Hartshorne, Chapter 1.5 Answers to exercises. REB 1994

5.1a This is the tacnode. The singular points are the points with $x^2 = x^4 + y^4$, $2x = 4x^3$, and $4y^3 = 0$, so (at least in characteristic 0) the only singular point is $(0, 0)$.

5.1b This is the node; singular point is $(0, 0)$.

5.1c This is the cusp; singular point is $(0, 0)$.

5.1d This is the triple point; singular point is $(0, 0)$.

5.2 The singular points of $f(x, y, z) = 0$ are given by $f = 0$, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, and $\frac{\partial f}{\partial z} = 0$.

5.2a This is the pinch point; singular points are where $xy^2 = z^2$, $y^2 = 0$, $2xy = 0$, and $2z = 0$, which is the line $y = z = 0$.

5.2b This is the conical double point; singular points are where $x^2 + y^2 = z^2$, $2x = 0$, $2y = 0$, and $2z = 0$, which is the point $(0, 0, 0)$.

5.2c This is the double line; singular points are where $xy + x^3 + y^3 = 0$, $y + 3x^2 = 0$, $x + 3y^2 = 0$, and $0 = 0$, which is the line $x = y = 0$.

5.3a If P is a point on Y then P is a nonsingular point of Y is equivalent to saying that one of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are nonzero at P, which is equivalent to saying that f has a term of degree 1 in x and y, which is equivalent to saying that $\mu_P(Y) = 1$.

5.3b The singularities in 1a, 1b, and 1c have multiplicity 2, and 1d has multiplicity 3.

5.4a f and g both vanish at only a finite number of points, so we can find a polynomial $h(y)$ which vanishes whenever f and g both vanish, so $h^n \in (f, g)$ for some n, so we can assume $n = 1$. The submodules of $O_P/(f, g)$ correspond to ideals of O_P containing f and g, so it is sufficient to show that $k[x, y]/(f, g)$ is finite dimensional (as its dimension is at least the length of $O_P/(f, g)$). But if we have polynomials $h_1(x)$ and $h_2(y)$ of degrees m and n in (f, g) then $k[x, y]/(f, g)$ has dimension at most that of $k[x, y]/(h_1, h_2)$ which is mn which is finite.

5.4b Put $P = (0, 0)$ and take any line L not in the tangent cone of Y. We can assume that L is the line $y = 0$, so the terms of lowest degree in f contain x^m (where m is the multiplicity of Y at P). Then $O_P/(f, g) = O_P/(y, x^m + \cdots) = O_Q/(x^m + \cdots)$ which has length m (where O_Q is the local ring of $Q = 0 \in A^1$).

5.4c We can assume that L is $y = 0$. If $z \neq 0$, the equation of the curve Y is $f(x) + y(*) = 0$ where f if a polynomial in x of some degree n. Then if x is a root of f of multiplicity m, we have $(L.Y)_(x, 0) = m$, so the sums of the intersection multiplicities along the x axis is the number of roots of f which is n. On the other hand, at the point $(0 : 1 : 0)$ the intersection multiplicity is $d - n$ as the equation for f is locally $z^{d-n} + \cdots + x(*) = 0$. So the sum of all intersection multiplicities is $n + d - n = d$.

5.5 If the characteristic p does not divide d we can use $x^d + y^d + z^d = 0$ Otherwise we can use $xy^{d-1} + yz^{d-1} + zx^{d-1} = 0$.