Chapter 11

11.1

(a) $V(x)=\frac{k}{2}x^2+\frac{k^2}{x}$

$V'=kx-\frac{k^2}{x^2}$ At equilibrium, $V'=0$

$x=k^{1/3}$

$V''=k+\frac{2k^2}{x^3}$

$V''\big|_{x=k^{1/3}}=k+2k=3k>0$ **Stable**

(b) $V(x)=kxe^{-bx}$

$V'=ke^{-bx}-bkxe^{-bx}$

At equilibrium $ke^{-bx}-bkxe^{-bx}=0$

$x=\frac{1}{b}$

$V''=-bke^{-bx}-bkxe^{-bx}+b^2kxe^{-bx}$

$V''\big|_{x=1/b}=-2bke^{-1}+bke^{-1}=-bke^{-1}<0$ **Unstable**

(c) $V(x)=k\left(x^4-b^2x^2\right)$

$V''=k\left(4x^3-2b^2x\right)$

At equilibrium $k\left(4x^3-2b^2x\right)=0$

$x=0,\quad \pm b/\sqrt{2}$

$V''=k\left(12x^2-2b^2\right)$

$V''\big|_{x=0}=-2kb^2<0$ **Unstable**

$V''\big|_{x=\pm b/\sqrt{2}}=k\left(6b^2-2b^2\right)=4kb^2>0$ **Stable**

(d) for case (a) $\omega^2=\frac{3k}{m}$ $T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m}{3k}}=\frac{2\pi}{\sqrt{3}}s$

for case (c) at $x=\pm b/\sqrt{2}$ $\omega^2=\frac{4kb^2}{m}$ $T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m}{4kb^2}}=\pi\, s$

11.2

$V(x,y)=k\left(x^2+y^2-2bx-4by\right)$

$$\frac{\partial V}{\partial x}=2k(x-b) \qquad \frac{\partial V}{\partial y}=2k(y-2b)$$

at equilibrium

$x=b \quad and \quad y=2b$

$$\frac{\partial^2 V}{\partial x^2}=2k \qquad \frac{\partial^2 V}{\partial x \partial y}=0 \qquad \frac{\partial^2 V}{\partial y^2}=2k$$

$k_{11}=2k>0 \qquad k_{12}=k_{21}=0 \qquad k_{22}=2k$

$$\begin{vmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{vmatrix}=4k^2-0>0$$

The equilibrium is stabe.

11.3

$$V(x)=-\frac{1}{2}kx^2$$

$$F(x)=-\frac{dV(x)}{dx}=kx=m\ddot{x}=m\dot{x}\frac{d\dot{x}}{dx}$$

$$kx\,dx=m\dot{x}\,d\dot{x}$$

$$\int_{x_0}^{x} kxdx=\int_0^v m\dot{x}\,d\dot{x} \qquad \frac{k}{2}\left(x^2-x_0^2\right)=m\frac{v^2}{2}$$

$$v=\frac{dx}{dt}=\sqrt{k/m}\left(x^2-x_0^2\right)^{1/2}$$

$$\int_{x_0}^{x}\frac{dx}{\left(x^2-x_0^2\right)^{1/2}}=\int_0^t \alpha\,dt; \qquad where\ \alpha=\sqrt{k/m}$$

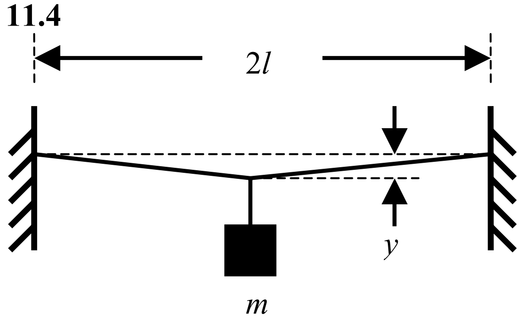
$$\ln\left(x+\sqrt{x^2-x_0^2}\right)-\ln x_0=\alpha t$$

$$x+\sqrt{x^2-x_0^2}=x_0e^{\alpha t}$$

$$x^2-x_0^2=x_0^2e^{2\alpha t}-2xx_0e^{\alpha t}+x^2$$

$$x=x_0\frac{e^{\alpha t}+e^{-\alpha t}}{2}=x_0\cosh\alpha t$$

11.4



Let the length of the unstretched, elastic cord be d. Then

$$d=2\sqrt{l^2+y^2}$$

$$V=\frac{1}{2}k\left(d-2l\right)^2-mgy$$

$$V=\frac{k}{2}\left(4l^2+4y^2-8l\sqrt{l^2+y^2}+4l^2\right)-mgy$$

$$V=2k\left(2l^2+y^2-2l\sqrt{l^2+y^2}\right)-mgy$$

The first term, $4kl^2$, is an additive constant to the potential energy, so with appropriate adjustment of the zero reference point …

$$V(y)=2k\left(y^2-2l\sqrt{l^2+y^2}\right)-mgy$$

$$\frac{dV(y)}{dy}=2k\left[2y-2yl\left(l^2+y^2\right)^{-1/2}\right]-mg$$

At equilibrium, the above expression is zero, so …

$$4ky-\frac{4kly}{\sqrt{l^2+y^2}}-mg=0$$

$$4ky-mg=\frac{4kly}{\sqrt{l^2+y^2}}$$

$$16k^2y^2-8kmgy+m^2g^2=\frac{16k^2l^2y^2}{l^2+y^2}$$

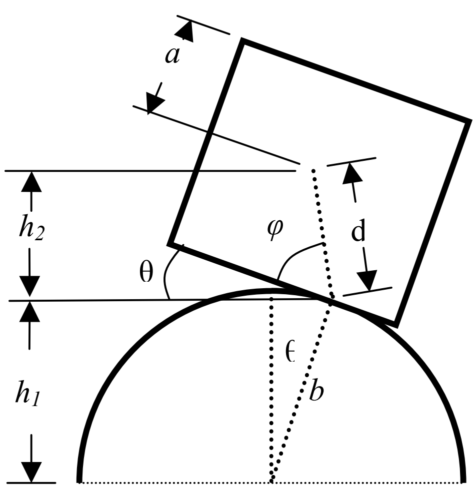
$$16k^2y^4-8kmgy^3+\left(16k^2l^2+m^2g^2-16k^2l^2\right)y^2-8kl^2mgy+l^2m^2g^2=0$$

$$\frac{y^4}{l^4}-\frac{mg}{2kl^4}y^3+\frac{m^2g^2}{16k^2l^4}y^2-\frac{mg}{2kl^2}y+\frac{m^2g^2}{16k^2l^2}=0$$

letting $u=\frac{y}{l}$ and $a=\frac{mg}{4kl}$

$$u^4-2au^3+a^2u^2-2au+a^2=0$$

11.5



$$V=mg\left(h_1+h_2\right)$$

$$h_1=b\cos\theta$$

$$h_2=d\sin\left(\theta+\varphi\right)$$

$$h_2=d\left(\sin\theta\cos\varphi+\cos\theta\sin\varphi\right)$$

$$\cos\varphi=\frac{b\theta}{d}\qquad\sin\varphi=\frac{a}{d}$$

$$h_2=b\theta\sin\theta+a\cos\varphi$$

$$V=mg\left[\left(a+b\right)\cos\theta+b\theta\sin\theta\right]$$

$$V'=mg\left[\left(a+b\right)\left(-\sin\theta\right)+b\sin\theta+b\theta\cos\theta\right]$$

$V' = mg\left[-a\sin\theta + b\theta\cos\theta\right]$

$V'' = mg\left[b\cos\theta - b\theta\sin\theta - a\cos\theta\right]$

$V_0'' = mg\left(b-a\right)$

Equilibrium …

Stable	Unstable
$a<b$	$a>b$

11.6

$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$

$V = mg\left[\left(a+b\right)\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + b\theta\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)\right]$

$V = mg\left[a + b - \frac{a-b}{2}\theta^2 + \frac{a-3b}{24}\theta^4 - \ldots\right]$

For a = b, $V = mg\left[2a - \frac{a}{12}\theta^4 + \ldots\right]$

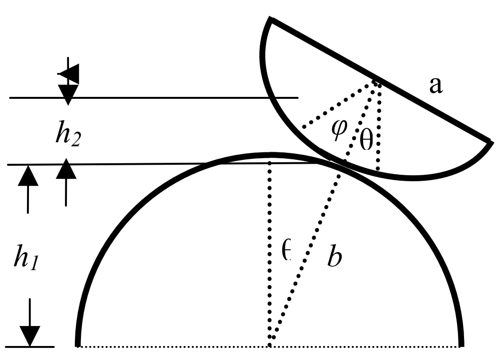
$V' = -\frac{mga}{3}\theta^3 + \ldots$ terms in higher order of θ

$V'' = -mga\theta^2 + \ldots$

$V''' = -2mga\theta + \ldots$

$V'''' = -2mga < 0$ $\quad\therefore$ Equilibrium is unstable

11.7



The center of mass (CM) of the hemisphere is $\frac{3}{8}a$ from the flat side (see Equation 8.1.8). The height of CM) above the point of contact between the two hemispheres is designated by h_2 in the figure. h_1 is the height of the point of contact above the ground.

$V = mg\left(h_1 + h_2\right) = mg\left[b\cos\theta + a\cos\theta - \frac{3}{8}a\cos\left(\theta+\varphi\right)\right]$ and $a\varphi = b\theta$

$$V = mg(h_1 + h_2) = mg\left[(a+b)\cos\theta - \frac{3}{8}a\cos\left(\theta + \frac{b\theta}{a}\right)\right]$$

$$V' = mg\left[-(a+b)\sin\theta + \frac{3a}{8}\left(\frac{a+b}{a}\right)\sin\left(\left(\frac{a+b}{a}\right)\theta\right)\right]$$ Equilibrium occurs at $\theta = 0^o$

$$V'' = mg\left[-(a+b)\cos\theta + \frac{3a}{8}\left(\frac{a+b}{a}\right)^2\cos\left(\left(\frac{a+b}{a}\right)\theta\right)\right]$$

$$V_0'' = mg\left[-(a+b) + \frac{3a}{8}\left(\frac{a+b}{a}\right)^2\right] = \frac{mg}{8a}(a+b)(3b-5a)$$

$V_0'' > 0 \quad for \;\; 3b > 5a$

Therefore, the equilibrium is stable for $a < \dfrac{3b}{5}$

11.8

From Problem 11.4, we have

$$V(y) = 2k\left(y^2 - 2l\sqrt{l^2 + y^2}\right) - mgy = 2k\left(y^2 - 2l^2\left(1 + \frac{y^2}{l^2}\right)^{\frac{1}{2}}\right) - mgy$$

Expanding the square root for small $\dfrac{y}{l}$ … $\left(1 + \dfrac{y^2}{l^2}\right)^{\frac{1}{2}} = 1 + \dfrac{1}{2}\dfrac{y^2}{l^2} - \dfrac{1}{8}\dfrac{y^4}{l^4} + ...$

$$V(y) \approx 2k\left[y^2 - 2l^2 - y^2 + \frac{y^4}{4l^2}\right] - mgy$$

$$V' \approx \frac{2k}{l^2}y^3 - mg$$

at equilibrium, $V' = 0 \quad \Rightarrow \quad y = \left(\dfrac{mgl^2}{2k}\right)^{\frac{1}{3}}$

$$V'' = \frac{6k}{l^2}y^2$$

$$V''\Big|_{y=\left(mgl^2/2k\right)^{\frac{1}{3}}} = \frac{6k}{l^2}\left(\frac{mgl^2}{2k}\right)^{\frac{2}{3}} = 6k^{\frac{1}{3}}\left(\frac{mg}{2l}\right)^{\frac{2}{3}}$$

$$\omega = \sqrt{\frac{V''}{m}} = \sqrt{6\left(\frac{k}{m}\right)^{\frac{1}{3}}\left(\frac{g}{2l}\right)^{\frac{2}{3}}} = \sqrt{6}\left(\frac{g}{2l}\right)^{\frac{1}{3}}\left(\frac{k}{m}\right)^{\frac{1}{6}}$$

11.9

From Problem 11.5, $V_0'' = mg(b-a)$

$$\omega = \sqrt{\frac{V_0''}{m}} = \sqrt{g(b-a)}$$

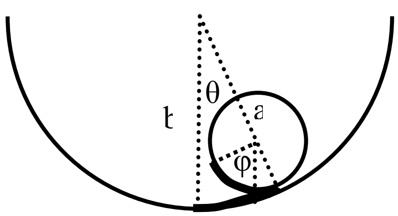
$$T_0 = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g(b-a)}}$$

11.10

From Problem 11.7, $V_0'' = \frac{mg}{8a}(a+b)(3b-5a)$

$$T_0 = 2\pi\sqrt{\frac{m}{V_0''}} = 4\pi\sqrt{\frac{2a}{g(a+b)(3b-5a)}}$$

11.11



$$T = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\dot{\varphi}^2$$
$$v_{cm} = (b-a)\dot{\theta}$$
$$I = \frac{2}{5}ma^2$$

The relationship between the angles, θ *and* φ (see Figure), can be determined from the condition that there is no slipping as the ball rolls in the hemisphere, so the length of roll measured along the ball, $a(\theta+\varphi)$, must equal the length of roll measured along the hemisphere, $b\theta$ … so we have …

$$b\theta = a(\theta+\varphi) \quad and \quad \therefore \; b\dot{\theta} = a(\dot{\theta}+\dot{\varphi})$$

and … $\dot{\varphi} = \left(\frac{b-a}{a}\right)\dot{\theta}$

$$T = \frac{1}{2}m(b-a)^2\dot{\theta}^2 + \frac{1}{2}\frac{2}{5}ma^2\frac{(b-a)^2\dot{\theta}^2}{a^2} = \frac{1}{2}m(b-a)^2\left(1+\frac{2}{5}\right)\dot{\theta}^2$$
$$V = -mg(b-a)\cos\theta$$
$$V' = mg(b-a)\sin\theta \quad equilibrium \;@\; \theta = 0^o$$

$$V'' = mg(b-a)\cos\theta \quad \therefore V_0'' = mg(b-a)$$

$$\omega = \sqrt{\frac{V_0''}{M}} = \sqrt{\frac{mg(b-a)}{\frac{7}{5}m(b-a)^2}} = \sqrt{\frac{5g}{7(b-a)}}$$

$$T_0 = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{7(b-a)}{5g}}$$

11.12

The potential energy of the satellite shaped like a "thin rod" is (See Example 11.2.2, Figure 11.2.1) …

$V = \int -G\frac{M_e\,dm}{r}$ but $dm = \frac{m}{2a}dx$ where *2a* is the length of the rod.

$$V = \int_{-a}^{a} -\frac{GM_e}{r}\frac{m}{2a}dx = -\frac{GM_e m}{2a}\int_{-a}^{a}\frac{dx}{r}$$

$$r = \left(r_0^{\,2} + x^2 + 2r_0\,x\cos\phi\right)^{\frac{1}{2}}$$

$r = \left(r_0^{\,2} + x^2\right)^{\frac{1}{2}}\left(1+\varepsilon\cos\phi\right)^{\frac{1}{2}}$ where $\varepsilon = \frac{2xr_0}{r_0^{\,2}+x^2}$

$$V = -\frac{GM_e m}{2a}\int_{-a}^{a}\frac{\left(1+\varepsilon\cos\phi\right)^{-\frac{1}{2}}dx}{\left(r_0^{\,2}+x^2\right)^{\frac{1}{2}}}$$

For $r_0 >> x$, $\quad r_0^{\,2} + x^2 \approx r_0^{\,2}$, $\quad \varepsilon \approx \frac{2x}{r_0}$ *and* $r \approx r_0\left(1+\varepsilon\cos\phi\right)^{\frac{1}{2}}$

Thus, for small ε, the expression for the potential energy, *V*, can be approximated …

$$V \approx -\frac{GM_e m}{2ar_0}\int_{-a}^{a}\left(1 - \frac{1}{2}\left(\frac{2x}{r_0}\right)\cos\phi + \frac{3}{8}\left(\frac{2x}{r_0}\right)^2\cos^2\phi\right)dx$$

$$V \approx -\frac{GM_e}{2ar_0}\left[2a + 0 + \frac{3\cos^2\phi}{2r_0^{\,2}}\,\frac{2a^3}{3}\right]$$

$$V \approx -\frac{GM_e}{r_0}\left[1 + \frac{a^2\cos^2\phi}{2r_0^{\,2}}\right]$$

$$V' \approx \frac{GM_e}{r_0}\left(\frac{a^2}{2r_0^{\,2}}\right)2\cos\phi\sin\phi = \frac{GM_e a^2}{2r_0^{\,3}}\sin 2\phi$$

Equilibrium @ $\phi = 0$

$$V'' \approx \frac{GM_e a^2}{r_0^3}\cos 2\phi \quad \text{and} \quad V_0' \approx \frac{GM_e a^2}{r_0^3}$$

$$M = I = \frac{1}{3}ma^2$$

$$\omega = \sqrt{\frac{V_0''}{M}} = \sqrt{\frac{3GM_e}{r_0^3}}$$

$$T_0 = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r_0^3}{3GM_e}}$$

11.13

The amplitude of the symmetric component is A_1 and the amplitude of the anti-symmetric component is A_2 (See Equations 11.3.19a through 11.3.20b).

$$A_1^2 = \frac{1}{4}\left[x_1(0) + x_2(0)\right]^2$$

$$A_1 = \frac{1}{2}\left[x_1(0) + x_2(0)\right] = \frac{A_0}{2}$$

$$A_2^2 = \frac{1}{4}\left[x_2(0) - x_1(0)\right]^2$$

$$A_2 = \frac{1}{2}\left[x_2(0) - x_1(0)\right] = \frac{A_0}{2} \qquad \therefore\ A_1 = A_2$$

From Equation 11.3.18 the solution for x_1 is …

$$x_1(t) = \frac{A_0}{2}\left(\cos\omega_1 t + \cos\omega_2 t\right)$$ (The phase δ_2 is 180°, which insures that $x_1(0) = A_0$)

$$x_1(t) = \frac{A_0}{2}\left(2\cos\frac{(\omega_1+\omega_2)t}{2}\cos\frac{(\omega_1-\omega_2)t}{2}\right)$$

Letting … $\bar{\omega} = \dfrac{\omega_1+\omega_2}{2}$ *and* $\Delta = \dfrac{\omega_1-\omega_2}{2}$

$$x_1(t) = A_0\left(\cos\bar{\omega}t\ \cos\Delta t\right)$$

From Equation 11.3.18, the solution for x_2 is …

$$x_2(t) = \frac{A_0}{2}\left(\cos\omega_1 t - \cos\omega_2 t\right)$$

$$x_2(t) = \frac{A_0}{2}\left(2\sin\frac{(\omega_1+\omega_2)t}{2}\sin\frac{(\omega_1-\omega_2)t}{2}\right)$$

$$x_2(t) = A_0\left(\sin\bar{\omega}t\sin\Delta t\right)$$

11.14

At time $t = 0$ and short times thereafter … $\cos\Delta t \approx 1$ and $\sin\Delta t \approx 0$. Thus, …
$x_1 \approx A_0 \cos\bar{\omega}t \quad and \quad x_2 \approx 0$. This situation occurs again when $\Delta t = 2\pi$.

$$\Delta = \frac{\omega_2 - \omega_1}{2} = \frac{1}{2}\left[\left(\frac{k+2k'}{m}\right)^{\frac{1}{2}} - \left(\frac{k}{m}\right)^{\frac{1}{2}}\right]$$

$$\Delta = \frac{1}{2}\left(\frac{k}{m}\right)^{\frac{1}{2}}\left[\left(1+\frac{2k'}{k}\right)^{\frac{1}{2}} - 1\right]$$

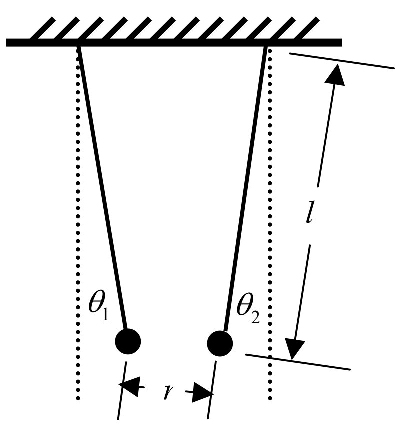
for $k' << k$, $\left(1+\frac{2k'}{k}\right)^{\frac{1}{2}} - 1 \approx 1 + \frac{1}{2}\left(\frac{2k'}{k}\right) + \ldots - 1 = \frac{k'}{k}$

$$\Delta = \frac{1}{2}\left(\frac{k}{m}\right)^{\frac{1}{2}}\left(\frac{k'}{k}\right)$$

$$T = \frac{2\pi}{\Delta} = 2\pi\left(\frac{m}{k}\right)^{\frac{1}{2}}\frac{2k}{k'} = \frac{2\pi}{\omega_1}\left(\frac{2k}{k'}\right)$$

$$T = T_1\left(\frac{2k}{k'}\right)$$

11.15



$$T = \frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right)$$

$$V = mgl\left[\left(1-\cos\theta_1\right) + \left(1-\cos\theta_2\right)\right] - \frac{k}{r}$$

$$r = r_0 + l\sin\theta_2 - l\sin\theta_1$$

$$\frac{\partial V}{\partial\theta_1} = mgl\sin\theta_1 - \frac{kl\cos\theta_1}{\left(r_0 + l\sin\theta_2 - l\sin\theta_1\right)^2}$$

$$\frac{\partial^2 V}{\partial\theta_1^2} = mgl\cos\theta_1 - \frac{kl\sin\theta_1}{\left(r_0 + l\sin\theta_2 - l\sin\theta_1\right)^2} - \frac{2kl^2\cos^2\theta_1}{\left(r_0 + l\sin\theta_2 - l\sin\theta_1\right)^3}$$

$$k_{11} = \left.\frac{\partial^2 V}{\partial\theta_1^2}\right|_{\theta_1=\theta_2=0} = mgl - \frac{2kl^2}{r_0^3}$$

$$\frac{\partial^2 V}{\partial\theta_2\partial\theta_1} = \frac{2kl^2\cos\theta_1\cos\theta_2}{\left(r_0 + l\sin\theta_2 - l\sin\theta_1\right)^3}$$

$$k_{12} = \left.\frac{\partial^2 V}{\partial\theta_1\partial\theta_2}\right|_{\theta_1=\theta_2=0} = \left.\frac{\partial^2 V}{\partial\theta_2\partial\theta_1}\right|_{\theta_1=\theta_2=0} = \frac{2kl^2}{r_0^{\,3}}$$

$$\frac{\partial V}{\partial\theta_2} = mgl\sin\theta_2 + \frac{kl\cos\theta_2}{\left(r_0 + l\sin\theta_2 - l\sin\theta_1\right)^2}$$

$$\frac{\partial^2 V}{\partial\theta_2^{\,2}} = mgl\cos\theta_2 - \frac{kl\sin\theta_2}{\left(r_0 + l\sin\theta_2 - l\sin\theta_1\right)^2} - \frac{2kl^2\cos^2\theta_2}{\left(r_0 + l\sin\theta_2 - l\sin\theta_1\right)^3}$$

$$k_{22} = \left.\frac{\partial^2 V}{\partial\theta_2^{\,2}}\right|_{\theta_1=\theta_2=0} = mgl - \frac{2kl^2}{r_0^{\,3}}$$

Thus, from Equation 11.3.37a or b, we have

$$V = \frac{1}{2}\left[k_{11}\theta_1^{\,2} + 2k_{12}\theta_1\theta_2 + k_{22}\theta_2^{\,2}\right]$$

But from Equation 11.3.9, for the coupled oscillator, we have

$$V = \frac{1}{2}kx_1^{\,2} + \frac{1}{2}k'\left(x_1^{\,2} - 2x_1x_2 + x_2^{\,2}\right) + \frac{1}{2}kx_1^{\,2}$$

The forms of the potential energy function are similar with …

$$k_{11} = k + k' = k_{22} \quad and \quad k_{12} = -k'$$

In the case here …

$$k = mgl \quad and \quad k' = -\frac{2kl^2}{r_0^{\,3}}$$

Chapter 11 (continued)

11.16

$$T=\frac{1}{2}m_1\dot{x}_1{}^2+\frac{1}{2}m_2\dot{x}_2{}^2$$
$$V=\frac{1}{2}k_1x_1{}^2+\frac{1}{2}k'\left(x_2-x_1\right)^2+\frac{1}{2}k_2x_2{}^2$$
$$L=T-V$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1}=m_1\ddot{x}_1 \qquad \frac{\partial L}{\partial x_1}=-k_1x_1+k'\left(x_2-x_1\right)$$
$$m_1\ddot{x}_1+k_1x_1-k'\left(x_2-x_1\right)=0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2}=m_2\ddot{x}_2 \qquad \frac{\partial L}{\partial x_2}=-k_2x_2-k'\left(x_2-x_1\right)$$
$$m_2\ddot{x}_2+k_2x_2+k'\left(x_2-x_1\right)$$

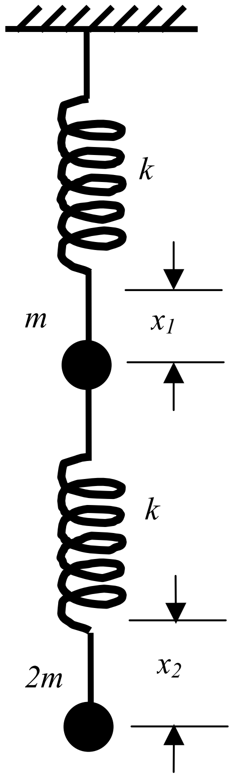
$$\begin{vmatrix}-m_1\omega^2+k_1+k' & -k' \\ -k' & -m_2\omega^2+k_2+k'\end{vmatrix}=0$$

$$m_1m_2\omega^4-\left[m_2\left(k_1+k'\right)+m_1\left(k_2+k'\right)\right]\omega^2+\left(k_1+k'\right)\left(k_2+k'\right)-k'^2=0$$
$$m_1m_2\omega^4-\left[m_2\left(k_1+k'\right)+m_1\left(k_2+k'\right)\right]\omega^2+k_1k_2\left(k_1+k_2\right)k'=0$$
$$\omega^2=\frac{m_2\left(k_1+k'\right)+m_1\left(k_2+k'\right)\pm\sqrt{\left[m_2\left(k_1+k'\right)+m_1\left(k_2+k'\right)\right]^2-4m_1m_2\left[k_1k_2+\left(k_1+k_2\right)k'\right]}}{2m_1m_2}=0$$

For $m_1=m,\ m_2=2m,\ k_1=k,\ k_2=2k,\ k'=2k$

$$\omega^2=\frac{2m\left(3k\right)+m\left(4k\right)\pm\sqrt{\left(6mk+4mk\right)^2-4\left(2m^2\right)\left(2k^2+6k^2\right)}}{2\left(2m^2\right)}$$
$$\omega^2=\frac{10}{4}\frac{k}{m}\pm\frac{6}{4}\frac{k}{m}$$
$$\omega=2\omega_0 \ \ and \ \ \omega_0 \ \ where \ \ \omega_0=\sqrt{\frac{k}{m}}$$

11.17



Note: As discussed in Section 3.2, the effect of any constant external force on a harmonic oscillator is to shift the equilibrium position. x_1 and x_2 are the positions of the harmonic oscillator masses away from their respective "shifted" equilibrium positions.

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2$$

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2$$

$$L = T - V$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = m\ddot{x}_1, \quad \frac{\partial L}{\partial x_1} = -kx_1 + k(x_2 - x_1)$$

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = 2m\ddot{x}_2, \quad \frac{\partial L}{\partial x_2} = -k(x_2 - x_1)$$

$$2m\ddot{x}_2 + kx_2 - kx_1 = 0$$

The secular equation (11.4.12) is thus

$$\begin{vmatrix} -m\omega^2 + 2k & -k \\ -k & -2m\omega^2 + k \end{vmatrix} = 0$$

$$2m^2\omega^4 - 5mk\omega^2 + 2k^2 + k^2 = 0$$

The eigenfrequencies are thus …

$$\omega^2 = \frac{5 \pm \sqrt{17}}{4}\left(\frac{k}{m}\right)$$

The homogeneous equations (Equations 11.4.10) for the two components of the j[th] eigenvector are …

$$\begin{pmatrix} -m\omega^2 + 2k & -k \\ -k & -2m\omega^2 + k \end{pmatrix}\begin{pmatrix} a_{1j} \\ a_{2j} \end{pmatrix} = 0$$

For the first eigenvector (the anti-symmetric mode, j = 1) …

Inserting $\omega_1^2 = \frac{5 + \sqrt{17}}{4}\left(\frac{k}{m}\right)$ into the first of the two homogeneous equations yields

$$\left[-\frac{5 + \sqrt{17}}{4}k + 2k\right]a_{11} = ka_{21}$$

$$a_{21} = \frac{3 - \sqrt{17}}{4}a_{11}$$

Letting $a_{11} = 1$, then $a_{21} = -0.281$ (Thus, in the anti-symmetric normal mode, the amplitude of the vibration of the second mass is 0.281 that of the first mass and 180^o out of phase with it.)

For the second eigenvector (the symmetric mode, j = 2) …

Inserting $\omega_2^{\,2}=\dfrac{5-\sqrt{17}}{4}\left(\dfrac{k}{m}\right)$ into the first of the two homogeneous equations yields

$$\left[-\frac{5-\sqrt{17}}{4}k+2k\right]a_{12}=ka_{22}$$

$$a_{22}=\frac{3+\sqrt{17}}{4}a_{12}$$

Letting a_{12} = 1, then a_{22} = 1.781 (Thus, in the symmetric normal mode, the amplitude of the vibration of the second mass is 1.781 that of the first mass and in phase with it.)

The two eigenvectors (Equation 11.4.13 and see accompanying Table) are …

$$Q_1=\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}\cos\left(\omega_1 t-\delta_1\right)=\begin{pmatrix}1\\-0.281\end{pmatrix}\cos\left(\omega_1\,t-\delta_1\right)$$

$$Q_2=\begin{pmatrix}a_{12}\\a_{22}\end{pmatrix}\cos\left(\omega_2 t-\delta_2\right)=\begin{pmatrix}1\\1.781\end{pmatrix}\cos\left(\omega_2 t-\delta_2\right)$$

11.18

$$T=\frac{1}{2}ml_1^{\,2}\dot{\theta}^2+\frac{1}{2}m\left(l_1\dot{\theta}+l_2\dot{\phi}\right)^2$$

$$V=-mgl_1\cos\theta-mg\left(l_1\cos\theta+l_2\cos\phi\right)$$

For small angular displacements …

$$L=T-V\approx\frac{m}{2}l_1^{\,2}\dot{\theta}^2+\frac{m}{2}\left(l_1\dot{\theta}+l_2\dot{\phi}\right)^2+2mgl_1\left(1-\frac{\theta^2}{2}\right)+mgl_2\left(1-\frac{\phi^2}{2}\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}=ml_1^{\,2}\ddot{\theta}+ml_1\left(l_1\ddot{\theta}+l_2\ddot{\phi}\right),\quad \frac{\partial L}{\partial\theta}=-2mgl_1\theta$$

$$2l_1\ddot{\theta}+l_2\ddot{\phi}+2g\theta=0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}}=ml_2\left(l_1\ddot{\theta}+l_2\ddot{\phi}\right),\quad \frac{\partial L}{\partial\phi}=-mgl_2\phi$$

$$l_1\ddot{\theta}+l_2\ddot{\phi}+g\phi=0$$

The secular equation (Equation 11.4.12) is …

$$\begin{vmatrix}-2l_1\omega^2+2g & -l_2\omega^2\\ -l_1\omega^2 & -l_2\omega^2+g\end{vmatrix}=0$$

$$2l_1l_2\omega^4-2g\left(l_1+l_2\right)\omega^2+2g^2-l_1l_2\omega^4=0$$

Solving for the eigenfrequencies ω^2 …

$$\omega^2=\frac{2g\left(l_1+l_2\right)\pm\sqrt{4g^2\left(l_1+l_2\right)^2-8l_1l_2g^2}}{2l_1l_2}=\frac{g}{l_1l_2}\left(l_1+l_2+\sqrt{l_1^{\,2}+l_2^{\,2}}\right)$$

The homogeneous equations (Equations 11.4.10) for the two components of the j[th] eigenvector are …

$$\begin{pmatrix} -2l_1\omega^2+2g & -l_2\omega^2 \\ -l_1\omega^2 & -l_2\omega^2+g \end{pmatrix}\begin{pmatrix} a_{1j} \\ a_{2j} \end{pmatrix}=0$$

Inserting the larger eigenfrequency (the (+) solution for ω^2 above) into the upper homogeneous equation yields the solution for the components of the 1st eigenvector …

$$\left(-2l_1\omega_1{}^2+2g\right)a_{11}-l_2\omega_1{}^2a_{21}=0$$

$$2ga_{11}\left[1-\frac{l_1+l_2+\sqrt{l_1^2+l_2^2}}{l_2}\right]=ga_{21}\frac{l_1+l_2+\sqrt{l_1^2+l_2^2}}{l_1}$$

$$a_{21}=a_{11}\frac{l_2-l_1-\sqrt{l_1^2+l_2^2}}{l_2}$$ for the higher frequency, anti-symmetric mode.

Inserting the smaller eigenfrequency (the (-) solution for ω^2) into the upper homogeneous equation yields the solution for the components of the 2nd eigenvector …

$$\left(-2l_1\omega_2{}^2+2g\right)a_{12}-l_2\omega_2{}^2a_{22}=0$$

$$a_{22}=a_{12}\frac{l_2-l_1+\sqrt{l_1^2+l_2^2}}{l_2}$$ for the lower frequency, symmetric mode.

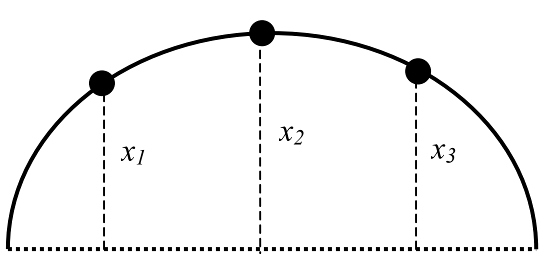
Again, we let $a_{11} = 1$ and $a_{21} = 1$, since only ratios of the components of a given eigen vector can be determined. The two eigenvectors are thus (Equation 11.4.12 and accompanying table)

$$Q_1=\begin{pmatrix} 1 \\ \dfrac{l_2-l_1-\sqrt{l_1^2+l_2^2}}{l_2} \end{pmatrix}\cos\left(\omega_1 t-\delta_1\right)$$ … anti-symmetric

$$Q_2=\begin{pmatrix} 1 \\ \dfrac{l_2-l_1+\sqrt{l_1^2+l_2^2}}{l_2} \end{pmatrix}\cos\left(\omega_2 t-\delta_2\right)$$ … symmetric

As a check, set $l_1=l_2=l$ and compare with the solution for Example 11.3.1.

11.19



$$T=\frac{1}{2}m\left(\dot{x}_1{}^2+\dot{x}_2{}^2+\dot{x}_3{}^2\right)$$

$$V=\frac{1}{2}k\left[x_1{}^2+\left(x_2-x_1\right)^2+\left(x_3-x_2\right)^2+x_3{}^2\right]$$

$$L=T-V$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}_i}-\frac{\partial L}{\partial x_i}=0$$

$$m\ddot{x}_1 + kx_1 - k\left(x_2 - x_1\right) = 0$$

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$m\ddot{x}_2 + k\left(x_2 - x_1\right) - k\left(x_3 - x_2\right) = 0$$

$$-kx_1 + m\ddot{x}_2 + 2kx_2 - kx_3 = 0$$

$$m\ddot{x}_3 + k\left(x_3 - x_2\right) + kx_3 = 0$$

$$-kx_2 + m\ddot{x}_3 + 2kx_3 = 0$$

The secular equation (Equation 11.4.12) is …

$$\begin{vmatrix} -m\omega^2 + 2k & -k & 0 \\ -k & -m\omega^2 + 2k & -k \\ 0 & -k & -m\omega^2 + 2k \end{vmatrix} = 0$$

$$\left(-m\omega^2 + 2k\right)^3 - 2k^2\left(-m\omega^2 + 2k\right) = 0$$

$$-m\omega^2 + 2k = 0, \quad or \quad \left(-m\omega^2 + 2k\right)^2 - 2k^2 = 0$$

$$\omega^2 = \frac{2k}{m} = 2\,\omega_0{}^2$$

$$-m\omega^2 + 2k = \pm\sqrt{2}\,k$$

$$\omega^2 = \left(2 \pm \sqrt{2}\right)\frac{k}{m} = \left(2 \pm \sqrt{2}\right)\omega_0{}^2$$

From Equation 11.5.17 ($N = 1$ and $n = 3$)

$$\omega_1 = 2\omega_0 \sin\frac{\pi}{8}$$

Because $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$

$$\omega_1 = 2\omega_0\sqrt{\frac{1-\cos\pi/4}{2}} = \sqrt{2}\omega_0\sqrt{1-\sqrt{2}/2}$$

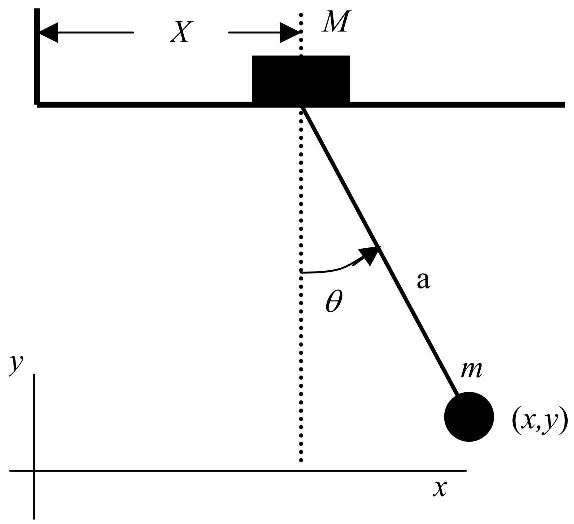
$$\omega_1 = \omega_0\sqrt{2-\sqrt{2}}$$

$$\omega_2 = 2\omega_0 \sin\frac{2\pi}{8} = 2\omega_0\frac{\sqrt{2}}{2} = \sqrt{2}\omega_0$$

$$\omega_3 = 2\omega_0 \sin\frac{3\pi}{8} = 2\omega_0\sqrt{\frac{1-\cos 3\pi/4}{2}}$$

$$\omega_3 = \sqrt{2}\omega_0\sqrt{1+\sqrt{2}/2} = \omega_0\sqrt{2+\sqrt{2}}$$

11.20



Generalized coordinates: $X,\ s\left(=a\theta\right)$

and

$$x = X + a\sin\theta \qquad y = a(1-\cos\theta)$$
$$\dot{x} = \dot{X} + a\dot{\theta}\cos\theta \qquad \dot{y} = a\dot{\theta}\sin\theta$$

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$
$$= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left[\left(\dot{X} + a\dot{\theta}\cos\theta\right)^2 + \left(a\dot{\theta}\sin\theta\right)^2\right]$$
$$V = mgy = mga\left(1-\cos\theta\right)$$

For small oscillations, in terms of the generalized coordinates X and s …

$$T \approx \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{X}+\dot{s}\right)^2 \qquad V \approx \frac{1}{2a}mgs^2$$
$$L = T - V \approx \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{X}+\dot{s}\right)^2 - \frac{1}{2a}mgs^2$$

Lagrange's equations of motion yield …

$$\ddot{X} + \ddot{s} + \frac{g}{a}s = 0 \qquad \left(M+m\right)\ddot{X} + m\ddot{s} = 0$$

Assuming …

$$X = Ae^{i\omega t} \qquad s = Be^{i\omega t}$$

we obtain the matrix equation …

$$\begin{pmatrix} \omega^2 & \omega^2 - g/a \\ \left(M+m\right)\omega^2 & m\omega^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0$$

Setting the determinant of the above matrix equal to zero yields…

$$\omega_1^{\,2} = 0\,, \qquad \omega_2^{\,2} = \frac{g\left(m+M\right)}{M}$$

The mode corresponding to ω_1 is …

$$\begin{pmatrix} 0 & -g/a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0 \qquad \text{implies that} \quad A = B = 0$$

Thus, mode 1 exhibits no oscillation! It is a pure translation with …

$\theta = 0$ and $X = A_1 t + A_2$

The mode corresponding to ω_2 is …

$$\omega_2^{\,2}A + \left(\omega_2^{\,2} - g/a\right)B = 0$$

or… $\dfrac{B}{A}=\dfrac{\omega_2^{\,2}}{\omega_2^{\,2}-g/a}=-\dfrac{m+M}{m}$

Setting $A=1$, we have for the 2[nd] mode …

$X=e^{i\omega_2 t}$ and $s=a\theta=-\dfrac{m+M}{m}e^{i\omega_2 t}$

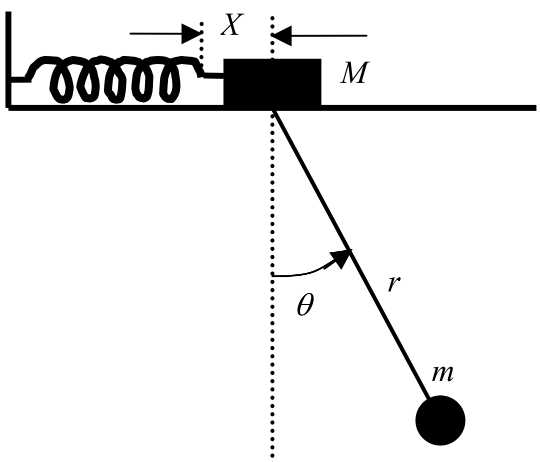
This mode corresponds to an oscillation about the CM where …

$(m+M)X=-ms(=-ma\theta)$

The normal mode vectors are …

$$Q_1=\begin{pmatrix}A_1 t+A_2\\ 0\end{pmatrix} \quad\text{and}\quad Q_2=\begin{pmatrix}1\\ -\dfrac{m+M}{m}\end{pmatrix}e^{i\omega_2 t}$$

11.21



(a) We can solve for the normal modes using Equation 11.4.9 …

$\left(\boldsymbol{K}-\omega^2\boldsymbol{M}\right)\boldsymbol{a}=0$

K and ***M*** are the potential energy and kinetic energy matrices respectively. ***a*** is a two-component vector whose elements are the amplitudes of the coordinates ***q***. The kinetic energy in Example 11.3.2, assuming small displacements from equilibrium, are …

$$T\approx\frac{1}{2}m\dot{X}^2+\frac{1}{2}m\left(\dot{X}^2+\left(r\dot{\theta}\right)^2+2\dot{X}r\dot{\theta}\right)$$

$$T\approx\frac{1}{2}2m\dot{X}^2+\frac{1}{2}m\left(2\dot{X}r\dot{\theta}\right)+\frac{1}{2}m\left(r\dot{\theta}\right)^2$$

or, in matrix form

$T=\dfrac{1}{2}\tilde{\boldsymbol{q}}\boldsymbol{M}\boldsymbol{q}$ where $\boldsymbol{q}=\begin{pmatrix}X\\ r\theta\end{pmatrix}$

Thus, $\boldsymbol{M}=m\begin{pmatrix}2&1\\1&1\end{pmatrix}$

The potential energy is …

$$V\approx\frac{1}{2}k\dot{X}^2+m\frac{g}{2r}\left(r\theta\right)^2$$

$$V\approx\frac{1}{2}\frac{k}{2m}\left[2m\dot{X}^2+\frac{2mg}{kr}m\left(r\theta\right)^2\right]=\frac{1}{2}m\omega_0^{\,2}\left[2\dot{X}^2+\left(r\theta\right)^2\right]$$

where $\omega_0^{\,2}=\dfrac{k}{M+m}=\dfrac{k}{2m}$

Thus $\boldsymbol{K} = m\omega_0^{\,2}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ The above matrix equation is thus …

$$\left[m\omega_0^{\,2}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - m\omega^2\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\right]\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2\left(\omega_0^{\,2}-\omega^2\right) & -\omega^2 \\ -\omega^2 & \omega_0^{\,2}-\omega^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

From the top equation in the matrix equation above, we get …

$$2\left(\omega_0^{\,2}-\omega^2\right)a_1 - \omega^2 a_2 = 0$$

$$\frac{a_2}{a_1} = \frac{2\left(\omega_0^{\,2}-\omega^2\right)}{\omega^2}$$

The amplitudes for the two normal modes can be found by setting $\omega^2 = \omega_1^{\,2} \;\; or \;\; \omega_2^{\,2}$.
In each case we set $a_1 = 1$, which we are free to do since only ratios of the amplitudes can be determined.

For $\omega^2 = \omega_1^{\,2} = \left(2-\sqrt{2}\right)\omega_0^{\,2}$ we obtain …

$$a_2 = \frac{2\left(\omega_0^{\,2}-\omega_1^{\,2}\right)}{\omega_1^{\,2}} = \frac{2\left[1-\left(2-\sqrt{2}\right)\right]}{\left(2-\sqrt{2}\right)} = \sqrt{2}$$

Thus, for mode 1, the lower frequency, symmetric mode …

$$\boldsymbol{q}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{i\omega_1 t}$$

For $\omega^2 = \omega_2^{\,2} = \left(2+\sqrt{2}\right)\omega_0^{\,2}$ we obtain …

$$a_2 = \frac{2\left(\omega_0^{\,2}-\omega_2^{\,2}\right)}{\omega_1^{\,2}} = \frac{2\left[1-\left(2+\sqrt{2}\right)\right]}{\left(2+\sqrt{2}\right)} = -\sqrt{2}$$

Thus, for mode 2, the higher frequency, anti-symmetric mode …

$$\boldsymbol{q}_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} e^{i\omega_2 t}$$

(b) In this case, $\dfrac{m}{m+M} << 1 \;\; or \;\; M >> m$. We also assume that the spring is "slack", i.e., $\dfrac{k}{M+m} << \dfrac{g}{r}$, an assumption not stated in the problem (which needs to be rectified in the next edition, I suppose)

Also, $2mg \neq kr$, so we let $\omega_0^{\,2} = \dfrac{k}{M+m} \approx \dfrac{k}{M} \;\; and \;\; \Omega_0^{\,2} = \dfrac{g}{r}$

The kinetic energy is

$$T \approx \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{X}^2 + \left(r\dot{\theta}\right)^2 + 2\dot{X}r\dot{\theta}\right)$$

$$T \approx \frac{1}{2}\left(M+m\right)\dot{X}^2 + \frac{1}{2}m\left(\left(r\dot{\theta}\right)^2 + 2\dot{X}r\dot{\theta}\right) \approx \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(2\dot{X}r\dot{\theta}\right) + \frac{1}{2}m\left(r\dot{\theta}\right)^2$$

and the ***M***-matrix is

$$\boldsymbol{M} \approx \begin{pmatrix} M & m \\ m & m \end{pmatrix}$$

The potential energy is

$$V \approx \frac{1}{2}k\dot{X}^2 + \frac{1}{2}m\frac{g}{r}\left(r\theta\right)^2 = \frac{1}{2}M\omega_0{}^2 + \frac{1}{2}m\Omega_0{}^2$$

The ***K***-matrix is thus

$$\boldsymbol{K} \approx \begin{pmatrix} \omega_0{}^2 M & 0 \\ 0 & \Omega_0{}^2 m \end{pmatrix}$$

To solve for ω^2, we set $\left|\boldsymbol{K} - \omega^2\boldsymbol{M}\right| = 0$

$$\begin{vmatrix} \left(\omega_0{}^2 - \omega^2\right)M & -\omega^2 m \\ -\omega^2 m & \left(\Omega_0{}^2 - \omega^2\right)m \end{vmatrix} = 0$$

which yields …

$$\left[\omega_0{}^2\Omega_0{}^2 + \omega^4 - \omega^2\Omega_0{}^2 - \omega^2\omega_0{}^2\right]mM - \omega^4 m^2 = 0$$

$$\omega^4\left(M-m\right) - \omega^2\left(\omega_0{}^2 + \Omega_0{}^2\right)M + \omega_0{}^2\Omega_0{}^2 M = 0$$

Neglecting m with respect to M and simplifying yields …

$$\omega^4 - \omega^2\left(\omega_0{}^2 + \Omega_0{}^2\right) + \omega_0{}^2\Omega_0{}^2 = 0$$

Solving for ω^2 …

$$\omega^2 = \frac{\left(\omega_0{}^2 + \Omega_0{}^2\right)}{2} \pm \frac{\sqrt{\left(\omega_0{}^2 + \Omega_0{}^2\right)^2 - 4\omega_0{}^2\Omega_0{}^2}}{2}$$

$$\omega^2 = \frac{\left(\omega_0{}^2 + \Omega_0{}^2\right)}{2} \pm \frac{\left(\omega_0{}^2 - \Omega_0{}^2\right)}{2}$$ Thus, we get …

$$\omega_1{}^2 = \omega_0{}^2 \quad and \quad \omega_2{}^2 = \Omega_0{}^2$$

Now, we solve for the amplitudes ***a*** of the normal mode vectors …

$$\left(\boldsymbol{K} - \omega^2\boldsymbol{M}\right)\boldsymbol{a} = 0$$

$$\begin{pmatrix} \left(\omega_0{}^2 - \omega^2\right)M & -\omega^2 m \\ -\omega^2 m & \left(\Omega_0{}^2 - \omega^2\right)m \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

Using the first of the above matrix equations for $\omega^2 = \omega_1{}^2 = \omega_0{}^2$ gives …

$$a_2 = 0 \quad and \quad a_1 = 1$$

Using the second equation for $\omega^2 = \omega_2{}^2 = \Omega_0{}^2$ gives …

$a_1 = 0 \;\; and \;\; a_2 = 1$

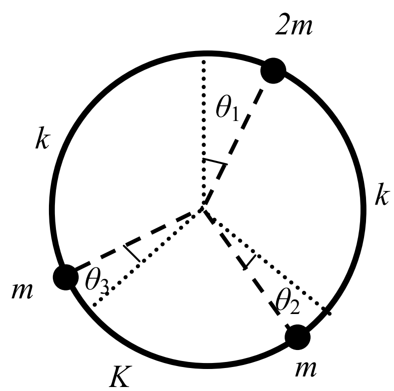
Thus, the normal modes are approximately …

$\boldsymbol{Q}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\omega_0 t} \quad and \quad \boldsymbol{Q}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\Omega_0 t}$

Note that we could have guessed this almost immediately. The above assumption is tantamount to omitting the cross elements $-\omega^2 m$. This completely eliminates the small coupling between the two oscillators, which reduces the matrix $\boldsymbol{K} - \omega^2 \boldsymbol{M}$ to the purely diagonal terms …

$\begin{pmatrix} \left(\omega_0{}^2 - \omega^2\right) M & 0 \\ 0 & \left(\Omega_0{}^2 - \omega^2\right) m \end{pmatrix}$, which leads directly to the above solution.

22.



We "scale" the force constants and masses to 1 unit, namely,

$m = 1 \;\; and \;\; k = 1$.

Let $k' = K \;\; and \;\; l = 1$

such that $3l$ = circumference

So: $T = \frac{1}{2}\left(2\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + \dot{\theta}_3{}^2\right)$

$$V = \frac{1}{2}(\theta_1 - \theta_2)^2 + \frac{1}{2}(\theta_2 - \theta_1)^2 + \frac{1}{2}(\theta_1 - \theta_3)^2 + \frac{1}{2}(\theta_3 - \theta_1)^2 + \frac{1}{2}K(\theta_2 - \theta_3)^2 + \frac{1}{2}K(\theta_3 - \theta_2)^2$$

Collecting terms …

$$V = \frac{1}{2}\left[4\theta_1^2 + 2(1+K)\theta_2{}^2 + 2(1+K)\theta_3{}^2 - 4\theta_1\theta_2 - 4\theta_1\theta_3 - 4K\theta_2\theta_3\right]$$

<u>***K*** - Matrix</u>

$$\boldsymbol{K} = \begin{pmatrix} 4 & -2 & -2 \\ -2 & 2+2K & -2K \\ -2 & -2K & 2+2K \end{pmatrix}$$

<u>***M*** - Matrix</u>

$$\boldsymbol{M} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The ttansformation matrix A that diagonalizes these matrices is made up of the three eigenvectors $\boldsymbol{Q}_{\mathrm{i}}$ whose amplitudes are $\boldsymbol{a}_{\mathrm{i}}$ … we guess that …

1. Uniform rotation

$$\theta_1 = \theta_2 = \theta_3 \;\; or \;\; \boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

2. Anti-symmetric oscillation of 2&3, while 1 remains fixed

$$\theta_1 = 0;\ \theta_3 = -\theta_2 \ \ or \ \ \boldsymbol{a}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

3. Anti-symmetric oscillation of 1&2 together with respect to 3

$$\theta_1 = 1;\ \theta_3 = \theta_2 \ \ or \ \ \boldsymbol{a}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Thus, $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$ *and* $\tilde{\boldsymbol{A}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$

We can now diagonalize $\boldsymbol{K}$and $\boldsymbol{M}$ …

$$\tilde{\boldsymbol{A}}\boldsymbol{K}\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 4 & -2 & -2 \\ -2 & 2+2K & -2K \\ -2 & -2K & 2+2K \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} = \boldsymbol{K}_{diag}$$

$$\boldsymbol{K}_{diag} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4+8K & \\ 0 & & 16 \end{pmatrix} \ \ Likewise \ \ \boldsymbol{M}_{diag} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

The eigenfrequencies are $\omega_i^2 = \left[\boldsymbol{K}_{diag} / \boldsymbol{M}_{diag}\right]_i$

$$\omega_2^2 = 0;\ \omega_2^2 = 2+4K;\ \omega_3^2 = 4$$

The general solution can be generated from the following table …

	$\underline{Q_1}$	$\underline{Q_2}$	$\underline{Q_3}$
θ_1	a_1	0	a_3
θ_2	a_1	a_2	$-a_3$
θ_3	a_1	$-a_2$	$-a_3$

where $\boldsymbol{Q}_1 = a_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cos\left(\omega_1 t - \delta_1\right),\ \boldsymbol{Q}_2 = a_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cos\left(\omega_2 t - \delta_2\right),\ \boldsymbol{Q}_3 = a_3 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cos\left(\omega_3 t - \delta_3\right)$

Thus,
$$\theta_1 = a_1 \cos\left(\omega_2 t - \delta_2\right) + a_3 \cos\left(\omega_3 t - \delta_3\right)$$
$$\theta_1 = a_1 \cos\left(\omega_2 t - \delta_2\right) + a_2 \cos\left(\omega_2 t - \delta_2\right) - a_3 \cos\left(\omega_3 t - \delta_3\right)$$
$$\theta_3 = a_1 \cos\left(\omega_2 t - \delta_2\right) - a_2 \cos\left(\omega_2 t - \delta_2\right) - a_3 \cos\left(\omega_3 t - \delta_3\right)$$

Initial conditions are: $\theta_1 = \theta_0 = 10^o$; $\theta_2 = \theta_3 = 0$; $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = 0$
The conditions generate 6 equations with 6 unknowns and solving gives …

$$\theta_1 = \frac{\theta_0}{2}\cos\omega_1 t + \frac{\theta_0}{2}\cos\omega_3 t$$

$$\theta_2 = \frac{\theta_0}{2}\cos\omega_1 t - \frac{\theta_0}{2}\cos\omega_3 t$$

$$\theta_2 = \frac{\theta_0}{2}\cos\omega_1 t - \frac{\theta_0}{2}\cos\omega_3 t$$

11.23 See Ex. 11.4.1, page 497-498

The amplitudes of the eigenvectors are …

$$\boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \boldsymbol{a}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \boldsymbol{a}_3 = \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}$$

K **- Matrix** | ***M*** **- Matrix**

$$\boldsymbol{K} = \begin{pmatrix} K & \text{-}K & 0 \\ \text{-}K & 2K & \text{-}K \\ 0 & \text{-}K & K \end{pmatrix} \qquad \boldsymbol{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$$

$$\boldsymbol{K}_{diag} = \tilde{\boldsymbol{A}}\boldsymbol{K}\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -2m/M & 1 \end{pmatrix}\begin{pmatrix} K & -K & 0 \\ -K & 2K & -K \\ 0 & -K & K \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2m/M \\ 1 & -1 & 1 \end{pmatrix}$$

$$\boldsymbol{K}_{diag} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2K & 0 \\ 0 & 2K + 8K\,m/M & 8K\,m^2/M^2 \end{pmatrix}$$

$$\boldsymbol{M}_{diag} = \tilde{\boldsymbol{A}}\boldsymbol{M}\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -2m/M & 1 \end{pmatrix}\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2m/M \\ 1 & -1 & 1 \end{pmatrix}$$

$$\boldsymbol{M}_{diag} = \begin{pmatrix} 2m + M & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m + 4m^2/M \end{pmatrix}$$

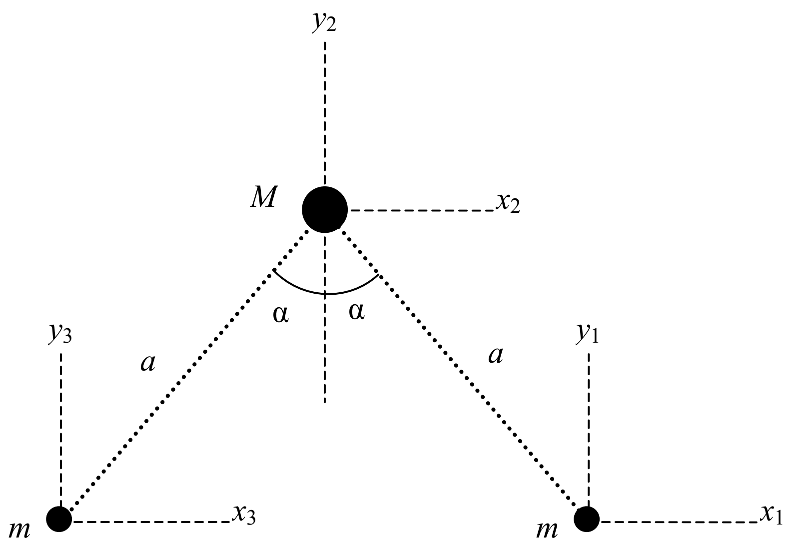
So, we have …

$\omega_1^2 = 0$, $\omega_2^2 = \dfrac{2K}{2m} = \dfrac{K}{m}$, and …

$$\omega_3^2 = \frac{2K + 8K\,m/M + 8K\,m^2/M^2}{2m + 4\,m^2/M} = \frac{K\left(1 + 4m/M + 4m^2/M^2\right)}{m\left(1 + 2m/M\right)}$$

$$\omega_3^{\ 2} = \frac{K}{m}\frac{\left(1+2\,m/M\right)^2}{\left(1+2\,m/M\right)} = \frac{K}{m}\left(1+2\,m/M\right)$$

11.24



Select coordinates (x_i , y_i) as shown, then …

$$T = \frac{1}{2}m\left(\dot{x}_1^{\ 2} + \dot{y}_1^{\ 2} + \dot{x}_3^{\ 2} + \dot{y}_3^{\ 2}\right) + \frac{1}{2}M\left(\dot{x}_2^{\ 2} + \dot{y}_2^{\ 2}\right)$$

The potential energy depends only on the compression (or stretching) of the two springs connecting each m to M (hydrogen to sulfur). Let δa_1 and δa_2 be incremental changes in the distances a_1 or $HS(1 \to 2)$ and a_2 or $HS(3 \to 2)$. We have …

$$\delta a_1 = \left(x_1 - x_2\right)\sin\alpha + \left(y_2 - y_1\right)\cos\alpha = \frac{1}{\sqrt{2}}\left(x_1 - x_2 + y_2 - y_1\right)$$

$$\delta a_2 = \left(x_2 - x_3\right)\sin\alpha + \left(y_2 - y_3\right)\cos\alpha = \frac{1}{\sqrt{2}}\left(x_2 - x_3 + y_2 - y_3\right)$$

$$V = \frac{1}{2}k\left[\left(\delta a_1\right)^2 + \left(\delta a_2\right)^2\right]$$

We can reduce the degrees of freedom from 6 to 3 by ignoring the two translational modes and the rotational mode. Thus we consider only vibrational modes. The coordinates must obey the following constraints …

No center of mass motion:

$$m\left(y_1 + y_3\right) + My_2 = 0 \quad and \quad m\left(x_1 + x_3\right) + Mx_2 = 0$$

No angular momentum about any point. We choose that point to be the sulfur atom (M)…

$$my_3 a\sin\alpha - my_1 a\sin\alpha - mx_3 a\sin\alpha - mx_1 a\sin\alpha = 0$$

$$y_3 - y_1 - x_3 - x_1 = 0$$

We introduce three generalized coordinates Q, q_1 and q_2 that should be close to what we guess would be normal modes …

$$Q = x_1 + x_3\,;\quad q_1 = x_1 - x_3\,;\quad q_2 = y_1 + y_3$$

Solve for x_i , y_i in terms of Q, q_1 and q_2 using the above 3 equations of constraint …

$$x_1 = \frac{1}{2}(Q + q_1);\quad x_2 = -\frac{m}{M}Q;\quad = x_1 - x_3\,;\quad x_3 = \frac{1}{2}(Q - q_1)$$

$$y_1 = \frac{1}{2}(q_2 - Q)\,;\quad y_2 = -\frac{m}{M}q_2\,;\qquad y_3 = \frac{1}{2}(Q + q_2)$$

Thus, the kinetic energy, in terms of these generalized coordinates, is …

$$T = \frac{1}{2}m\left(1 + \frac{m}{M}\right)\dot{Q}^2 + \frac{1}{4}m\dot{q}_1^{\,2} + \frac{1}{4}m\frac{\mu}{M}\dot{q}_2^{\,2}$$

where $\mu = M + 2m$ is the mass of the H_2S molecule

The potential energy is …

$$V = \frac{1}{2}k\left(1 + \frac{m}{M}\right)^2 Q^2 + \frac{1}{8}kq_1^{\,2} + \frac{1}{8}k\frac{\mu^2}{M^2}q_2^{\,2} - \frac{1}{4}k\frac{\mu}{M}q_1 q_2$$

Note, that in the Lagrangian $L = T - V$, the only cross term is one involving $q_1 q_2$. therefore, Q is a normal mode with eigenfrequency given by the ratio …

$$\omega_Q^{\,2} = K_Q / M_Q = \frac{k}{m}\left(1 + \frac{m}{M}\right)$$

Constructing the residual 2x2 ***K*** and ***M*** matrices involving only q_1 and q_2 terms, which we will call ***Kq*** and ***Mq*** gives …

$$\boldsymbol{Kq} = \frac{1}{4}\begin{pmatrix} 1 & -\mu/M \\ -\mu/M & \mu^2/M^2 \end{pmatrix} \quad and \quad \boldsymbol{Mq} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & \mu/M \end{pmatrix}$$

We have omitted the factors k and m, which we'll replace in the final solution, remembering that the eigenfrequencies that we find as a solution to $\left|\boldsymbol{Kq} - \omega^2\boldsymbol{Mq}\right| = 0$ will be multiples of k / m.

$$\left|\boldsymbol{Kq} - \omega^2\boldsymbol{Mq}\right| = \frac{1}{4}\begin{vmatrix} 1 - 2\omega^2 & -\mu/M \\ -\mu/M & \mu/M\left(\mu/M - 2\omega^2\right) \end{vmatrix} = 0$$

which reduces to …

$2\omega^2\left[2\omega^2 - \left(1 + \mu/M\right)\right] = 0$ which has the non-trivial solution …

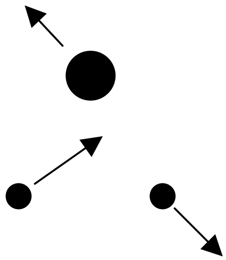
$\omega_{1,2}^{\;\;2} = \frac{1}{2}\left(1 + \mu/M\right)\frac{k}{m}$ in which, we have put back the factor k / m. These two modes are "degenerate."

Plugging ω^2 back into the matrix equation $\left(\boldsymbol{Kq} - \omega^2 \boldsymbol{Mq}\right)\boldsymbol{q} = 0$ gives …

$q_1 = -q_2 \quad or \quad -x_1 + x_3 = y_1 + y_3$

which can be satisfied in a variety of ways, for example, with $x_1 = x_3 \quad and \quad y_1 = -y_3$, etc

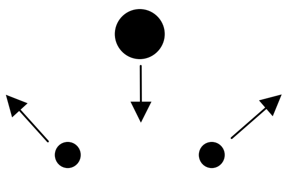
Pictorially, the three normal modes are …



Q-mode: ant-symmetric about y-axis:

$x_1 = x_3 \quad and \quad y_1 = -y_3$

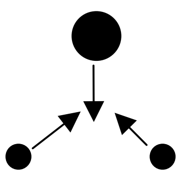
$$\omega_Q^{\ 2} = \frac{k}{m}\left(1 + \frac{m}{M}\right)$$



Breathing mode: symmetric about y-axis:

$x_1 = -x_3 \quad and \quad y_1 = y_3$

$$\omega_1^{\ 2} = \frac{k}{2m}\left(1 + \mu/M\right)$$



Stretching mode: symmetric about y-axis:

$x_1 = -x_3 \quad and \quad y_1 = y_3$

$$\omega_2^{\ 2} = \frac{k}{2m}\left(1 + \mu/M\right)$$

11.25

(a) Plug each of these functions into the wave equation and it is satisfied!

(b)
$$Q = q_1 + q_2 = e^{i\omega t}\, e^{-ikx} + e^{i(\omega+\Delta\omega)t}\, e^{-i(k+\Delta k)x}$$
$$= e^{i(\omega+\Delta\omega/2)t}\, e^{-i(k+\Delta k/2)x}\left[e^{-i[(\Delta\omega)t-(\Delta k)x]/2} + e^{+i[(\Delta\omega)t-(\Delta k)x]/2}\right]$$

The real part of the above is …

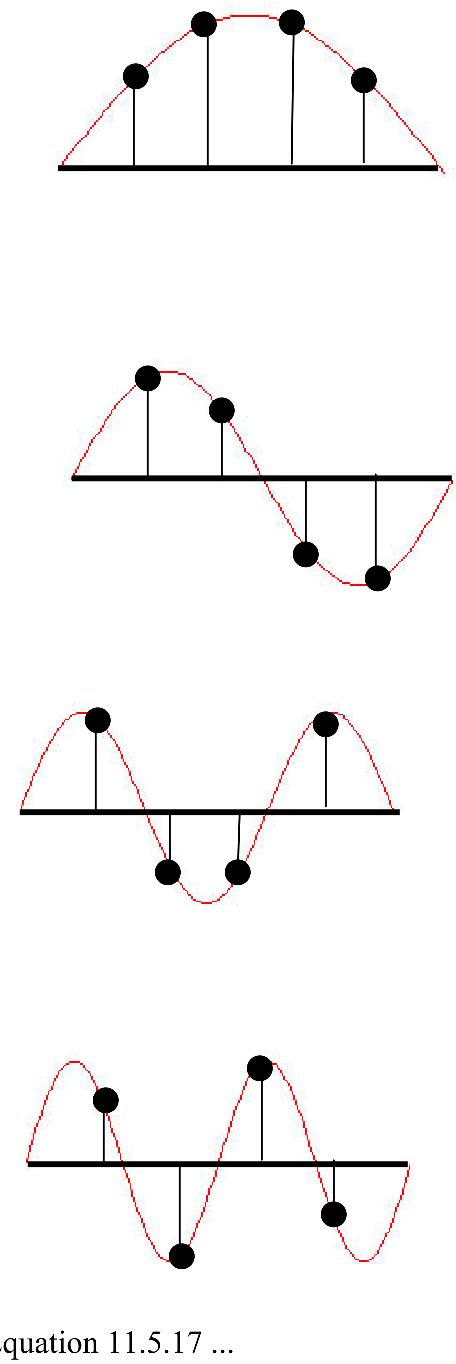
$$Q = 2\cos\left[\frac{(\Delta\omega)t - (\Delta k)x}{2}\right]\cos\left[\left(\omega + \frac{\Delta\omega}{2}\right)t - \left(k + \frac{\Delta k}{2}\right)x\right]$$
$$\approx 2\cos\left[\frac{(\Delta\omega)t - (\Delta k)x}{2}\right]\cos\left(\omega t - kx\right)$$

(c) The group speed is …

$$u_g = \frac{dx}{dt}$$ (the phase of the amplitude remains the same)

$$u_g = \frac{\Delta\omega}{\Delta k}$$

11.26



From Equation 11.5.17 ...

$$\frac{\omega_N}{\omega_1} = \frac{\sin\frac{N\pi}{10}}{\sin\frac{\pi}{10}}, \quad \frac{\omega_2}{\omega_1} = \frac{\sin\frac{2\pi}{10}}{\sin\frac{\pi}{10}} = 1.28, \quad \frac{\omega_3}{\omega_1} = \frac{\sin\frac{3\pi}{10}}{\sin\frac{\pi}{10}} = 2.62, \quad \frac{\omega_4}{\omega_1} = \frac{\sin\frac{4\pi}{10}}{\sin\frac{\pi}{10}} = 3.08$$

11.27

$F = k\,\Delta l$ is the tension in the cord

$d = \dfrac{l+\Delta l}{n+1}$ is its stretched length

From the equation following Equation 11.5.7…

$$K = \frac{F}{d}$$

From Equation 11.6.4c …

$$v^2 = \frac{k}{m}\left(l+\Delta l\right)\Delta l$$

$$v_{trans} = \left(\frac{k}{m}\right)^{\frac{1}{2}}\left[\left(l+\Delta l\right)\Delta l\right]^{\frac{1}{2}} \quad \text{for transverse waves}$$

For longitudinal vibrations, we use Y, the tension in the cord per unit stretched length

$$Y = \frac{k\,\Delta l}{\Delta l/\left(l+\Delta l\right)} = k\left(l+\Delta l\right)$$

$$K = \frac{Y}{d} \qquad \text{(Equation 11.6.8)}$$

$$v^2 = \frac{kd^2}{m/n} = \frac{k\left(l+\Delta l\right)nd}{m} \qquad \text{(Equation 11.6.9a)}$$

$$nd \approx \left(n+1\right)d = l+\Delta l$$

$$v_{long} = \left(\frac{k}{m}\right)^{\frac{1}{2}}\left(l+\Delta l\right)$$

11.28

From Equation 11.6.7b …

$$v = \left(\frac{F}{\mu}\right)^{\frac{1}{2}}$$

As in problem 11.27, $F = k\,\Delta l$

$$v_{trans} = \left(\frac{k\,\Delta l}{\mu}\right)^{\frac{1}{2}}$$

From Equation 11.6.9b …

$v = \left(\dfrac{Y}{\mu}\right)^{\frac{1}{2}}$ and from problem 11.27 … $Y = k\left(l+\Delta l\right)$ so … $v_{long} = \left(\dfrac{k\left(l+\Delta l\right)}{\mu}\right)^{\frac{1}{2}}$

11.29

The general solution to the wave equation (Equation 11.6.10) that yields a standing wave of any arbitrary shape can be obtained as a linear combination of standing sine waves of a form given by Equation 11.6.14, i.e.,

$$y(x,t) = \sum_{n=1}^{\infty} \left(A_n \sin \omega_n t + B_n \cos \omega_n t \right) \sin\left(\frac{2\pi x}{\lambda_n} \right)$$

where $\omega_n = \frac{2\pi}{T_n}$

since the speed of a wave is …

$$v = \frac{\lambda_n}{T_n} = \frac{\omega_n \lambda_n}{2\pi} = \left(\frac{F_0}{\mu} \right)^{\frac{1}{2}}$$

we have …

$$\omega_n = \left(\frac{F_0}{\mu} \right)^{\frac{1}{2}} \frac{2\pi}{\lambda_n}$$

But the wavelength of the standing wave is constrained by the fixed endpoints of the string, i.e. …

$\lambda_n = \frac{2l}{n}$, so …

$$\omega_n = \left(\frac{F_0}{\mu} \right)^{\frac{1}{2}} \frac{n\pi}{l}$$

Now, at t = 0, the wave starts from rest in the configuration specified, so…

$$y(x,0) = \sum B_n \sin \frac{n\pi x}{l}$$

From the discussion of Fourier analysis in Appendix G or in Section 3.9, the Fourier coefficients are given by …

$$B_n = \frac{2}{l} \int_0^l y(x,0) \sin \frac{n\pi x}{l} dx$$

Since the string starts from rest, we have …

$$\left. \frac{\partial y(x,t)}{\partial t} \right|_{t=0} = \sum \omega_n A_n \sin \frac{n\pi x}{l} \equiv 0$$

Therefore, $A_n = 0$

The initial configuration is shown in the Figure P11.29. Thus …

$$y(x,0) = \frac{2a}{l} x \qquad 0 < x < \frac{l}{2}$$

$$y(x,0)=\frac{2a}{l}\left(l-x\right) \qquad \frac{l}{2}<x<l$$

We can now determine B_n …

$$B_n=\frac{2}{l}\left[\frac{2a}{l}\int_0^{l/2} x\sin\frac{n\pi x}{l}dx+\frac{2a}{l}\int_{l/2}^{l}\left(l-x\right)\sin\frac{n\pi x}{l}dx\right]$$

Using integration tables, we obtain the result …

$$B_n=\left[\left(-1\right)^{\frac{n-1}{2}}\right]\frac{8a}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad n=1,3,5,...$$

$$B_n=0 \qquad n=2,4,6,...$$

Thus, the general solution is …

$$y(x,t)=\frac{8a}{\pi^2}\left(\cos\frac{\pi vt}{l}\sin\frac{\pi x}{l}-\frac{1}{9}\cos\frac{3\pi vt}{l}\sin\frac{3\pi x}{l}+\frac{1}{25}\cos\frac{5\pi vt}{l}\sin\frac{5\pi x}{l}-...\right)$$

None of the harmonics which have a node at the midpoint have been stimulated … only the odd harmonics have been excited.

11.30

By analogy with the generation of the traveling sine wave of Equation 11.6.14 from Equation 11.6.13, we get …

$$y(x,t)=\frac{4a}{\pi^2}\left[\left(\sin\frac{\pi\left(x+vt\right)}{l}+\sin\frac{\pi\left(x-vt\right)}{l}\right)-\frac{1}{9}\left(\sin\frac{3\pi\left(x+vt\right)}{l}+\sin\frac{3\pi\left(x-vt\right)}{l}\right)\right.$$
$$\left.+\frac{1}{25}\left(\sin\frac{5\pi\left(x+vt\right)}{l}+\sin\frac{5\pi\left(x-vt\right)}{l}\right)...\right]$$
