Hartshorne, Chapter 1 Answers to exercises. REB 1994

1.1a $k[x,y]/(y-x^2)$ is identical with its subring $k[x]$.

1.1b $A(\mathbf{Z}) = k[x, 1/x]$ which contains an invertible element not in k and is therefore not a polynomial ring over k.

1.1c Any nonsingular conic in P^2 can be reduced to the form $xy + yz + zx = 0$ and this curve is isomorphic to P^1. (Proof: choose any 3 points on the conic, and choose coordinates so that these points are $(1:0:0), (0:1:0), (0,0,1)$; this means the conic must have the equation $cxy + ayz + bzx = 0$, with a, b, c all nonzero (otherwise the conic is singular). Then multiplying x, y, z by a, b, c shows that the conic has equation $xy + yz + zx = 0$. Hence all nonsingular conics are isomorphic to this one, and as it is easy to find one isomorphic to P^1 they all are.) Therefore (regular function on a conic) = (regular functions on the conic $xy + yz + zx = 0$ - some hyperplane) = (regular functions on P^1 - 1 or 2 points) = $A(Y)$ or $A(z)$. The ring is $A(Y)$ if and only if the conic $ax^2 + bxy + cy^2$+(terms of degree < 2) intersects the line at infinity in exactly one point, which happens if and only if $b^2 = 4ac$.

1.2 Y is isomorphic to A^1 and is therefore an affine variety of dimension 1, and $A(Y) = k[x]$. $I(Y)$ is generated by $Y - X^2, Z - X^3$.

1.3 $xy = x$, so $x = 0$ or $z = 1$. $x^2 = yz$, so $x = 0, y = 0$ or $x = 0, z = 0$, or $z = 1, x^2 = y$. Therefore Y is the union of 2 lines and a parabola. The prime ideals are generated by x, y or x, z or $z - 1, x^2 - y$.

1.4 The line $x = y$ is closed in A^2 but not in $A^1 \times A^1$ (at least if k is infinite).

1.5 B is a finitely generated algebra over k and has no nilpotents. If $x_1, ...x_n$ is a set of generators for B then $B = k[x_1, \ldots, x_n]/I$ for some ideal I, and $\sqrt{I} = I$ as B has no nilpotents. Hence $I(V(I)) = I$ by the nullstellensatz, so that B is the coordinate ring of $V(I)$ in A^n.

1.6 Put $U \subset X$, U open, X irreducible. Then $X = (X - U) \cup \bar{U}$, so $\bar{U} = X$, so U is dense in X. If $U \subset C_1 \cup C_2$, then $X = \bar{U} = \bar{C}_1 \cup \bar{C}_2 = C_1 \cup C_2$, so C_1 or C_2 contains U, so U is irreducible.

1.7a (i) is equivalent to (iii) by taking complements. (ii) implies (iv) is trivial. (i) implies (ii) because if some set contains no smallest closed subset then we can choose an infinite descending chain $C_1 \supset C_2 \supset \cdots$ using Zorn's lemma. The proof that (iii) is equivalent to (iv) is similar.

1.7b If U is any open cover of X, apply (a)(iv) to the unions of the finite subsets of U.

1.7c Follows from (a)(iv).

1.7d X Noetherian and Hausdorff implies X Hausdorff and every subset Noetherian implies X Hausdorff and every subset compact implies X compact and every subset closed implies X compact and discrete implies X finite and discrete.

1.8 Let H have ideal (f). As Y is not contained in H, f is neither a unit nor a zero divisor in the coordinate ring B of Y. Therefore by 1.11A every minimal prime P containing f has height 1. By 1.8A $\dim(B/P) = r - 1$. If X is an irreducible component of $Y \cup H$ then the ideal of X is a minimal prime ideal P of B containing f and the coordinate ring of X is B/P.

1.9 The dimension of any component of $Z(a)$ = transcendence degree of its function field. This function field contains $x_1, \ldots, x_n$ and the algebraic relations between these are a consequence of the r generators of a. Therefore the dimension of any component is at least $n-$number of generators of $a \geq n - r$.

1.10a If $Y_0 \subset Y_1 \subset \cdots \subset Y_n$ is a chain of irreducible closed subsets of Y, then $\bar{Y}_0 \subset \bar{Y}_1 \subset \cdots \subset \bar{Y}_n$ is a chain of irreducible closed subsets of X.

1.10b By (a), $\dim(X) \geq \sup \dim(U_i)$. If $X_0 \subset \cdots X_n$ is a sequence of irreducible closed subsets of X with X_0 a point, choose some set U in the cover with $X_0 \in U$. Then by 1.6 $X_i \cap U$ is irreducible and dense in X_i and therefore not contained in X_{i-1}. Hence $X_0 \cap U \subset X_1 \cap U \subset \cdots \subset X_n \cap U$ is a sequence of closed strictly increasing irreducible subsets of U, so $\dim(X) \leq \dim U \leq \sup \dim U_i$.

1.10c $X = \{u, v\}$ (a 2 point set) with open sets $\emptyset, \{u\} = U, X$.

1.10d If $Y_0 \subset \cdots \subset Y_n$ is a chain of closed irreducible subsets of Y and $Y \neq X$, then we can add X to the end of this chain to see that $\dim(X) \geq \dim(Y) + 1$ so either $\dim(X) = \infty$ or $\dim(X) > \dim(Y)$.

1.10e The set of positive integers, closed sets those of the form $\{1, 2, 3, \ldots, n\}$.

1.11 $t \to (t^3, t^4, t^5)$ is a homeomorphism from A^1 to Y, so $\dim(Y) = 1$, so P has height 2. No element of the ideal of P has homogeneous components of degree 0 or 1, and the possible homogeneous components of degree 2 form a vector space of dimension 3, so P needs at least 3 generators. (P is generated by $x^2y - z^2, zx - y^2, x^3 - zy$.)

1.12 $f(x, y) = y^4 + y^2 + x^2(x-1)^2$.