|z| = \sqrt{z\bar{z}}. \tag{7.1.16}$$

The magnitude of a complex number generalizes the idea of the absolute value of a real number, and in fact reduces to the absolute value when z is a real number. Moreover, note that if $z = a + bi$, with either $a \neq 0$ or $b \neq 0$, we may now write

$$z^{-1} = \frac{\bar{z}}{|z|^2}. \tag{7.1.17}$$

Although (7.1.12) and (7.1.17) are useful expressions, in most situations the easiest way to simplify a quotient of two complex numbers is to multiply numerator and denominator by the conjugate of the denominator.

Example $\dfrac{2+i}{1+i} = \dfrac{2+i}{1+i}\dfrac{1-i}{1-i} = \dfrac{2-2i+i+1}{1+1} = \dfrac{3-i}{2} = \dfrac{3}{2} - \dfrac{1}{2}i.$

Definition Given a complex number $z = a + bi$, we call a the *real part* of z, denoted $\Re(z)$, and b the *imaginary part* of z, denoted $\Im(z)$.

Because of this definition, we call the horizontal axis of the complex plane the *real axis* and the vertical axis the *imaginary axis*. In this way we may identify the real number line with the the real axis of the complex plane. A complex number of the form bi, where b is a real number, lies on the imaginary axis of the complex plane and is said to be *purely imaginary*. However, we should be careful with our interpretation of this terminology: a purely imaginary number is just as "real", in the ordinary sense of real, as a real number, in the same way that an irrational number is just as "rational", in the sense of reasonable, as a rational number. Note that with this terminology, the conjugate $\bar{z}$ of a complex number z is the point in the complex plane obtained by reflecting z about the real axis.

Example If $z = 3 + 6i$, then $\Re(z) = 3$, $\Im(z) = 6$, $\bar{z} = 3 - 6i$, and

$$|z| = \sqrt{9 + 36} = \sqrt{45} = 3\sqrt{5}.$$

With the above definitions, we may work with the arithmetic and algebra of complex numbers in the same way we work with real numbers. For example, for any complex numbers z and w,

$$z + w = w + z$$

and

$$zw = wz.$$

You will be asked to verify these and other standard properties of the complex numbers in Problem 7 at the end of this section.

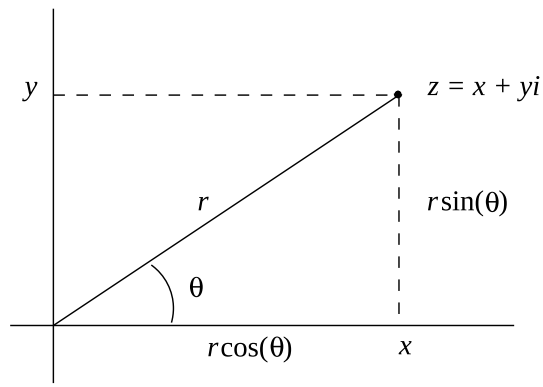

Figure 7.1.2 Polar coordinates for a complex number

Polar notation

When we write a complex number z in the form $z = x + yi$, we refer to x and y as the *rectangular* or *Cartesian coordinates* of z. We now consider another method of representing complex numbers. Let us begin with a complex number $z = x + yi$ written in rectangular form. Assume for the moment that x and y are not both 0. If we let θ be the angle between the real axis and the line segment from $(0, 0)$ to (x, y), measured in the counterclockwise direction, then z is completely determined by the two numbers z and θ. We call θ the *argument* of z and denote it by $\arg(z)$. Geometrically, if we are given $|z|$ and θ, we can locate z in the complex plane by taking the line segment of length $|z|$ lying on the positive real axis, with a fixed endpoint at the origin, and rotating it counterclockwise through an angle θ; the final resting point of the rotating endpoint is the location of z. Algebraically, if $z = x + yi$ is a complex number with $r = |z|$ and $\theta = \arg(z)$, then

$$x = r\cos(\theta) \tag{7.1.18}$$

and

$$y = r\sin(\theta). \tag{7.1.19}$$

Together, r and θ are called the *polar coordinates* of z. See Figure 7.1.2.

Example If $|z| = 2$ and $\arg(z) = \frac{\pi}{6}$, then

$$z = 2\cos\left(\frac{\pi}{6}\right) + 2\sin\left(\frac{\pi}{6}\right) = \sqrt{3} + i.$$

Example If $z = 1 - i$, then $|z| = \sqrt{2}$ and $\arg(z) = -\frac{\pi}{4}$.

Note that in the last example we could have taken $\arg(z) = \frac{7\pi}{4}$, or, in fact,

$$\arg(z) = -\frac{\pi}{4} + 2n\pi$$

for any integer n. In particular, there are an infinite number of possible values for $\arg(z)$ and we will let $\arg(z)$ stand for any one of these values. At the same time, it is often important to choose $\arg(z)$ in a consistent fashion; to this end, we call the value of $\arg(z)$ which lies in the interval $(-\pi, \pi]$ the *principal value* of $\arg(z)$ and denote it by $\operatorname{Arg}(z)$. For our example, $\operatorname{Arg}(z) = -\frac{\pi}{4}$.

In general, if we are given a complex number in rectangular coordinates, say $z = x+yi$, then, as we can see from Figure 7.1.2, the polar coordinates $r = |z|$ and $\theta = \operatorname{Arg}(z)$ are determined by

$$r = \sqrt{x^2 + y^2} \tag{7.1.20}$$

and

$$\tan(\theta) = \frac{y}{x}, \tag{7.1.21}$$

where the latter holds only if $x \neq 0$. If $x = 0$ and $y \neq 0$, then z is purely imaginary and hence lies on the imaginary axis of the complex plane. In that case, $\theta = \frac{\pi}{2}$ if $y > 0$ and $\theta = \frac{\pi}{2}$ if $y < 0$. If both $x = 0$ and $y = 0$, then z is completely specified by the condition $r = 0$ and θ may take on any value.

Note that, since the range of the arc tangent function is $(-\frac{\pi}{2}, \frac{\pi}{2})$, the condition

$$\tan(\theta) = \frac{y}{x}$$

only implies that

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

if $x > 0$, that is, if θ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Example Suppose $z = -1 - \sqrt{3}i$. Then

$$|z| = \sqrt{1+3} = 2$$

and, if $\theta = \operatorname{Arg}(z)$,

$$\tan(\theta) = \frac{-\sqrt{3}}{-1} = -\sqrt{3}.$$

Since z lies in the third quadrant, we have

$$\operatorname{Arg}(z) = -\frac{2\pi}{3}.$$

Now suppose z_1 and z_2 are two nonzero complex numbers with $|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$, and $\arg(z_2) = \theta_2$. Then

$$z_1 = r_1\cos(\theta_1) + r_1\sin(\theta_1)i = r_1(\cos(\theta_1) + \sin(\theta_1)i)$$

and

$$z_2 = r_2\cos(\theta_2) + r_2\sin(\theta_2)i = r_2(\cos(\theta_2) + \sin(\theta_2)i).$$

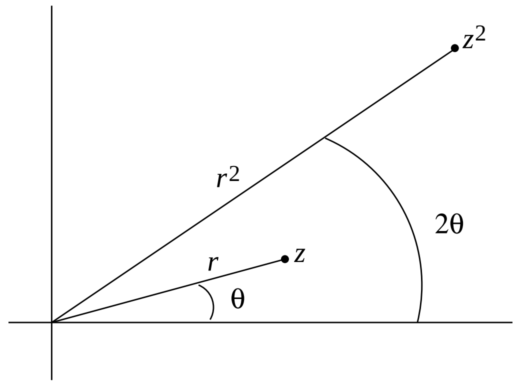

Figure 7.1.3 Geometry of z and z^2 in the complex plane

Hence

$$\begin{aligned} z_1 z_2 &= (r_1\cos(\theta_1) + r_1\sin(\theta_1)i)(r_2\cos(\theta_2) + r_2\sin(\theta_2)i) \\ &= r_1 r_2(\cos(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)i + \sin(\theta_1)\cos(\theta_2)i - \sin(\theta_1)\sin(\theta_2)) \\ &= r_1 r_2(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + (\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\cos(\theta_2))i) \\ &= r_1 r_2(\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2)i). \end{aligned}$$

It follows that

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2| \tag{7.1.22}$$

and

$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2). \tag{7.1.23}$$

In other words, the magnitude of the product of two complex numbers is the product of their respective magnitudes and the argument of the product of two complex numbers is the sum of their respective arguments.

In particular, for any complex number z, $|z^2| = |z|^2$ and $\arg(z^2) = 2\arg(z)$. More generally, for any positive integer n,

$$|z^n| = |z|^n \tag{7.1.24}$$

and

$$\arg(z^n) = n\arg(z). \tag{7.1.25}$$

See Figure 7.1.3.

If z is a complex number with $|z| = r$ and $\arg(z) = \theta$, then

$$z = r(\cos(\theta) + \sin(\theta))$$

and

$$\bar{z} = r(\cos(\theta) - \sin(\theta)i) = r(\cos(\theta) + \sin(-\theta)i). \tag{7.1.26}$$

Hence

$$|\bar{z}| = |z| \tag{7.1.27}$$

and

$$\arg(\bar{z}) = -\arg(z), \tag{7.1.28}$$

in agreement with our previous observation that $\bar{z}$ is obtained from z by reflection about the real axis.

If z_1 and z_2 are two nonzero complex numbers with $|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$, and $\arg(z_2) = \theta_2$, then

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{z_1\bar{z}_2}{z_2\bar{z}_2} \\ &= \frac{r_1r_2(\cos(\theta_1-\theta_2)+\sin(\theta_1-\theta_2)i)}{r_2^2} \\ &= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+\sin(\theta_1-\theta_2)i).\end{aligned}$$

Hence

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{7.1.29}$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2. \tag{7.1.30}$$

In other words, the magnitude of the quotient of two complex numbers is the quotient of their respective magnitudes and the argument of the quotient of two complex numbers is the difference of their respective arguments.

Example Let $z = 2(\cos(\frac{\pi}{12}) + \sin(\frac{\pi}{12})i)$ and $w = 3(\cos(\frac{\pi}{6}) + \sin(\frac{\pi}{6})i)$. Then

$$\begin{aligned}zw &= 6\left(\cos\left(\frac{\pi}{12}+\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{12}+\frac{\pi}{6}\right)i\right) \\ &= 6\left(\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)i\right) \\ &= 6\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) \\ &= 3\sqrt{2}+3\sqrt{2}i.\end{aligned}$$

Also,

$$\begin{aligned}\frac{z}{w} &= 6\left(\cos\left(\frac{\pi}{12}-\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{12}-\frac{\pi}{6}\right)i\right) \\ &= 6\left(\cos\left(-\frac{\pi}{12}\right)+\sin\left(-\frac{\pi}{12}\right)i\right) \\ &= 6\left(\cos\left(\frac{\pi}{12}\right)-\sin\left(\frac{\pi}{12}\right)i\right) \\ &= 5.796-1.553i,\end{aligned}$$

where we have rounded the real and imaginary parts to three decimal places.

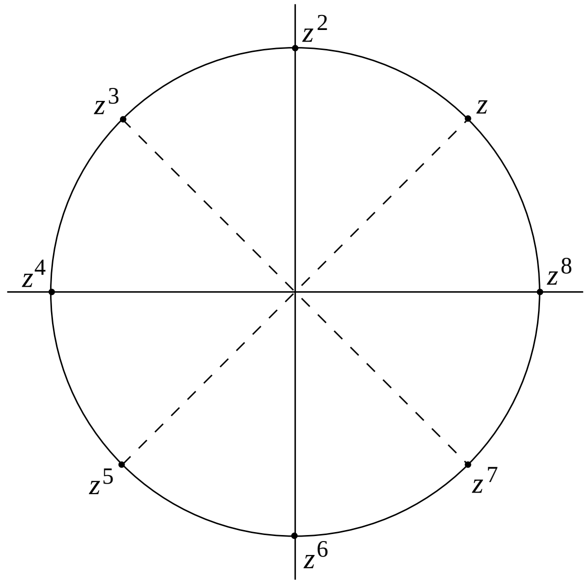

Figure 7.1.4 Powers of $z = \cos(\frac{\pi}{4}) + \sin(\frac{\pi}{4})i$

Example Let

$$z = \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)i = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$

Since $|z| = 1$ and $\arg(z) = \frac{\pi}{4}$, z is a point on the unit circle centered at the origin, one-eighth of the way around the circle from $(1, 0)$ (see Figure 7.1.4). Then

$$\begin{aligned}
z^2 &= \cos\left(2 \cdot \frac{\pi}{4}\right) + \sin\left(2 \cdot \frac{\pi}{4}\right)i = \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)i = i,\\
z^3 &= \cos\left(3 \cdot \frac{\pi}{4}\right) + \sin\left(3 \cdot \frac{\pi}{4}\right)i = \cos\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right)i = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,\\
z^4 &= \cos\left(4 \cdot \frac{\pi}{4}\right) + \sin\left(4 \cdot \frac{\pi}{4}\right)i = \cos(\pi) + \sin(\pi)i = -1,\\
z^5 &= \cos\left(5 \cdot \frac{\pi}{4}\right) + \sin\left(5 \cdot \frac{\pi}{4}\right)i = \cos\left(\frac{5\pi}{4}\right) + \sin\left(\frac{5\pi}{4}\right)i = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\\
z^6 &= \cos\left(6 \cdot \frac{\pi}{4}\right) + \sin\left(6 \cdot \frac{\pi}{4}\right)i = \cos\left(\frac{3\pi}{2}\right) + \sin\left(\frac{3\pi}{2}\right)i = -i,\\
z^7 &= \cos\left(7 \cdot \frac{\pi}{4}\right) + \sin\left(7 \cdot \frac{\pi}{4}\right)i = \cos\left(\frac{7\pi}{4}\right) + \sin\left(\frac{7\pi}{4}\right)i = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,\\
z^8 &= \cos\left(8 \cdot \frac{\pi}{4}\right) + \sin\left(8 \cdot \frac{\pi}{4}\right)i = \cos(2\pi) + \sin(2\pi)i = 1,
\end{aligned}$$

and

$$z^9 = zz^8 = (z)(1) = z.$$

Hence each successive power of z is obtained by rotating the previous power through an angle of $\frac{\pi}{4}$ on the unit circle centered at the origin; after eight rotations, the point has returned to where it started. See Figure 7.1.4. Notice in particular that z is a root of the polynomial

$$P(z) = z^8 - 1.$$

In fact, z^n is a solution of $z^8 - 1 = 0$ for any positive integer n since

$$(z^n)^8 - 1 = (z^8)^n - 1 = 1^n - 1 = 1 - 1 = 0.$$

Thus there are eight distinct roots of $P(z)$, namely, z, z^2, z^3, z^4, z^5, z^6, z^7, and z^8, only two of which, $z^4 = -1$ and $z^8 = 1$, are real numbers.

Problems

1. Evaluate the following if $w = 3 - 4i$ and $z = -2 + 7i$.

   (a) $w + z$
   (b) $w - z$
   (c) $3w - 2z$
   (d) $\bar{w}$
   (e) zw
   (f) $\dfrac{z}{w}$
   (g) $|z|$
   (h) $\dfrac{z^2 - w}{z + w}$
   (i) $\Re(z - w)$
   (j) $\Im(3z + w)$

2. Find the real and imaginary parts of each of the following.

   (a) $\dfrac{1}{i}$
   (b) $\dfrac{3}{1 + 2i}$
   (c) $\dfrac{3 - 4i}{-2 + 3i}$
   (d) $(1 + i)^3$

3. For each of the following, write the given z in rectangular coordinates and plot it in the complex plane.

   (a) $|z| = 3$, $\text{Arg}(z) = \dfrac{\pi}{2}$
   (b) $|z| = 5$, $\text{Arg}(z) = \dfrac{2\pi}{3}$
   (c) $|z| = 0.5$, $\text{Arg}(z) = -\dfrac{3\pi}{4}$
   (d) $|z| = 2$, $\text{Arg}(z) = \pi$

4. For each of the following, find $|z|$ and $\text{Arg}(z)$ and plot z in the complex plane.

   (a) $z = -i$
   (b) $z = -5$
   (c) $z = 1 + i$
   (d) $z = -1 - i$
   (e) $z = 2 + 2\sqrt{3}i$
   (f) $z = \sqrt{3} - i$

5. Suppose w and z are complex numbers with $|w| = 3$, $\text{Arg}(w) = \frac{\pi}{6}$, $|z| = 2$, and $\text{Arg}(z) = -\frac{\pi}{3}$. Find both polar and rectangular coordinates for each of the following.

   (a) w^2
   (b) z^3
   (c) wz
   (d) $\dfrac{w}{z}$

(e) $\dfrac{z}{w^2}$

(f) w^5

6. Find all the roots of the polynomial $P(z) = z^6 - 1$ and plot them in the complex plane.

7. Let $v = a_1 + b_1 i$, $w = a_2 + b_2 i$, and $z = a_3 + b_3 i$ be complex numbers. Verify each of the following.

   (a) $v + w = w + v$

   (b) $vw = wv$

   (c) $v(w + z) = vw + vz$

   (d) $(v + w) + z = v + (w + z)$

   (e) $v(wz) = (vw)z$

   (f) $(w + z)^2 = w^2 + 2wz + z^2$

8. Suppose z is a complex number with $|z| = r$ and $\arg(z) = \theta$.

   (a) Let w be a complex number with $|w| = \sqrt{r}$ and $\arg(w) = \frac{\theta}{2}$. Show that $w^2 = z$.

   (b) Let v be a complex number with $|v| = \sqrt{r}$ and $\arg(v) = \frac{\theta}{2} + \pi$. Show that $v^2 = z$.

   (c) From (a) and (b) we see that every nonzero complex number has two distinct square roots. Find the square roots, in rectangular form, of $1 + \sqrt{3}i$ and -9.