Difference Equations to Differential Equations

Section 2.3

Limits And The Notion Of Continuity

Of particular interest in mathematics and its applications to the physical world are functional relationships in which the dependent variable changes continuously with changes in the independent variable. Intuitively, changing continuously means that small changes in the independent variable do not produce abrupt changes in the dependent variable. For example, a small change in the radius of a circle does not produce an abrupt change in the area of the circle; we would say that the area of the circle changes continuously with the radius of the circle. Similarly, a small change in the height from which some object is dropped will result in a related small change in the object's terminal velocity; hence terminal velocity is a continuous function of height. On the other hand, when an electrical switch is closed, there is an abrupt change in the current flowing through the circuit; the current flow through the circuit is not a continuous function of time. The purpose of this section is to introduce the terminology and concepts that will give us a proper mathematical basis for discussing continuity in the next section.

To begin our study of continuity, we will first look at two examples of functions which are not continuous. In this way we will discover what properties to exclude when forming our definition of a continuous function.

Example Consider the function H defined by

$$H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0. \end{cases} \tag{2.3.1}$$

This function, known as the *Heaviside function*, might be used in connection with modeling the current passing through a switch which is open until time $t = 0$ and then closed. The graph of this function consists of two horizontal half-lines with a vertical gap of unit length at the origin, as shown in Figure 2.3.1. Since this function has a break in its graph at 0, its output changes abruptly as t passes from negative values to positive values. In fact, if $t < 0$, $H(t) = 0$ no matter how close t is to 0, whereas if $t > 0$, $H(t) = 1$ no matter how close t is to 0. Hence, near 0, small changes in t may result in sudden changes in $H(t)$. We say that H has a discontinuity at $t = 0$.

In this section we will develop the language and notation necessary to describe this situation mathematically. In particular, note that for any sequence $\{t_n\}$ with $t_n < 0$ for all n and $\lim_{n\to\infty} t_n = 0$, we have

$$\lim_{n\to\infty} H(t_n) = 0$$

since $H(t_n) = 0$ for all n. We say that the limit of $H(t)$ as t approaches 0 from the left is 0, which we denote by

$$\lim_{t\to 0^-} H(t) = 0.$$

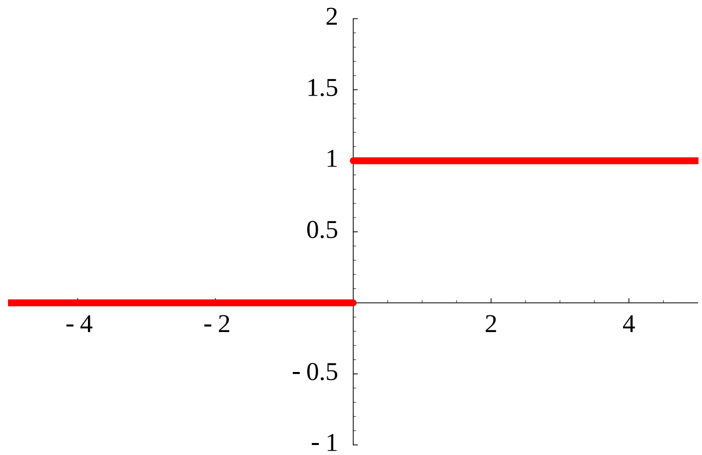

Figure 2.3.1 Graph of the Heaviside function

However, for any sequence $\{t_n\}$ with $t_n > 0$ for all n and $\lim_{n\to\infty} t_n = 0$, we have

$$\lim_{n\to\infty} H(t_n) = 1$$

since $H(t_n) = 1$ for all n. We say that the limit of $H(t)$ as t approaches 0 from the right is 1, which we denote by

$$\lim_{t\to 0^+} H(t) = 1.$$

Since these two limiting values do not agree, we say that $H(t)$ does not have a limit as t approaches 0. Hence, in this case, the discontinuity of H at 0 is characterized by the absence of a limiting value for $H(t)$ at 0. In the next section we will make the existence of a limiting value one of the criteria for a function to be continuous at a point.

Example Now consider the function

$$g(x) = \begin{cases} -x, & \text{if } x < 0, \\ 1, & \text{if } x = 0, \\ x, & \text{if } x > 0. \end{cases}$$

As with the previous example, this function does not change continuously as x passes from negative to positive values. However, the discontinuity arises in a different manner. Note that if $\{x_n\}$ is a sequence with $x_n < 0$ for all n and $\lim_{n\to\infty} x_n = 0$, then

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty}(-x_n) = -\lim_{n\to\infty} x_n = 0.$$

Thus

$$\lim_{x\to 0^-} g(x) = 0.$$

Similarly, if $\{x_n\}$ is a sequence with $x_n > 0$ for all n and $\lim_{n\to\infty} x_n = 0$, then

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} x_n = 0.$$

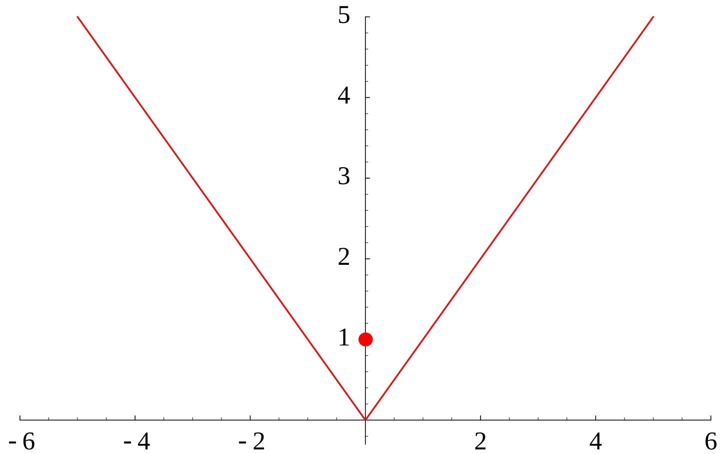

Figure 2.3.2 Graph of $y = g(x)$

Thus

$$\lim_{x \to 0^+} g(x) = 0.$$

Hence in this case $g(x)$ does have a limiting value as x approaches 0 and we can write

$$\lim_{x \to 0} g(x) = 0.$$

However, there is still an sudden change in the value of the function at 0 because $g(0) = 1$, not 0. Graphically, this shows up as a hole in the graph of g at the origin, as shown in Figure 2.3.2. Thus the abrupt change in values of $g(x)$ results not from the lack of a limiting value as x approaches 0, but rather from the fact that

$$g(0) = 1 \neq 0 = \lim_{x \to 0} g(x).$$

This illustrates another type of behavior that we will have to exclude in our definition of continuity.

These examples illustrate two ways in which a function may fail to be continuous. The definition which we will discuss in Section 2.4 essentially says that a function is continuous if it does not have either of the problems that we have seen with H and g. However, before pursuing this question further, we must first introduce the notion of a limit for a function defined on an interval of real numbers. We have already seen the pattern for this definition in the previous examples. Namely, in order to define, for some function f, the limit of $f(x)$ as x approaches some number c, we consider sequences $\{x_n\}$ that converge to c and ask if the sequence $\{f(x_n)\}$ has a limit. Hence we reduce our new question to the old problem of limits of sequences that we considered back in Section 1.2. However, we must be careful about two points. First, there will always be more than one sequence $\{x_n\}$ which converges to a given point c. As we saw in the examples, in order to understand the behavior of a function near c, we must take into account how the function behaves on all possible sequences that converge to c. Second, we want the limit to describe what is happening to the function for values of x close to c, but not equal to c. Thus we must restrict the sequences $\{x_n\}$ to those for which $x_n \neq c$ for all values of n. With these ideas in mind, we now have the following definition.

Definition Let I be an open interval and let c be a point in I. Let J be the set consisting of all points of I except the point c; that is, $J = \{x \mid x \text{ is in } I, x \neq c\}$. Suppose J is in the domain of the function f. We say the *limit* of $f(x)$ as x approaches c is L, denoted

$$\lim_{x \to c} f(x) = L, \tag{2.3.2}$$

if for every sequence $\{x_n\}$ of points in J we have

$$\lim_{n \to \infty} f(x_n) = L \tag{2.3.3}$$

whenever

$$\lim_{n \to \infty} x_n = c. \tag{2.3.4}$$

In other words, to determine the value of $\lim\limits_{x \to c} f(x)$, we ask for the limit of the sequence $\{f(x_n)\}$, where $\{x_n\}$ is any sequence in J which is approaching c. If $\{f(x_n)\}$ approaches L for all such sequences , then L is the limit of $f(x)$ as x approaches c.

We define one-sided limits in a similar fashion. Namely, if J is an open interval of the form (c, b) in the domain of f, then we say the *limit of $f(x)$ as x approaches c from the right* is L, denoted

$$\lim_{x \to c^+} f(x) = L, \tag{2.3.5}$$

if for every sequence $\{x_n\}$ of points in J we have

$$\lim_{n \to \infty} f(x_n) = L \tag{2.3.6}$$

whenever

$$\lim_{n \to \infty} x_n = c. \tag{2.3.7}$$

Similarly, if J is an open interval of the form (a, c) in the domain of f, then we say the *limit of $f(x)$ as x approaches c from the left* is L, denoted

$$\lim_{x \to c^-} f(x) = L, \tag{2.3.8}$$

if for every sequence $\{x_n\}$ of points in J we have

$$\lim_{n \to \infty} f(x_n) = L \tag{2.3.9}$$

whenever

$$\lim_{n \to \infty} x_n = c. \tag{2.3.10}$$

Note that the existence of a one-sided limit only requires that the limiting value of $\{f(x_n)\}$ be the same for all sequences $\{x_n\}$ which approach c from the same side, whereas the existence of a limit requires that the limiting value of be the same for all sequences $\{x_n\}$ which approach c. In particular, this means that if

$$\lim_{x \to c} f(x) = L$$

then we must have both

$$\lim_{x \to c^+} f(x) = L$$

and

$$\lim_{x \to c^-} f(x) = L.$$

Not surprisingly, this also works in the other direction; in general, we have

$$\lim_{x \to c} f(x) = L \text{ if and only if both } \lim_{x \to c+} f(x) = L \text{ and } \lim_{x \to c^-} f(x) = L. \tag{2.3.11}$$

Since the above definitions are all in terms of limits of sequences, we may use all the properties of limits of sequences developed in Section 1.2 when discussing the limit of a function defined on an interval of real numbers.

Example Consider the constant function $f(x) = 2$ for all x. To compute, for example, $\lim_{x \to 3} f(x)$, we need to compute $\lim_{n \to \infty} f(x_n)$ for an arbitrary sequence $\{x_n\}$ with $\lim_{n \to \infty} x_n = 3$. For such a sequence, we have

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} 2 = 2.$$

Hence

$$\lim_{x \to 3} f(x) = 2.$$

In fact, it should be easy to see that for any value of c

$$\lim_{x \to c} f(x) = \lim_{x \to c} 2 = 2,$$

and, more generally, for any constant k,

$$\lim_{x \to c} k = k.$$

Example Suppose $f(x) = x$. To find $\lim_{x \to 5} f(x)$, first let $\{x_n\}$ be any sequence with $\lim_{n \to \infty} x_n = 5$. Then

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n = 5,$$

so

$$\lim_{x \to 5} f(x) = 5.$$

In fact, we could replace 5 by an arbitrary c in this computation and obtain the general result that

$$\lim_{x \to c} x = c. \tag{2.3.12}$$

Example To find $\lim_{x \to 6} x^2$, let $\{x_n\}$ be any sequence with $\lim_{n \to \infty} x_n = 6$. Then

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n^2 = \left(\lim_{n \to \infty} x_n\right)^2 = 6^2 = 36.$$

Thus

$$\lim_{x \to 6} x^2 = 36.$$

Again we can generalize this statement by replacing 6 by an arbitrary c, in which case we have

$$\lim_{x \to c} x^2 = c^2.$$

Moreover, we may replace the power 2 by any rational number p for which x^p and c^p are defined and have

$$\lim_{x \to c} x^p = c^p. \tag{2.3.13}$$

Example Let $f(x) = 4x^3 - 6x^2 + x - 7$. To find $\lim_{x \to 2} f(x)$, let $\{x_n\}$ be any sequence with $\lim_{n \to \infty} x_n = 2$. Then

$$\begin{aligned}
\lim_{n \to \infty} f(x_n) &= \lim_{n \to \infty} (4x_n^3 - 6x_n^2 + x_n - 7) \\
&= 4\left(\lim_{n \to \infty} x_n\right)^3 - 6\left(\lim_{n \to \infty} x_n\right)^2 + \lim_{n \to \infty} x_n - 7 \\
&= (4)(2^3) - (6)(2^2) + 2 - 7 = 3.
\end{aligned}$$

Hence

$$\lim_{x \to 2} f(x) = 3,$$

which is just $f(3)$.

Example Now let f be an arbitrary polynomial, say,

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

for some constants $a_0, a_1, a_2, \ldots, a_m$. If $\{x_n\}$ is any sequence with $\lim_{n \to \infty} x_n = c$, then

$$\begin{aligned}
\lim_{n \to \infty} f(x_n) &= \lim_{n \to \infty} (a_m x_n^m + a_{m-1} x_n^{m-1} + \cdots + a_2 x_n^2 + a_1 x_n + a_0) \\
&= a_m \left(\lim_{n \to \infty} x_n\right)^m + a_{m-1} \left(\lim_{n \to \infty} x_n\right)^{m-1} + \ldots + a_2 \left(\lim_{n \to \infty} x_n\right)^2 \\
&\qquad + a_1 \left(\lim_{n \to \infty} x_n\right) + a_0 \\
&= a_m c^n + a_{m-1} c^{m-1} + \cdots + a_2 c^2 + a_1 c + a_0 \\
&= f(c).
\end{aligned}$$

Hence, for any polynomial f and any real number c,

$$\lim_{x\to c} f(x) = f(c).$$

The previous example is important enough to state as a proposition.

Proposition If f is a polynomial and c is any real number, then

$$\lim_{x\to c} f(x) = f(c). \tag{2.3.14}$$

If we combine this result with our result about the limits of quotients in Section 1.2, we have the following proposition.

Proposition If f and g are both polynomials and c is any real number for which $g(c) \neq 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}. \tag{2.3.15}$$

In short, if h is a rational function and h is defined at c, then the value of the limit of $h(x)$ as x approaches c is simply the value of h at c. That is,

$$\lim_{x\to c} h(x) = h(c) \tag{2.3.16}$$

for any rational function h which is defined at c.

Example Using our result about polynomials, we have

$$\lim_{x\to 2}(3x^4 - 6x + 12) = (3)(2^4) - (6)(2) + 12 = 48.$$

Example Using our result about rational functions, we have

$$\lim_{x\to 3} \frac{3x+4}{2x^2+2x-1} = \frac{(3)(3)+4}{(2)(3^2)+(2)(3)-1} = \frac{13}{23}.$$

Example Now consider

$$\lim_{x\to 2} \frac{x^2-4}{x-2}.$$

Note that our result about the limits of rational functions does not apply here since the denominator is 0 at $x = 2$. However, since the numerator is also 0 at $x = 2$, the numerator and the denominator must have a common factor of $x - 2$. Canceling this common factor will simplify the problem and enable us to evaluate the limit. That is,

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4.$$

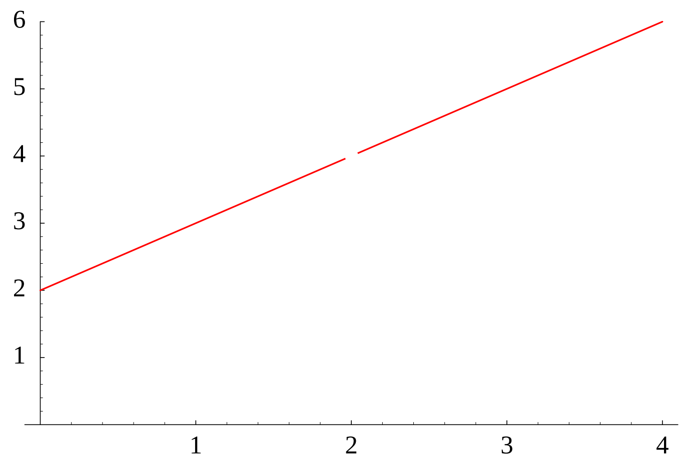

Figure 2.3.3 Graph of $f(x) = \frac{x^2-4}{x-2}$

Although technical, it is worth noting that the functions

$$f(x) = \frac{x^2 - 4}{x - 2}$$

and

$$g(x) = x + 2$$

are different functions. In particular, f is not defined at $x = 2$, whereas g is. However, for every point $x \neq 2$, $f(x) = g(x)$. As a result,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x),$$

since the limits depend only on the values of f and g for points close to, but not equal to, 2. See the graph of f in Figure 2.3.3.

Example As another example of the technique used in the previous example, we have

$$\begin{aligned}
\lim_{t \to -1} \frac{t^2 - 1}{t^3 + 1} &= \lim_{t \to -1} \frac{(t+1)(t-1)}{(t+1)(t^2 - t + 1)} \\
&= \lim_{t \to -1} \frac{t - 1}{t^2 - t + 1} \\
&= \frac{-1 - 1}{1 + 1 + 1} = -\frac{2}{3}.
\end{aligned}$$

In the last two examples, we have used the algebraic fact that if c is a root of a polynomial $f(x)$, then $x - c$ is a factor of $f(x)$. In particular, this means that if both the numerator and the denominator of a rational function are 0 at $x = c$, then they have a common factor of $x - c$. However, if the numerator is not 0 at c, but the limit of the denominator is 0, then the limit will not exist. For example, if

$$f(x) = \frac{1}{x^2},$$

then $\lim_{x\to 0} f(x)$ does not exist, since dividing 1 by x_n^2, where $\{x_n\}$ is a sequence with $\lim_{n\to\infty} x_n = 0$, will always result in a sequence of positive numbers which are growing without bound. Borrowing from the notation we developed in Section 1.2, we may write

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

As before, we must be careful to remember that this notation means that, although the function does not have a limit as x approaches 0, the value of the function grows without any bound as x approaches 0. Similarly, since

$$\frac{1}{x} > 0$$

when $x > 0$, we have

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

and, since

$$\frac{1}{x} < 0$$

when $x < 0$,

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

However, since

$$f(x) = \frac{1}{x}$$

behaves differently as x approaches 0 from the right than it does when x approaches 0 from the left, all we can say about the limit of $f(x)$ as x approaches 0 is that it does not exist.

Graphically, for a given function f,

$$\lim_{x\to c^-} f(x) = \infty$$

or

$$\lim_{x\to c^-} f(x) = -\infty$$

tells us that the graph of f will approach the vertical line $x = c$ asymptotically as x approaches c from the left. The graph will go off along $x = c$ in the positive direction in the first case and in the negative direction in the second case. Similar remarks hold for x approaching c from the right when

$$\lim_{x\to c^+} f(x) = \infty$$

or

$$\lim_{x\to c^+} f(x) = -\infty.$$

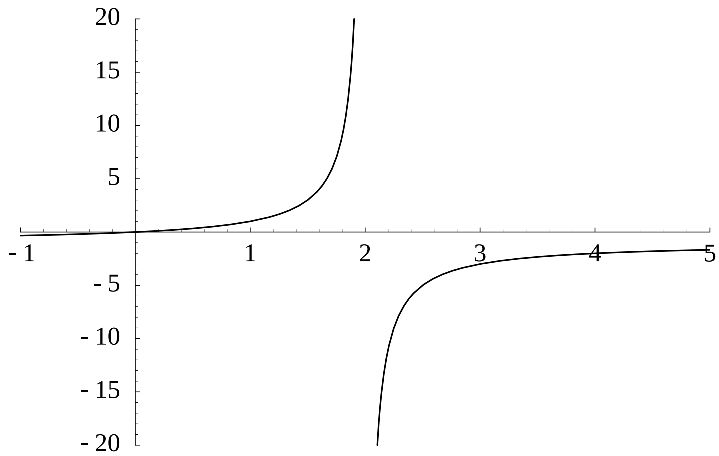

Figure 2.3.4 Graph of $y = \frac{x}{2-x}$

Example Since $2 - x > 0$ when $x < 2$ and $2 - x > 0$ when $x > 2$, it follows that

$$\lim_{x \to 2^-} \frac{x}{2-x} = \infty$$

and

$$\lim_{x \to 2^+} \frac{x}{2-x} = -\infty.$$

It follows that the line $x = 2$ is a vertical asymptote for the graph of

$$y = \frac{x}{2-x},$$

with the curve going off in the positive direction from the left and in the negative direction from the right. See Figure 2.3.4

The next examples illustrate the use of one-sided limits, using (2.3.8), in determining the existence of certain limits.

Example Suppose

$$g(z) = \begin{cases} z^2 + 1, & \text{if } z \leq 1, \\ 3z + 4, & z > 1. \end{cases}$$

Then, since $g(z) = z^2 + 1$ when $z < 1$,

$$\lim_{z \to 1^-} g(z) = \lim_{z \to 1^-} (z^2 + 1) = 2$$

and, since $g(z) = 3z + 4$ when $z > 1$,

$$\lim_{z \to 1^+} g(z) = \lim_{z \to 1^+} (3z + 4) = 7.$$

Since these limits are not the same, we know from (2.3.11) that $g(z)$ does not have a limiting value as z approaches 1. Graphically, we see this as a break in the graph of g at

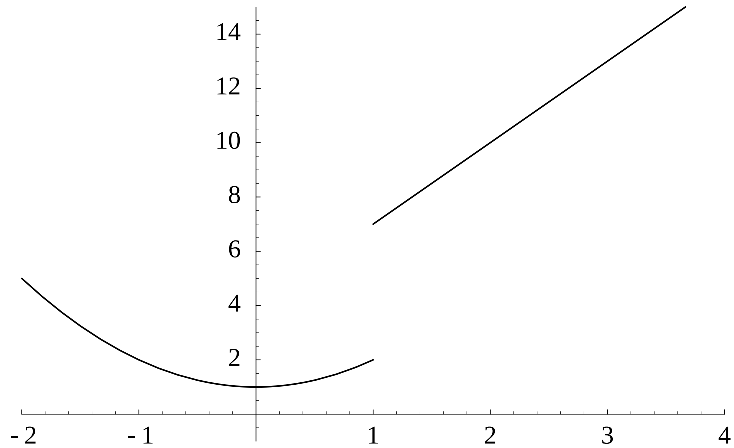

Figure 2.3.5 Graph of $y = g(z)$

$z = 1$, as shown in Figure 2.3.5. Note, however, that for any $c \neq 1$, $\lim_{z \to c} g(z) = g(c)$. For example

$$\lim_{z \to -4} g(z) = \lim_{z \to -4} (z^2 + 1) = 17.$$

Example Now consider

$$h(t) = \begin{cases} 2t + 3, & \text{if } t \leq 2, \\ 2t^2 - 1, & \text{if } t > 2. \end{cases}$$

Then

$$\lim_{t \to 2^-} h(t) = \lim_{t \to 2^-} (2t + 3) = 7$$

and

$$\lim_{t \to 2^+} h(t) = \lim_{t \to 2^+} (2t^2 - 1) = 7.$$

In this case both one-sided limits are equal to 7, so we have, using (2.3.11),

$$\lim_{t \to 2} h(t) = 7.$$

Graphically, the graph of h does not have a break at $t = 2$, even though the formula for computing $h(t)$ changes at this point. See Figure 2.3.6.

We may also use limits to inquire into the behavior of the values of a function f as x increases, or decreases, without bound. This leads to the following definition.

Definition Suppose f is a function defined on an interval J of the form (a, ∞). We say that the *limit of* $f(x)$ *as* x *approaches* ∞ is L, denoted

$$\lim_{x \to \infty} f(x) = L, \tag{2.3.17}$$

if for every sequence $\{x_n\}$ in J we have

$$\lim_{n \to \infty} f(x_n) = L \tag{2.3.18}$$

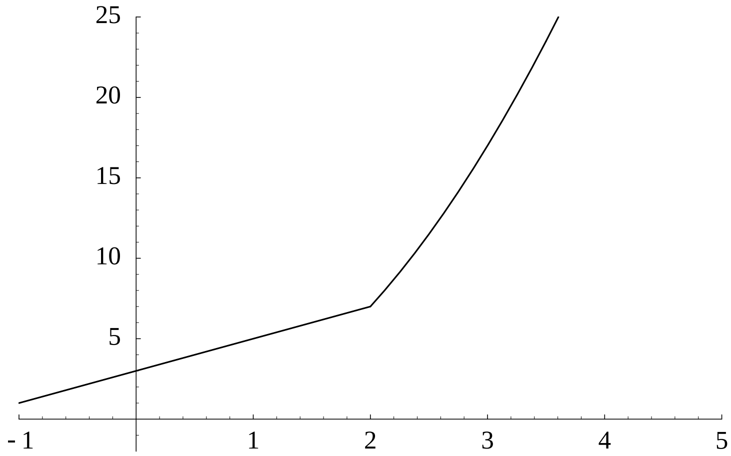

Figure 2.3.6 Graph of $y = h(t)$

whenever

$$\lim_{n \to \infty} x_n = \infty. \tag{2.3.19}$$

Similarly, suppose f is a function defined on an interval J of the form $(-\infty, b)$. We say that the *limit of* $f(x)$ *as* x *approaches* $-\infty$ is L, denoted

$$\lim_{x \to -\infty} f(x) = L, \tag{2.3.20}$$

if for every sequence $\{x_n\}$ in J we have

$$\lim_{n \to \infty} f(x_n) = L \tag{2.3.21}$$

whenever

$$\lim_{n \to \infty} x_n = -\infty. \tag{2.3.22}$$

Example Suppose $\{x_n\}$ is a sequence and $\lim\limits_{n \to \infty} x_n = \infty$. Given $\epsilon > 0$, there must exist an integer N such that

$$x_n > \frac{1}{\epsilon}$$

whenever $n > N$. Hence

$$\frac{1}{x_n} < \epsilon$$

whenever $n > N$. That is,

$$\lim_{n \to \infty} \frac{1}{x_n} = 0.$$

Since this true for any such sequence $\{x_n\}$, it follows that

$$\lim_{n \to \infty} \frac{1}{x} = 0.$$

In a similar fashion, we may show that

$$\lim_{n\to-\infty} \frac{1}{x} = 0.$$

With these two basic limits, it is possible to compute limits of these types for any rational function using the same techniques we used in Section 1.2. Namely, given a rational function, dividing numerator and denominator by the highest power appearing in the denominator simplifies the expression to a form where the limit may be evaluated easily.

Example $\displaystyle\lim_{x\to\infty} \frac{3x^2+4x-6}{2x^2-6x+2} = \lim_{x\to\infty} \frac{3+\frac{4}{x}-\frac{6}{x^2}}{2-\frac{6}{x}+\frac{2}{x^2}} = \frac{3}{2}.$

Example $\displaystyle\lim_{x\to\infty} \frac{3x^2-6x}{4x^3+2} = \lim_{x\to\infty} \frac{\frac{3}{x}-\frac{6}{x^2}}{4+\frac{2}{x^3}} = \frac{0}{4} = 0.$

Example We have

$$\lim_{x\to-\infty} \frac{4x^3-3}{2x^2+6} = \lim_{x\to-\infty} \frac{4x-\frac{3}{x^2}}{2+\frac{6}{x^2}} = -\infty,$$

since the denominator is approaching 2 while the numerator decreases without bound as x goes to $-\infty$. Note that, as usual, although the limit does not exist, we make use of this notation to indicate the manner in which the limit fails to exist.

Graphically, $\lim_{x\to\infty} f(x) = L$ tells us that the graph of $y = f(x)$ approaches the horizontal line $y = L$ asymptotically as x increases without bound. Similarly, $\lim_{x\to-\infty} f(x) = L$ tells us that the graph of $y = f(x)$ approaches the horizontal line $y = L$ asymptotically as x decreases without bound.

Example Since

$$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1+\frac{1}{x^2}} = 0$$

and

$$\lim_{x\to-\infty} \frac{x}{x^2+1} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{1+\frac{1}{x^2}} = 0,$$

we know that the line $y = 0$, that is, the x-axis, is a horizontal asymptote for the graph of

$$y = \frac{x}{x^2+1}.$$

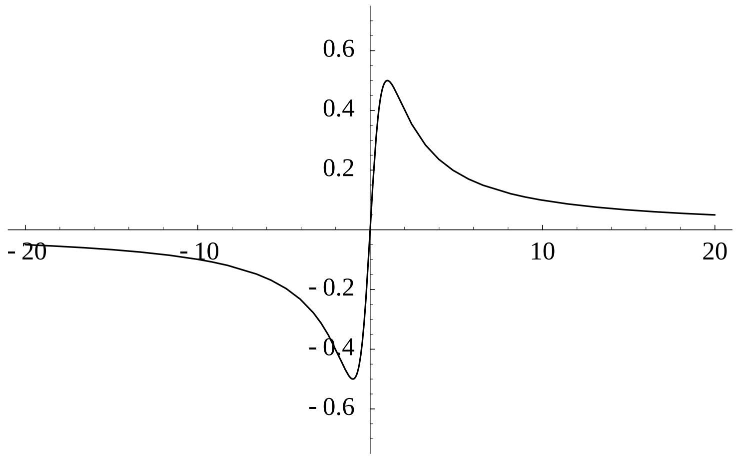

Figure 2.3.7 Graph of $y = \frac{x}{x^2+1}$

Moreover, since

$$\frac{x}{x^2+1} > 0$$

for $x > 0$ and

$$\frac{x}{x^2+1} < 0$$

for $x < 0$, we know that the approach to the x-axis is from above as x increases and from below as x decreases. See Figure 2.3.7.

The following proposition summarizes the basic properties of limits. These are essentially restatements of the properties of limits of sequences that we discussed in Section 1.2. The fact that they hold here follows from the way we have defined limits in this section in terms of limits of sequences. Moreover, the properties listed in this proposition also hold for one-sided limits.

Proposition Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, where L and M are real numbers and c is a real number, ∞, or $-\infty$. Then

$$\lim_{x \to c} kf(x) = kL \text{ for any constant } k, \tag{2.3.23}$$

$$\lim_{x \to c} (f(x) + g(x)) = L + M, \tag{2.3.24}$$

$$\lim_{x \to c} (f(x) - g(x)) = L - M, \tag{2.3.25}$$

$$\lim_{x \to c} (f(x)g(x)) = LM, \tag{2.3.26}$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \tag{2.3.27}$$

and, provided p is a rational number for which $(f(x))^p$ and L^p are defined,

$$\lim_{x \to c} (f(x))^p = L^P. \tag{2.3.28}$$

Example Using (2.3.28), we have

$$\lim_{x\to 4}\sqrt{x^2+3} = \sqrt{\lim_{x\to 4}(x^2+3)} = \sqrt{19}.$$

Example Using the fact hat

$$\sqrt{x^2} = |x| = \begin{cases} -x, & \text{if } x < 0, \\ x, & \text{if } x \geq 0, \end{cases}$$

we have

$$\begin{aligned}\lim_{x\to\infty}\frac{4x}{\sqrt{x^2+1}} &= \lim_{x\to\infty}\frac{4}{\dfrac{\sqrt{x^2+1}}{x}} \\ &= \lim_{x\to\infty}\frac{4}{\dfrac{\sqrt{x^2+1}}{\sqrt{x^2}}} \\ &= \lim_{x\to\infty}\frac{4}{\sqrt{\dfrac{x^2+1}{x^2}}} \\ &= \lim_{x\to\infty}\frac{4}{\sqrt{1+\dfrac{1}{x^2}}} \\ &= 4\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to-\infty}\frac{4x}{\sqrt{x^2+1}} &= \lim_{x\to-\infty}\frac{4}{\dfrac{\sqrt{x^2+1}}{x}} \\ &= \lim_{x\to-\infty}\frac{4}{\dfrac{\sqrt{x^2+1}}{-\sqrt{x^2}}} \\ &= \lim_{x\to-\infty}\frac{4}{-\sqrt{\dfrac{x^2+1}{x^2}}} \\ &= \lim_{x\to-\infty}\frac{4}{-\sqrt{1+\dfrac{1}{x^2}}} \\ &= -4.\end{aligned}$$

Hence the lines $y = 4$ and $y = -4$ are both horizontal asymptotes for the graph of

$$y = \frac{4x}{\sqrt{x^2+1}}.$$

See Figure 2.3.8.

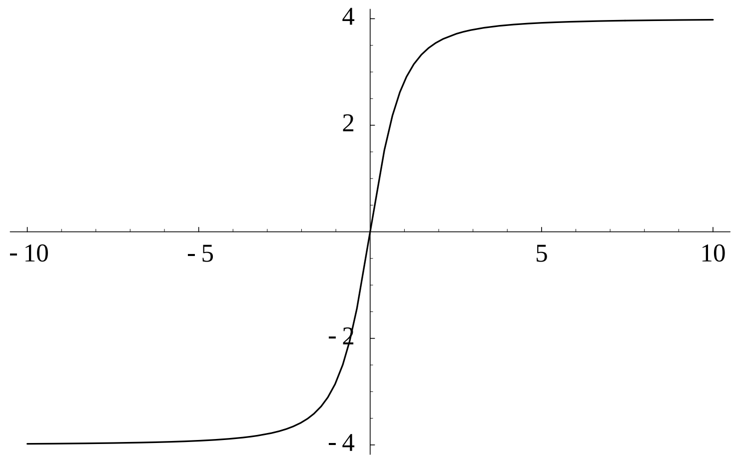

Figure 2.3.8 Graph of $y = \frac{4x}{\sqrt{x^2+1}}$

Problems

1. Evaluate the following limits.

(a) $\lim_{x\to 2}(4x^2 - 3x)$ (b) $\lim_{x\to 3}(x^3 - 2x + 3)$

(c) $\lim_{t\to 1}\dfrac{t^2-3}{t+5}$ (d) $\lim_{z\to -2}\dfrac{z+2}{z^2+3}$

2. Evaluate the following limits.

(a) $\lim_{x\to 2}\dfrac{x^3-8}{x+2}$ (b) $\lim_{x\to 3}\dfrac{x^2-x-6}{x-3}$

(c) $\lim_{s\to -1}\dfrac{s^3+1}{s^4-1}$ (d) $\lim_{x\to 1}\dfrac{\sqrt{x}-1}{x-1}$

(e) $\lim_{t\to 2}\dfrac{t^3-8}{t-2}$ (f) $\lim_{x\to 1}\dfrac{x^3-1}{x^2-1}$

(g) $\lim_{u\to 4}\dfrac{u}{(u-4)^2}$ (h) $\lim_{x\to -2}\dfrac{x}{x+2}$

3. Evaluate the following limits.

(a) $\lim_{x\to 1^+}(3x^2+4)$ (b) $\lim_{x\to 3^-}\dfrac{1}{x-3}$

(c) $\lim_{x\to 3^+}\dfrac{1}{x-3}$ (d) $\lim_{t\to -2^+}\dfrac{t}{t+2}$

(e) $\lim_{t\to -2^-}\dfrac{t}{t+2}$ (f) $\lim_{x\to 10^-}\dfrac{x^2-9x-10}{x^2-8x-20}$

4. Evaluate the following limits.

(a) $\lim_{x\to 2^+}\lfloor x\rfloor$ (b) $\lim_{x\to 2^-}\lfloor x\rfloor$

(c) $\lim_{x\to 3^-}\lceil x\rceil$ (d) $\lim_{x\to 3^+}\lceil x\rceil$

(e) $\lim_{x\to 0^+} \lfloor \cos(x) \rfloor$

(f) $\lim_{x\to 0^+} \lceil \sin(x) \rceil$

5. Suppose

$$g(z) = \begin{cases} 3z - 1, & \text{if } z < 2, \\ 7 - z, & \text{if } z \geq 2. \end{cases}$$

   (a) Sketch the graph of g.
   (b) Find $\lim_{z\to 2^-} g(z)$.
   (c) Find $\lim_{z\to 2^+} g(z)$.
   (d) Does $\lim_{z\to 2} g(z)$ exist? If so, what is its value?

6. Suppose

$$h(t) = \begin{cases} 2t + 1, & \text{if } t \leq 1, \\ 3t - 1, & \text{if } t > 1. \end{cases}$$

   (a) Sketch the graph of h.
   (b) Find $\lim_{t\to 1^-} h(t)$.
   (c) Find $\lim_{t\to 1^+} h(t)$.
   (d) Does $\lim_{t\to 1} h(t)$ exist? If so, what is its value?

7. Evaluate the following limits.

   (a) $\lim_{x\to\infty} (3x + 4)$
   (b) $\lim_{x\to\infty} \dfrac{x^3 + 3x - 1}{2x^3 - x^2 + 21}$
   (c) $\lim_{u\to\infty} \dfrac{u^4 + 3u - 6}{3u^2 + 1}$
   (d) $\lim_{z\to\infty} \dfrac{4z^2 - 3z + 10}{2z^3 + 14z + 9}$
   (e) $\lim_{x\to -\infty} \dfrac{x^5 - 6x + 13}{x^2 + 18x - 25}$
   (f) $\lim_{x\to\infty} \dfrac{2\sqrt{x} + 3}{\sqrt{x} + 2}$
   (g) $\lim_{v\to\infty} \sqrt{\dfrac{2v + 1}{v - 2}}$
   (h) $\lim_{t\to\infty} \dfrac{\sqrt{t + 1}}{t + 3}$
   (i) $\lim_{x\to\infty} \dfrac{3x + 1}{\sqrt{4x^2 + 5}}$
   (j) $\lim_{x\to -\infty} \dfrac{3x + 1}{\sqrt{4x^2 + 5}}$

8. Let

$$f(x) = \frac{2x}{x - 4}.$$

   Find $\lim_{x\to 4^-} f(x)$, $\lim_{x\to 4^+} f(x)$, $\lim_{x\to -\infty} f(x)$, and $\lim_{x\to\infty} f(x)$. Use this information to sketch the graph of f.

9. Discuss $\lim_{x\to \frac{\pi}{2}^-} \tan(x)$, $\lim_{x\to \frac{\pi}{2}^+} \tan(x)$, and $\lim_{x\to \frac{\pi}{2}} \tan(x)$.

10. Do $\lim_{x\to\infty} \sin(\pi x)$ and $\lim_{n\to\infty} \sin(\pi n)$ denote the same thing? Discuss.

11. (a) Explain why

$$-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$$

for all x .

(b) Use part (a) to find $\lim_{x\to\infty} \frac{\sin(x)}{x}$.