Hartshorne, Chapter 1.7 Answers to exercises. REB 1994

7.1a The polynomials of degree m in $N+1$ variables restricted to the image of P^n in P^N give the polynomials of degree md in $n+1$ variables. Hence the Hilbert polynomial of the embedding of P^n in P^N is $f(dk)$ where $f(k) = \binom{k+n}{n}$ is the Hilbert polynomial of P^n (embedded in itself). So the Hilbert polynomial of the d-tuple embedding is $\binom{dk+n}{n} = (dk)^n/n! + \cdots$ so the degree of the embedding is d^n.

7.1b Similarly we find that the Hilbert polynomial of the Segre embedding of $P^r \times P^s$ is the product of the Hilbert polynomials of P^r and P^s, which is

$$\binom{k+r}{r}\binom{k+s}{s} = (k^r/r! + \cdots)(k^s/s! + \cdots) = \binom{r+s}{r} k^{r+s}/(r+s)! + \cdots$$

so the degree of the Segre embedding is $\binom{r+s}{r}$.

7.2a This follows from the fact that the Hilbert polynomial $P_{P^n}(k) = \binom{n+k}{n}$ has constant term 1.

7.2bc By 7.6c, $P_H(k) = \binom{k+n}{n} - \binom{k-d+n}{n}$, whose value at 0 is $1 - \binom{n-d}{n} = 1 - (-1)^n\binom{d-1}{n}$.

7.2d The Hilbert polynomial of this complete intersection is

$$\binom{k+3}{3} - \binom{k-a+3}{3} - \binom{k-b+3}{3} + \binom{k-a-b+3}{3}$$

whose constant term is $1 - \binom{3-a}{3} - \binom{3-b}{3} + \binom{3-a-b}{3}$ which is $1 - (ab(a+b-4)/2+1)$.

7.2e The Hilbert polynomial of $Y \times Z$ is the product of the Hilbert polynomials of Y and Z, from which the result follows easily.

7.3 We can assume that P is $(0,0) \in A^2$, and we can assume that if f if the function defining Y then $f(x,y) = y+$(terms of degree at least 2). By Ex. 5.4 the only line whose intersection multiplicity with Y at P is the line $y=0$. In general the mapping takes $(x_0 : x_1 : x_2) \in Y$ to the point $(f_0(x_0,x_1,x_2) : f_1(x_0,x_1,x_2) : f_2(x_0,x_1,x_2))$ where $f_i = \frac{\partial f}{\partial x_i}$, which is well defined as long as one of the 3 numbers $f_i(x_0,x_1,x_2)$ is nonzero, i.e., the point P is nonsingular.

7.4 By Ex. 5.4, any line not tangent to Y and not passing through a singular point meets Y in exactly d distinct points. As Y has only a finite number of singular points, the lines intersection at least one of these form a proper closed subset of P^{2*} (in fact a union of lines). By 7.3 the lines tangent to Y are also contained in a proper closed subset of P^{2*}, so there is a nonempty open subset U of lines in P^{2*} intersecting Y in exactly d points.

7.5a We can assume that any point P of multiplicity at least d is $(0,0)$. But then the equation $f(x,y)$ defining Y has all terms of degree exactly d, so it is a product of linear factors, which is not possible if Y is irreducible of degree greater than 1.

7.5b As in 7.5a we can assume that the equation defining Y is of the form $f(x,y) + g(x,y) = 0$ where f is homogeneous of degree $d-1$ and g is homogeneous of degree d. If we make the substitution $t = y/x$ we find that $y = -f(t,1)/g(t,1)$, $x = yt$ gives an inverse rational map so that Y is birational to A^1.

7.6 Any linear variety obviously has degree 1 (by calculating its Hilbert polynomial). Assume that Y has degree 1. Then by 7.6b, Y is irreducible (as all components of Y have the same dimension). By theorem 7.7 if H is any hyperplane then $Y \cap H$ also has degree 1 (or $Y \subset H$, in which case Y is linear by induction on n). Therefore $Y \cap H$ is linear for every hyperplane H, and therefore for every linear variety H. In particular if $p, q \in Y$, then the intersection of Y with the line pq is linear and therefore is the line joining p and q. Hence Y contains any line joining two of its points, and is therefore linear.

7.7a We show that X is birational to the cone on Y which will show that X is irreducible and of dimension $r+1$. We choose a hyperplane "at infinity" in P^n not containing P or Y, and map X to the cone on Y by taking any line $PQ\backslash$(point at infinity) to the affine line on the cone over Y by taking Q to Q, P to the vertex of the cone. We can define a rational inverse in the obvious way.

7.7b We prove this when Y is any closed algebraic set, not necessarily reducible. If Y has dimension 0 then it is a union of d points and X is a union of at most $d-1$ lines, so the result is true in this case. If Y has dimension > 0 choose a generic hyperplane H containing P, which can be chosen to intersect Y transversely at generic points of the intersection as Y is nonsingular at P. The intersection of Y and H has degree at most $d \times \deg(H) = d$ by 7.7. The intersection of X with H is the union of the set of lines joining P and $H \cap Y$ which has degree less than d by induction on $\dim(Y)$. Again by 7.7, the degree of X is equal to the degree of $X \cap H$ as all components of $X \cap H$ have multiplicity 1 in the intersection (as H is generic). Hence the degree of X is less than that of Y. (Note that the intersection $X \cap H$ of an irreducible algebraic set X with a generic hyperplane H need not be irreducible! But see remark 7.9.1 on p. 245 of Hartshorne.)

7.8 Applying 7.7 to Y^r shows that Y is contained in a degree 1 variety H of dimension $r+1$ in P^n, which by 7.6 is a linear variety and therefore isomorphic to P^{r+1}.