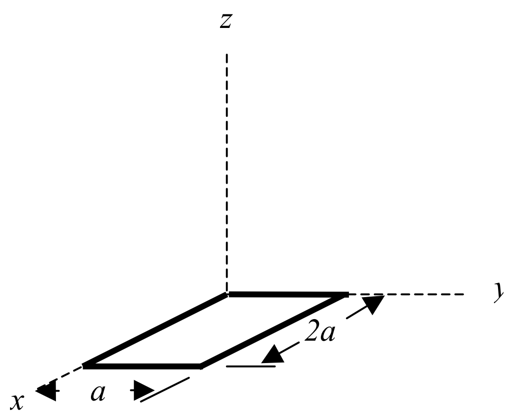
CHAPTER 9
MOTION OF RIGID BODIES
IN THREE DIMENSIONS

9.1 (a) $I_{xx} = \int \left(y^2 + z^2 \right) dm$



$dm = \rho dxdy$ and $m = 2a^2 \rho$

$$I_{xx} = \int_{y=0}^{y=a} \int_{x=0}^{x=2a} \left(y^2 + 0 \right) \rho dxdy$$

$$= \int_0^a 2a\rho y^2 dy$$

$$I_{xx} = \frac{2}{3} \rho a^4 = \frac{ma^2}{3}$$

$$I_{yy} = \int \left(x^2 + z^2 \right) dm = \int_{y=0}^{y=a} \int_{x=0}^{x=2a} x^2 \rho dxdy$$

$$= \int_0^a \frac{8a^3 \rho}{3} dy$$

$$I_{yy} = \frac{8a^4 \rho}{3} = \frac{4ma^2}{3}$$

From the perpendicular axis theorem:

$$I_{zz} = I_{xx} + I_{yy} + \frac{5ma^2}{3}$$

$$I_{xy} = I_{yx} = -\int xydm = -\int_{y=0}^{y=a} \int_{x=0}^{x=2a} xy\rho dxdy$$

$$= -\int_0^a \frac{4a^2 \rho}{2} ydy$$

$$I_{xy} = I_{yx} = -\rho a^4 = -\frac{ma^2}{2}$$

$$I_{xz} = I_{zx} = -\int xzdm = 0 = I_{yz} = I_{zy}$$

(b) $\cos\alpha = \frac{2}{\sqrt{5}}, \quad \cos\beta = \frac{1}{\sqrt{5}}, \quad \cos\gamma = 0$

From equation 9.1.10 …

$$I = \frac{ma^2}{3}\left(\frac{4}{5}\right) + \frac{4ma^2}{3}\left(\frac{1}{5}\right) + 2\left(-\frac{ma^2}{2}\right)\left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right)$$

$$= \frac{2}{15} ma^2$$

(c) $\vec{\omega} = \omega\left(\hat{i}\cos\alpha + \hat{j}\cos\beta \right) = \frac{\omega}{\sqrt{5}}\left(2\hat{i} + \hat{j} \right)$

From equation 9.1.29 …

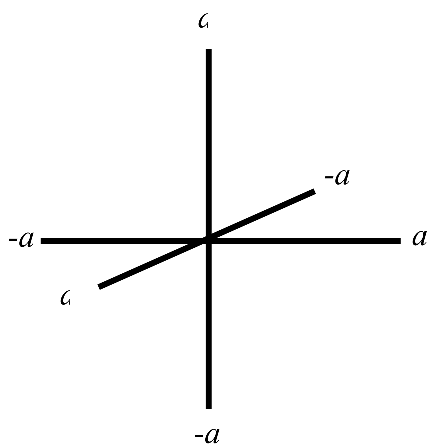
$$\vec{L} = \hat{i}\left[\frac{2\omega}{\sqrt{5}}\cdot\frac{ma^2}{3} + \frac{\omega}{\sqrt{5}}\left(-\frac{ma^2}{2}\right)\right] + \hat{j}\left[\frac{2\omega}{\sqrt{5}}\left(-\frac{ma^2}{2}\right) + \frac{\omega}{\sqrt{5}}\cdot\frac{4ma^2}{3}\right]$$

$$\vec{L} = \frac{\omega ma^2}{6\sqrt{5}}\left(\hat{i} + 2\hat{j}\right)$$

(d) From equation 9.1.32:

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{\omega}{2\sqrt{5}}\cdot\frac{\omega ma^2}{6\sqrt{5}}\left(2+2\right) = \frac{1}{15}ma^2\omega^2$$

9.2



(a) $\vec{\omega} = \frac{\omega}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$

$$I_{xx} = 2I_{rod} = 2\frac{m(2a)^2}{12} = \frac{2}{3}ma^2 = I_{yy} = I_{zz}$$

$I_{xy} = -\int xydm = 0$ since, for each rod, either x or y or both are 0. The same is true for the other products of inertia.

From equation 9.1.29:

$$\vec{L} = \frac{2}{3}ma^2\cdot\frac{\omega}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$$

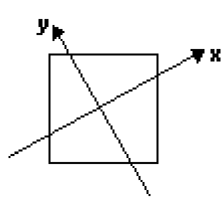
From equation 9.1.32,

$$T = \frac{1}{2}\frac{\omega}{\sqrt{3}}\frac{2ma^2\omega}{3\sqrt{3}}\left(1^2 + 1^2 + 1^2\right) = \frac{ma^2\omega^2}{3}$$

(b) From equation 9.1.10, with the moments of inertia equal to $\frac{2ma^2}{3}$ and the products of inertia equal to 0:

$$I = \frac{2ma^2}{3}\left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right) = \frac{2}{3}ma^2$$

(c)



For the x-axis being any axis through the center of the lamina and in the plane of the lamina, and the y-axis also in the plane of the lamina …

$I_{xx} = I_{yy}$ due to the similar geometry of the mass distributions with respect to the x- and y-axes.

From the perpendicular axis theorem:

$$I_{zz} = I_{xx} + I_{yy}$$

$$I_{zz} = 2I_{xx}$$

From Table 8.3.1, $I_{zz} = \frac{ma^2}{6}$

$$I_{xx} = \frac{ma^2}{12}$$

9.3 (a) From equation 9.2.13, $\tan 2\theta = \dfrac{2I_{xy}}{I_{xx} - I_{yy}}$

From Prob. 9.1, $I_{xx} = \dfrac{ma^2}{3}$, $I_{yy} = \dfrac{4ma^2}{3}$, $I_{xy} = -\dfrac{ma^2}{2}$

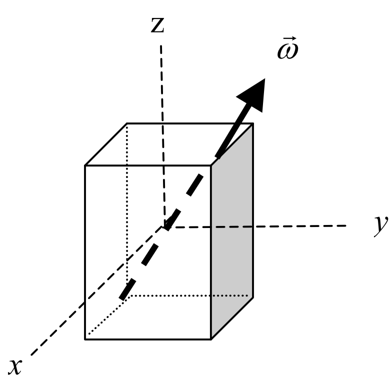
$$\tan 2\theta = 1$$

$$2\theta = 45^\circ \qquad \theta = 22.5^\circ$$

The 1-axis makes an angle of 22.5° with the x-axis.

(b) From symmetry, the principal axes are parallel to the sides of the lamina and perpendicular to the lamina, respectively.

9.4 (a) From symmetry, the coordinate axes are principal axes.



From Table 8.3.1:

$$I_1 = \frac{m}{12}\left[(2a)^2 + (3a)^2\right] = \frac{13}{12}ma^2$$

$$I_2 = \frac{m}{12}\left[a^2 + (3a)^2\right] = \frac{10}{12}ma^2$$

$$I_3 = \frac{m}{12}\left[a^2 + (2a)^2\right] = \frac{5}{12}ma^2$$

$$\vec{\omega} = \frac{\omega}{\sqrt{14}}(\hat{e}_1 + 2\hat{e}_2 + 3\hat{e}_3)$$

From equation 9.2.5,

$$T = \frac{1}{2}\left[\frac{13}{12}ma^2 \cdot \frac{\omega^2}{14} + \frac{10}{12}ma^2 \cdot \frac{4\omega^2}{14} + \frac{5}{12}ma^2 \cdot \frac{9\omega^2}{14}\right]$$

$$= \frac{7}{24}ma^2\omega^2$$

(b) From equation 9.2.4,

$$\vec{L} = \hat{e}_1\left(\frac{13}{12}ma^2 \cdot \frac{\omega}{\sqrt{14}}\right) + \hat{e}_2\left(\frac{10}{12}ma^2 \cdot \frac{2\omega}{\sqrt{14}}\right) + \hat{e}_3\left(\frac{5}{12}ma^2 \cdot \frac{3\omega}{\sqrt{14}}\right)$$

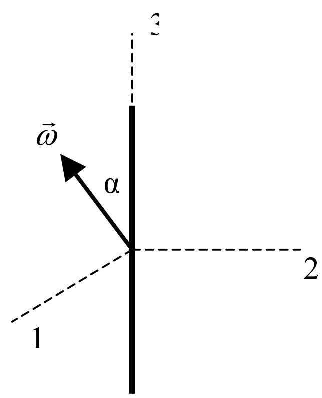
$$\vec{L} = \frac{ma^2\omega}{12\sqrt{14}}(13\hat{e}_1 + 20\hat{e}_2 + 15\hat{e}_3)$$

$$\cos\theta = \frac{\vec{\omega}\cdot\vec{L}}{|\omega||L|} = \frac{\left[(1)(13) + (2)(20) + (3)(15)\right]}{(1^2 + 2^2 + 3^2)^{\frac{1}{2}} \cdot (13^2 + 20^2 + 15^2)^{\frac{1}{2}}}$$

$$= \frac{98}{(11{,}116)^{\frac{1}{2}}} = 0.9295$$

$$\theta = 21.6^\circ$$

9.5 (a) Select coordinate axes such that the axis of the rod is the 3-axis, its center is at the



origin, and $\vec{\omega}$ lies in the 1, 3 plane.

From Table 8.3.1, $I_1 = I_2 = \frac{ml^2}{12}$, $I_3 = 0$

$\vec{\omega} = \omega\left(\hat{e}_1 \sin\alpha + \hat{e}_3 \cos\alpha\right)$

From equation 9.2.4,

$$\vec{L} = \frac{ml^2}{12}\omega\left(\hat{e}_1 \sin\alpha + 0 + 0\right) = \hat{e}_1 \frac{ml^2\omega}{12}\sin\alpha$$

$\vec{L}$ is perpendicular to the rod, and $\left|\vec{L}\right| = \frac{ml^2\omega}{12}\sin\alpha$

(b) Since $\vec{\omega}$ is constant, from equations. 9.3.5 …

$$N_1 = 0 + 0$$

$$N_2 = 0 + \left(\omega\cos\alpha\right)\left(\omega\sin\alpha\right)\left(\frac{ml^2}{12}\right)$$

$$N_3 = 0 + 0$$

$$\vec{N} = \hat{e}_2 \frac{ml^2\omega^2}{12}\sin\alpha\cos\alpha$$

$\vec{N}$ is perpendicular to the rod ($\hat{e}_3$ direction) and to $\vec{L}$ ($\hat{e}_1$ direction), and

$$\left|\vec{N}\right| = \frac{ml^2\omega^2}{12}\sin\alpha\cos\alpha$$

9.6 From Problem 9.4 … $\vec{\omega} = \frac{\omega}{\sqrt{14}}\left(\hat{e}_1 + 2\hat{e}_2 + 3\hat{e}_3\right)$

$I_1 = \frac{13}{12}ma^2$, $I_2 = \frac{10}{12}ma^2$, and $I_3 = \frac{5}{12}ma^2$

From eqns. 9.3.5:

$$N_1 = 0 + \frac{\omega^2}{14}(2)(3)\frac{ma^2}{12}(5-10) = -\frac{ma^2\omega^2}{28}(5)$$

$$N_2 = 0 + \frac{\omega^2}{14}(3)(1)\frac{ma^2}{12}(13-5) = \frac{ma^2\omega^2}{28}(4)$$

$$N_3 = 0 + \frac{\omega^2}{14}(1)(2)\frac{ma^2}{12}(10-13) = -\frac{ma^2\omega^2}{28}$$

$$\left|\vec{N}\right| = \frac{ma^2\omega^2}{28}\left(5^2 + 4^2 + 1^2\right)^{\frac{1}{2}} = \frac{ma^2\omega^2}{28}\sqrt{42}$$

9.7 Multiplying equations 9.3.5 by ω_1, ω_2, and ω_3,respectively …

$$0 = I_1\dot{\omega}_1\omega_1 + \omega_1\omega_2\omega_3\left(I_3 - I_2\right)$$

$$0 = I_2\dot{\omega}_2\omega_2 + \omega_1\omega_2\omega_3\left(I_1 - I_3\right)$$

$0 = I_3\dot{\omega}_3\omega_3 + \omega_1\omega_2\omega_3\left(I_2 - I_1\right)$

Adding, $0 = I_1\dot{\omega}_1\omega_1 + I_2\dot{\omega}_2\omega_2 + I_3\dot{\omega}_3\omega_3$

$0 = \frac{d}{dt}\left[\frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)\right]$

From equation 9.2.5, $0 = \frac{d}{dt}\left[T_{rot}\right]$

$T_{rot} = constant$

Multiplying equations 9.3.5 by $I_1\omega_1$, $I_2\omega_2$, and $I_3\omega_3$, respectively:

$0 = I_1^2\dot{\omega}_1\omega_1 + I_1\omega_1\omega_2\omega_3\left(I_3 - I_2\right)$

$0 = I_2^2\dot{\omega}_2\omega_2 + I_2\omega_1\omega_2\omega_3\left(I_1 - I_3\right)$

$0 = I_3^2\dot{\omega}_3\omega_3 + I_3\omega_1\omega_2\omega_3\left(I_2 - I_1\right)$

Adding, $0 = I_1^2\dot{\omega}_1\omega_1 + I_2^2\dot{\omega}_2\omega_2 + I_3^2\dot{\omega}_3\omega_3$

$0 = \frac{1}{2}\frac{d}{dt}\left(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2\right)$

From equation 9.2.4, $0 = \frac{d}{dt}\left(L^2\right)$

$L^2 = constant$

9.8 From equations 9.3.5 for zero torque …

$0 = I_1\dot{\omega}_1 + \omega_2\omega_3\left(I_3 - I_2\right)$

$0 = I_2\dot{\omega}_2 + \omega_1\omega_3\left(I_1 - I_3\right)$

From the perpendicular axis theorem, $I_3 = I_1 + I_2$

$0 = I_1\dot{\omega}_1 + \omega_2\omega_3 I_1$

$0 = I_2\dot{\omega}_2 - \omega_1\omega_3 I_2$

Multiplying by $\frac{\omega_1}{I_1}$ and $\frac{\omega_2}{I_2}$, respectively:

$0 = \omega_1\dot{\omega}_1 + \omega_1\omega_2\omega_3$

$0 = \omega_2\dot{\omega}_2 - \omega_1\omega_2\omega_3$

Adding, $0 = \omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2 = \frac{1}{2}\frac{d}{dt}\left(\omega_1^2 + \omega_2^2\right)$

$\omega_1^2 + \omega_2^2 = constant$

If $I_1 = I_2$, from the third of Euler's equations … $\left(I_3\dot{\omega}_3 = 0\right)$

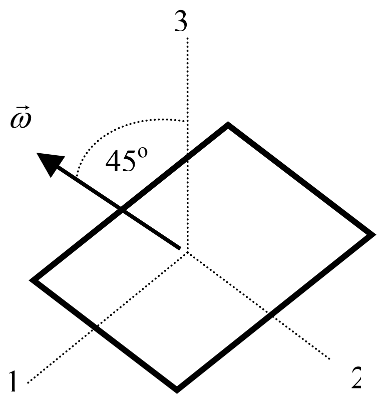
$\omega_3 = constant$

9.9 (a) From symmetry … $I_s = I_3$ and $I_1 = I_2 = I$

From the perpendicular axis theorem, $I_3 = I_1 + I_2$ or $I_s = 2I$

From eqn 9.5.8, $\Omega = \left(2 - 1\right)\omega\cos\alpha$

For $\alpha = 45^\circ$, $\Omega = \frac{\omega}{\sqrt{2}}$



For $\frac{2\pi}{\omega} = 1\text{s}$, $T_1 = \frac{2\pi}{\Omega} = \sqrt{2}\,s = 1.414\text{s}$

T_1 is the period of precession of $\vec{\omega}$ about $\hat{e}_3$.

From equation 9.6.12,

$$\dot{\phi} = \omega\left[1 + \left(\frac{I_s^2}{I^2} - 1\right)\cos^2\alpha\right]^{\frac{1}{2}} = \omega\left[1 + \left(\frac{2^2}{1^2} - 1\right)\left(\frac{1}{\sqrt{2}}\right)^2\right]^{\frac{1}{2}}$$

$$\dot{\phi} = \omega\sqrt{\frac{5}{2}}$$

$$T_2 = \frac{2\pi}{\dot{\phi}} = \sqrt{0.4}\,s = 0.632\text{s}$$

T_2 is the period of wobble of $\hat{e}_3$ about $\vec{L}$.

(b) $$I_1 = I_2 = I = \frac{m}{12}\left(a^2 + \frac{a^2}{16}\right) = \frac{17}{16}\cdot\frac{ma^2}{12}$$

$$I_3 = I_s = \frac{m}{12}\left(a^2 + a^2\right) = 2\cdot\frac{ma^2}{12}$$

From equation 9.5.8, $\Omega = \left(\frac{2}{\frac{17}{16}} - 1\right)\omega\frac{1}{\sqrt{2}} = \frac{15}{17}\frac{\omega}{\sqrt{2}}$

$$T_1 = \frac{2\pi}{\Omega} = \frac{17}{15}\sqrt{2}\,s = 1.603\text{s}$$

From equation 9.6.12, $\dot{\phi} = \omega\left[1 + \left(\frac{2^2\cdot 16^2}{17^2} - 1\right)\left(\frac{1}{\sqrt{2}}\right)^2\right]^{\frac{1}{2}}$

$$\dot{\phi} = 1.5072\,\omega$$

$$T_2 = \frac{2\pi}{\dot{\phi}} = \frac{1}{1.5072}s = 0.663\text{s}$$

9.10 From equation 9.6.10, $\tan\theta = \frac{I}{I_s}\tan\alpha$.

Since $I_s > I$, $\theta < \alpha$ and the angle between the axis of rotation $(\vec{\omega})$ and the invariable line $(\vec{L})$ is $\alpha - \theta$.

From your favorite table of trigonometric identities …

$$\tan(\alpha - \theta) = \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta}$$

$$\tan(\alpha-\theta)=\frac{\left(1-\frac{I}{I_s}\right)\tan\alpha}{1+\frac{I}{I_s}\tan^2\alpha}$$

$$\alpha-\theta=\tan^{-1}\left[\frac{(I_s-I)\tan\alpha}{I_s+I\tan^2\alpha}\right]$$

9.11 $\alpha-\theta$ is a maximum for $\frac{I_s}{I}$ a maximum.

For $\frac{I_s}{I}=2$, $\tan(\alpha-\theta)=\frac{\tan\alpha}{2+\tan^2\alpha}$

$$\frac{d\tan(\alpha-\theta)}{d\tan\alpha}=\frac{1}{2+\tan^2\alpha}-\frac{2\tan^2\alpha}{\left(2+\tan^2\alpha\right)^2}=\frac{2-\tan^2\alpha}{\left(2+\tan^2\alpha\right)^2}$$

At the maximum, $\frac{d\tan(\alpha-\theta)}{d\tan\alpha}=0=2-\tan^2\alpha$

$$\alpha=\tan^{-1}\sqrt{2}=54.7^\circ$$

$$\tan(\alpha-\theta)\le\frac{\left(1-\frac{1}{2}\right)\sqrt{2}}{1+\frac{1}{2}\left(\sqrt{2}\right)^2}=\frac{\sqrt{2}}{4}$$

$$\alpha-\theta\le\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)=19.5^\circ$$

9.12 (a) From Problem 9.10, for $I_s>I$, the angle between $\vec{\omega}$ and $\vec{L}$ is …

$$\alpha-\theta=\tan^{-1}\left[\frac{(I_s-I)\tan\alpha}{I_s+I\tan^2\alpha}\right]$$

From Problem 9.9(a), $I_s=2I$ and $\tan\alpha=\tan 45^\circ=1$

$$\alpha-\theta=\tan^{-1}\frac{1}{(2+1)}=\tan^{-1}\frac{1}{3}=18.4^\circ$$

(b) From Prob. 9.9(b), $I_s=2\cdot\frac{ma^2}{12}$ and $I=\frac{17}{16}\cdot\frac{ma^2}{12}$

$$\alpha-\theta=\tan^{-1}\left[\frac{\left(2-\frac{17}{16}\right)}{2+\frac{17}{16}}\right]=\tan^{-1}\frac{15}{49}=17.0^\circ$$

9.13 From Example 9.6.2, $\alpha = 0.2''$ and $\frac{I_s}{I} = 1.00327$

It follows from equation 9.6.10 that, for $I_s > I$, the angle between $\vec{\omega}$ and $\vec{L}$ is:

$$\alpha - \theta = \alpha - \tan^{-1}\left(\frac{I}{I_s}\tan\alpha\right)$$

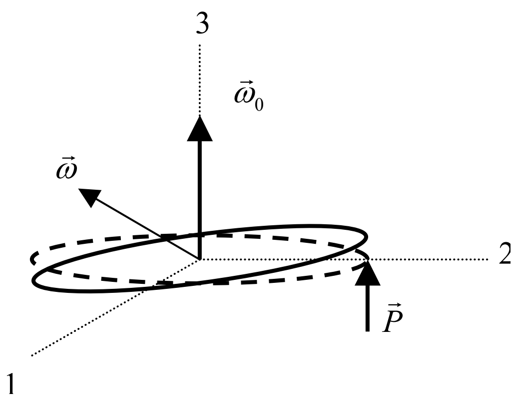
For α so small,

$$\tan\alpha \approx \alpha \text{, and } \tan^{-1}\left(\frac{I}{I_s}\tan\alpha\right) \approx \frac{I}{I_s}\tan\alpha \approx \frac{I\alpha}{I_s}$$

$$\alpha - \theta \approx \alpha - \frac{I\alpha}{I_s} = \frac{I_s - I}{I_s}\alpha$$

$$\alpha - \theta \approx \frac{.00327}{1.00327}0.2 = 0.00065 \text{ arcsec}$$

9.14 From Table 8.3.1, $I_s = \frac{ma^2}{2}$ and $I = \frac{ma^2}{4}$



$\vec{\omega}_\circ = \omega\hat{e}_3$ and $\vec{\hat{P}} = \frac{ma\omega}{4}\hat{e}_3$

Selecting the 2-axis through the point of impact, during the collision …

$$\int \vec{N}dt = \vec{r} \times \vec{p} = a\hat{e}_2 \times \frac{ma\omega}{4}\hat{e}_3 = \frac{ma^2\omega}{4}\hat{e}_1$$

The only non-zero component of $\vec{N}$ is N_1. During the collision $\omega_2 = 0$, so integrating the first of Euler's equations 9.3.5 …

$$\int N_1 dt = I_1\omega_1$$

Immediately after the collision,

$$\omega_1 = \frac{ma^2\omega}{4}\left(\frac{4}{ma^2}\right) = \omega$$

(ω_3 still is equal to ω, and $\omega_2 = 0$)

After the collision, $\vec{N} = 0$. From Prob. 9.8, with $I_1 = I_2$, for $\vec{N} = 0$,

$\omega_3 = constant$ and $\omega_1^2 + \omega_2^2 = constant$.

$$\tan\alpha = \frac{\left(\omega_1^2 + \omega_2^2\right)^{\frac{1}{2}}}{\omega_3} = constant$$

Using the values of ω_i *immediately after the collision* …

$$\tan\alpha = \frac{\left(\omega^2 + 0\right)^{\frac{1}{2}}}{\omega} = 1$$

$\alpha = 45^\circ$

From equation 9.5.8, with $\vec{\omega} = \vec{\omega}_\circ + \omega_1\hat{e}_1 = \omega\left(\hat{e}_1 + \hat{e}_3\right)$:

$$\Omega = (2-1)\left(\sqrt{2}\omega\right)\left(\cos 45^\circ\right) = \omega$$

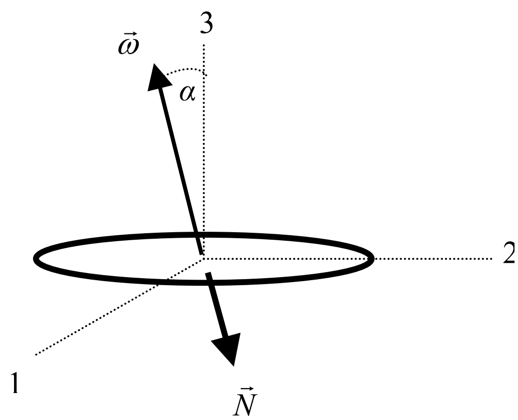
From equation 9.6.12 …

$$\dot{\phi} = \sqrt{2}\omega\left[1 + \left(2^2 - 1\right)\left(\cos^2 45^\circ\right)\right]^{\frac{1}{2}} = \omega\sqrt{5}$$

9.15 $\vec{N} = -c\vec{\omega} = -c\left(\omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3\right)$

Say the 3 axis is the symmetry axis, $I_3 = I_s$, $I_1 = I_2 = I$

For the third component of equations 9.3.5 …



$$-c\omega_3 = I_s\dot{\omega}_3 + 0$$

$$\frac{\dot{\omega}_3}{\omega_3} = -\frac{c}{I_s}$$

$$\ln\frac{\omega_3}{\left(\omega_3\right)_\circ} = -\frac{c}{I_s}t$$

$$\omega_3 = \left(\omega_3\right)_\circ e^{-\frac{ct}{I_s}}$$

For the first two components of equations 9.3.5 …

$-c\omega_1 = I\dot{\omega}_1 + \omega_2\omega_3\left(I_s - I\right)$, and

$-c\omega_2 = I\dot{\omega}_2 + \omega_1\omega_3\left(I - I_s\right)$

Rearranging terms and multiplying by ω_1 and ω_2 respectively …

$$\dot{\omega}_1\omega_1 + \frac{c}{I}\omega_1^2 + \omega_1\omega_2\omega_3\left(I_s - I\right) = 0$$

$$\dot{\omega}_2\omega_2 + \frac{c}{I}\omega_2^2 - \omega_1\omega_2\omega_3\left(I_s - I\right) = 0$$

Adding, $\dot{\omega}_1\omega_1 + \dot{\omega}_2\omega_2 + \frac{c}{I}\left(\omega_1^2 + \omega_2^2\right) = 0$

$$\frac{1}{\left(\omega_1^2 + \omega_2^2\right)}\frac{d}{dt}\left(\omega_1^2 + \omega_2^2\right) = -\frac{2c}{I}$$

$$\ln\frac{\omega_1^2 + \omega_2^2}{\left(\omega_1^2 + \omega_2^2\right)_\circ} = -\frac{2c}{I}t$$

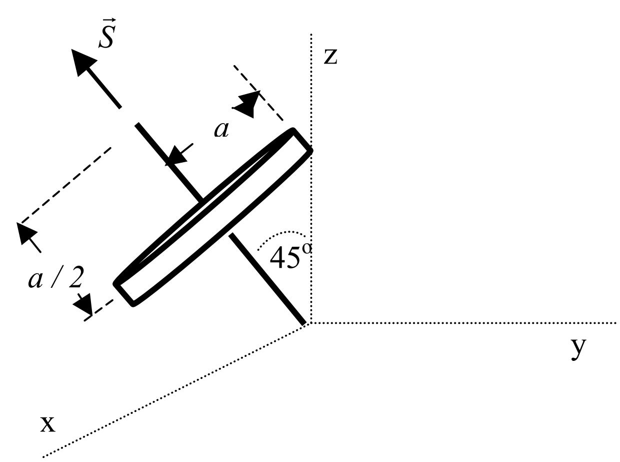
$$\left(\omega_1^2 + \omega_2^2\right) = \left(\omega_1^2 + \omega_2^2\right)_\circ e^{-\frac{2ct}{I}}$$

$$\left(\omega_1^2 + \omega_2^2\right)^{\frac{1}{2}} = \left(\omega_1^2 + \omega_2^2\right)_\circ^{\frac{1}{2}} e^{-\frac{ct}{I}}$$

$$\tan\alpha = \frac{\left(\omega_1^2 + \omega_2^2\right)^{\frac{1}{2}}}{\omega_3} = \left(\tan\alpha_\circ\right)e^{-ct\left(\frac{1}{I} - \frac{1}{I_s}\right)}$$

For $I_s > I$, $\left(\frac{1}{I} - \frac{1}{I_s}\right) > 0$ and α decreases with time.

9.16 (a) From table 8.3.1… $I_s = \frac{ma^2}{2}$ about the symmetry (spin) axis.



$I_{cm\ disk} = m\frac{a^2}{4}$ about an axis through the center of mass of the disk and ⊥ to the spin axis. From the parallel axis theorem, equation 8.3.21, …

$I_{disk} = \frac{ma^2}{4} + m\left(\frac{a}{2}\right)^2$ about the pivot point.

$I_{rod} = \left(\frac{m}{2}\right)\frac{a^2}{3}$ moment of inertia of the rod about the pivot point. Thus …

$$I = I_{disc} + I_{rod} = \left[\frac{ma^2}{4} + m\left(\frac{a}{2}\right)^2\right] + \left(\frac{m}{2}\right)\frac{a^2}{3} = \frac{2}{3}ma^2$$

From equation 9.7.10, $\dot{\phi} = \frac{I_s S + \left(I_s^2 S^2 - 4MglI\cos\theta\right)^{\frac{1}{2}}}{2I\cos\theta}$

$$\dot{\phi} = \frac{1}{2\cos\theta}\left\{\frac{I_s}{I}S \pm \left[\left(\frac{I_s}{I}\right)^2 S^2 - \frac{4Mgl\cos\theta}{I}\right]^{\frac{1}{2}}\right\}$$

$$\frac{I_s}{I} = \frac{\frac{ma^2}{2}}{\frac{2ma^2}{3}} = \frac{3}{4}, \quad \cos\theta = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$\frac{Ml}{I} = \frac{\left(m + \frac{m}{2}\right)\left(\frac{a}{2}\right)}{\frac{2ma^2}{3}} = \frac{9}{8a} = \frac{9}{80}cm^{-1}$$

$$\dot{\phi} = \frac{1}{\sqrt{2}}\left\{\frac{3}{4}(900\,rpm) \pm \left[\left(\frac{3}{4}\right)^2 (900)^2 - 4\left(\frac{9}{80}cm^{-1}\right)\left(980\frac{cm}{s^2}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{60\,rpm}{2\pi s^{-1}}\right)^2\right]^{\frac{1}{2}}\right\}$$

$\dot{\phi} = 15.1\,rpm$ or $939\,rpm$

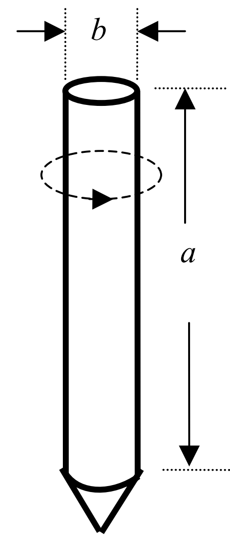
(b) From equation 9.7.12, $I_s^2 S^2 \geq 4MglI$

$$\frac{I}{I_s} = \frac{4}{3}, \ \frac{Ml}{I_s} = \frac{\left(\frac{3m}{2}\right)\left(\frac{a}{2}\right)}{\frac{ma^2}{2}} = \frac{3}{2a} = \frac{3}{20} cm^{-1}$$

$$S^2 \geq \frac{4Mgl}{I_s} \cdot \frac{I}{I_s} = 4\left(\frac{3}{20} cm^{-1}\right)\left(980\, cm \cdot s^{-2}\right)\left(\frac{4}{3}\right)\left(\frac{60\, rpm}{2\pi\, s^{-1}}\right)^2$$

$$S \geq 267\, rpm$$

9.17 (Neglecting the pencil head) From Table 8.3.1,



$$I_s = \frac{m\left(\frac{b}{2}\right)^2}{2} = \frac{mb^2}{8}$$

The moment of inertia of the pencil about an axis thru its center of mass and ⊥ to its symmetry axis from equation

$$8.3.26 \ldots \ I_c = m\left[\frac{\left(\frac{b}{2}\right)^2}{4} + \frac{a^2}{12}\right] = m\left(\frac{b^2}{16} + \frac{a^2}{12}\right)$$

From the parallel axis theorem, $I = I_c + m\left(\frac{a}{2}\right)^2 = m\left(\frac{b^2}{16} + \frac{a^2}{3}\right)$

From equation 9.7.12, $S^2 \geq \frac{4MglI}{I_s^2}$

$$S^2 \geq \frac{4mg\left(\frac{a}{2}\right)m\left(\frac{b^2}{16} + \frac{a^2}{3}\right)}{\frac{m^2 b^4}{64}}$$

$$S \geq \frac{16}{b^2}\left[\frac{ga}{2}\left(\frac{b^2}{16} + \frac{a^2}{3}\right)\right]^{\frac{1}{2}} = \frac{16}{(1)^2}\left[\frac{(980)(20)}{2}\left(\frac{1^2}{16} + \frac{20^2}{3}\right)\right]^{\frac{1}{2}} rad \cdot s^{-1}$$

$$S \geq 18{,}294\, rad \cdot s^{-1} = 2910\, rps$$

9.18 $I_s = I_{rim} + I_{spokes} + I_{hub}$

$$I_{rim} = m_{rim} a^2 = \frac{ma^2}{2}$$

$I_{spokes} = m_{spokes}\frac{a^3}{3} = \frac{ma^2}{12}$, assuming the spokes to be thin rods

$I_{hub} = 0$, assuming its radius is small

From the perpendicular axis theorem, equation 8.3.14, $I = \frac{I_s}{2}$

From equation 9.10.14, $S^2 > \frac{Imga}{I_s\left(I_s + ma^2\right)}$

$$S > \left[\frac{1}{2}\frac{mga}{\left(\frac{7ma^2}{12}\right) + ma^2}\right]^{\frac{1}{2}} = \left(\frac{6g}{19a}\right)^{\frac{1}{2}}$$

For rolling without slipping, $v = aS$

$$v > \left(\frac{6ga}{19}\right)^{\frac{1}{2}} = \left[\frac{6 \times 32 \times \left(\frac{30}{2}\right)}{19 \times 12}\right]^{\frac{1}{2}} ft \cdot s^{-1} = 3.55\, ft \cdot s^{-1}$$

If the spokes and hub are neglected, $I_s = \frac{ma^2}{2}$

$$S > \left[\frac{1}{2}\frac{mga}{\left(\frac{ma^2}{2}\right) + ma^2}\right]^{\frac{1}{2}} = \left(\frac{g}{3a}\right)^{\frac{1}{2}}$$

$$v > \left(\frac{ga}{3}\right)^{\frac{1}{2}} = \left[\frac{32 \times \left(\frac{30}{2}\right)}{3 \times 12}\right]^{\frac{1}{2}} ft \cdot s^{-1} = 3.65\, ft \cdot s^{-1}$$

9.19 From equations 9.3.5 with $\vec{N} = 0$ …

$$I_1\dot{\omega}_1 + \left(I_3 - I_2\right)\omega_2\omega_3 = 0$$
$$I_2\dot{\omega}_2 + \left(I_1 - I_3\right)\omega_1\omega_3 = 0$$
$$I_3\dot{\omega}_3 + \left(I_2 - I_1\right)\omega_1\omega_2 = 0$$

Differentiating the first equation with respect to t:

$$I_1\ddot{\omega}_1 + \left(I_3 - I_2\right)\left(\omega_2\dot{\omega}_3 + \dot{\omega}_2\omega_3\right) = 0$$

From the second and third equations:

$$\dot{\omega}_2 = \frac{\left(I_3 - I_1\right)}{I_2}\omega_1\omega_3, \text{ and } \dot{\omega}_3 = \frac{\left(I_1 - I_2\right)}{I_3}\omega_1\omega_2$$

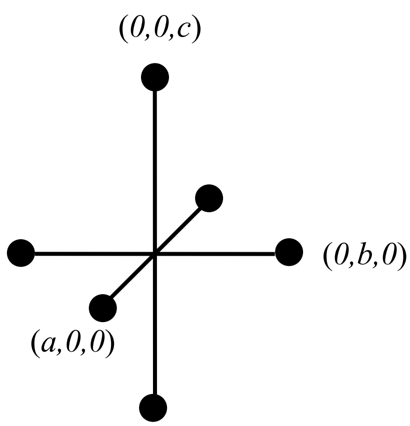
$$I_1\ddot{\omega}_1 + (I_3 - I_2)\left[\frac{(I_1 - I_2)}{I_3}\omega_1\omega_2^2 + \frac{(I_3 - I_1)}{I_2}\omega_1\omega_3^2\right] = 0$$

$$\ddot{\omega}_1 + K_1\omega_1 = 0\,,\ K_1 = -\omega_2^2\left[\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}\right] + \omega_3^2\left[\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\right]$$

(a) For ω_3 large and $\omega_2 = 0$, $K_1 > 0$ so $\ddot{\omega}_1 + K_1\omega_1 = 0$ is the harmonic oscillator equation. ω_1 oscillates, but remains small. Motion is stable for initial rotation about the 3 axis if the 3 axis is the principal axis having the largest or smallest moment of inertia.

(b) For $\omega_3 = 0$ and ω_2 large, $K_1 < 0$ so $\ddot{\omega}_1 + K_1\omega_1 = 0$ is the differential equation for exponential growth of ω_1 with time: $\omega_1 = Ae^{\sqrt{k_1}t} + Be^{-\sqrt{k_1}t}$. Motion is unstable for the initial rotation mostly about the principal axis having the median moment of inertia.

9.20 $I_{xy} = \sum_i m_i x_i y_i = 0$ since either x_i or y_i is zero for all six particles. Similarly, all the other products of inertia are zero. Therefore the coordinate axes are principle axes.



$$I_{xx} = \sum_i m_i\left(y_i^2 + z_i^2\right) = m\left[0 + 0 + b^2 + (-b)^2 + c^2 + (-c)^2\right]$$

$$I_{xx} = 2m\left(b^2 + c^2\right)$$
$$I_{yy} = 2m\left(a^2 + c^2\right)$$
$$I_{zz} = 2m\left(a^2 + b^2\right)$$

$$\vec{I} = 2m\begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}$$

$$\vec{\omega} = \frac{\omega}{\left(a^2 + b^2 + c^2\right)^{\frac{1}{2}}}\left(a\hat{e}_1 + b\hat{e}_2 + c\hat{e}_3\right) = \frac{\omega}{\left(a^2 + b^2 + c^2\right)^{\frac{1}{2}}}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

From equation 9.1.28, $\vec{L} = \vec{I}\vec{\omega}$

$$\vec{L} = \frac{2m\omega}{\left(a^2 + b^2 + c^2\right)^{\frac{1}{2}}}\begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\vec{L} = \frac{2m\omega}{\left(a^2+b^2+c^2\right)^{\frac{1}{2}}}\begin{bmatrix} a\left(b^2+c^2\right) \\ b\left(a^2+c^2\right) \\ c\left(a^2+b^2\right) \end{bmatrix}$$

From equation 9.1.32, $T = \frac{1}{2}\vec{\omega}\cdot\vec{L}$

$$T = \frac{1}{2}\frac{2m\omega^2}{\left(a^2+b^2+c^2\right)}\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} a\left(b^2+c^2\right) \\ b\left(a^2+c^2\right) \\ c\left(a^2+b^2\right) \end{bmatrix}$$

$$T = \frac{m\omega^2}{a^2+b^2+c^2}\left[a^2\left(b^2+c^2\right)+b^2\left(a^2+c^2\right)+c^2\left(a^2+b^2\right)\right]$$

$$T = \frac{2m\omega^2}{a^2+b^2+c^2}\left(a^2b^2+a^2c^2+b^2c^2\right)$$

9.21 For Problem 9.1:

$$\vec{I} = ma^2\begin{pmatrix} \frac{1}{3} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{4}{3} & 0 \\ 0 & 0 & \frac{5}{3} \end{pmatrix}, \qquad \vec{\omega} = \frac{\omega}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{L} = \vec{I}\,\vec{\omega} = \frac{ma^2\omega}{\sqrt{5}}\begin{pmatrix} \frac{1}{3} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{4}{3} & 0 \\ 0 & 0 & \frac{5}{3} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{ma^2\omega}{\sqrt{5}}\begin{pmatrix} \frac{2}{3}-\frac{1}{2} \\ -1+\frac{4}{3} \\ 0 \end{pmatrix} = \frac{ma^2\omega}{\sqrt{5}}\begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \\ 0 \end{pmatrix} = \frac{ma^2\omega}{6\sqrt{5}}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\frac{\omega}{\sqrt{5}}\frac{ma^2\omega}{6\sqrt{5}}\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \frac{ma^2\omega^2}{60}\left(2+2+0\right) = \frac{ma^2\omega^2}{15}$$

For Problem 9.4:

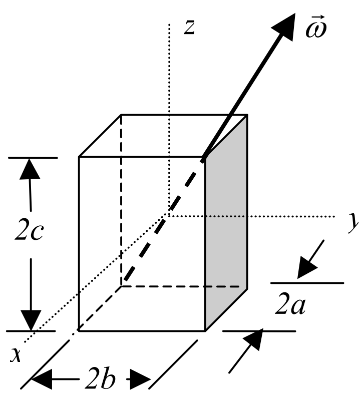
$$\vec{I} = \frac{ma^2}{12}\begin{pmatrix} 13 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \qquad \vec{\omega} = \frac{\omega}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\vec{L} = \vec{I}\vec{\omega} = \frac{ma^2\omega}{12\sqrt{14}}\begin{pmatrix} 13 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{ma^2\omega}{12\sqrt{14}}\begin{pmatrix} 13 \\ 20 \\ 15 \end{pmatrix}$$

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\frac{\omega}{\sqrt{14}}\frac{ma^2\omega}{12\sqrt{14}}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 13 \\ 20 \\ 15 \end{pmatrix}$$

$$= \frac{ma^2\omega^2}{24(14)}(13+40+45) = \frac{7}{24}ma^2\omega^2$$

9.22 Since the coordinate axes are axes of symmetry, they are principal axes and all products of inertia are zero.



From Table 8.3.1,

$$I_{xx} = \frac{m}{12}\left[(2b)^2 + (2c)^2\right] = \frac{m}{3}\left(b^2 + c^2\right)$$

$$I_{yy} = \frac{m}{3}\left(a^2 + c^2\right), \qquad I_{zz} = \frac{m}{3}\left(a^2 + b^2\right)$$

$$\vec{I} = \frac{m}{3}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

$$\vec{\omega} = \frac{\omega}{\left(a^2+b^2+c^2\right)^{\frac{1}{2}}}\left(a\hat{e}_1 + b\hat{e}_2 + c\hat{e}_3\right) = \frac{\omega}{\left(a^2+b^2+c^2\right)}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

From equation 9.1.28, $\vec{L} = \vec{I}\vec{\omega}$

$$\vec{L} = \frac{m\omega}{3\left(a^2+b^2+c^2\right)^{\frac{1}{2}}}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\vec{L} = \frac{m\omega}{3\left(a^2+b^2+c^2\right)^{\frac{1}{2}}}\begin{bmatrix} a\left(b^2+c^2\right) \\ b\left(a^2+c^2\right) \\ c\left(a^2+b^2\right) \end{bmatrix}$$

From equation 9.1.32, $T = \frac{1}{2}\vec{\omega}^T\cdot\vec{L}$

$$T=\frac{1}{2}\frac{m\omega^2}{3\left(a^2+b^2+c^2\right)}\begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}a\left(b^2+c^2\right)\\ b\left(a^2+c^2\right)\\ c\left(a^2+b^2\right)\end{bmatrix}$$

$$T=\frac{m\omega^2}{6\left(a^2+b^2+c^2\right)}\left[a^2\left(b^2+c^2\right)+b^2\left(a^2+c^2\right)+c^2\left(a^2+b^2\right)\right]$$

$$T=\frac{m\omega^2}{3\left(a^2+b^2+c^2\right)}\left(a^2b^2+a^2c^2+b^2c^2\right)$$

With the origin at one corner, from the parallel axis theorem:

$$I_{xx}=\frac{m\left(b^2+c^2\right)}{3}+m\left(b^2+c^2\right)=\frac{4m}{3}\left(b^2+c^2\right)$$

$$I_{yy}=\frac{4m}{3}\left(a^2+c^2\right),\qquad I_{zz}=\frac{4m}{3}\left(a^2+b^2\right)$$

$$I_{xy}=-\int xydm=-\int xy\rho dV$$

$$I_{xy}=-\rho\int_{x=0}^{x=2a}\int_{y=0}^{y=2b}\int_{z=0}^{z=2c}xydxdydz=-8\rho a^2b^2c$$

$m=\rho(2a)(2b)(2c)=8\rho abc$, so $I_{xy}=-mab$

$I_{xy}=-mac$, $\qquad I_{yz}=-mbc$

$$\vec{I}=m\begin{bmatrix}\frac{4}{3}\left(b^2+c^2\right) & -ab & -ac\\ -ab & \frac{4}{3}\left(a^2+c^2\right) & -bc\\ -ac & -bc & \frac{4}{3}\left(a^2+b^2\right)\end{bmatrix}$$

9.23 (See Figure 9.7.1)

$$L_z=\left(L_{x'}\right)_z+\left(L_{y'}\right)_z+\left(L_{z'}\right)_z$$

$$L_z=L_{y'}\sin\theta+L_{z'}\cos\theta=\left(I\dot{\phi}\sin\theta\right)\sin\theta+\left(I_sS\right)\cos\theta$$

9.24 (See Figure 9.7.1) If the top precesses without nutation, it must do so at $\theta=\theta_\circ$ where $V\left(\theta_\circ\right)$ is a minimum of $V(\theta)$ …

$$\left.\frac{dV(\theta)}{d\theta}\right|_{\theta=\theta_\circ}=0$$

$$V(\theta)=\frac{\left(L_z-L_{z'}\cos\theta\right)^2}{2I\sin^2\theta}+mgl\cos\theta \qquad \text{(See equation 9.8.7)}$$

$$\left.\frac{dV}{d\theta}\right|_{\theta=\theta_\circ} = \frac{-\cos\theta_\circ\left(L_z - L_{z'}\cos\theta_\circ\right)^2 + L_{z'}\sin^2\theta_\circ\left(L_z - L_{z'}\cos\theta_\circ\right)}{I\sin^3\theta_\circ} - mgl\sin\theta_\circ = 0$$

let $\gamma = L_z - L_{z'}\cos\theta_\circ$

Then $\gamma^2\cos\theta_\circ - \gamma L_{z'}\sin^2\theta_\circ + mglI\sin^4\theta_\circ = 0$ and solving for γ

$$\gamma = \frac{L_{z'}\sin^2\theta_\circ}{2\cos\theta_\circ}\left[1 \pm \left(1 - \frac{4mglI\cos\theta_\circ}{L_{z'}^2}\right)^{\frac{1}{2}}\right]$$

now $L_{z'}$ is large since $\dot\psi$ is large and the precession rate is small, so we can expand the term in square root above and use the (-) solution since γ must be positive …

$$\gamma \approx \frac{L_{z'}\sin^2\theta_\circ}{2\cos\theta_\circ}\left[1 - 1 + \frac{2mglI\cos\theta_\circ}{L_{z'}^2}\right]$$

$$\gamma = L_z - L_{z'}\cos\theta_\circ \approx \frac{mglI\sin^2\theta_\circ}{L_{z'}}$$

From equations 9.7.2, 9.7.5 and 9.7.7 …

$L_z = I\dot\phi^2\sin^2\theta_0 + I_s\left(\dot\phi\cos\theta_0 + \dot\psi\right)\cos\theta_0$ and …

$L_{z'}\cos\theta_0 = I_s(\dot\phi\cos\theta_0 + \dot\psi)\cos\theta_0$ so …

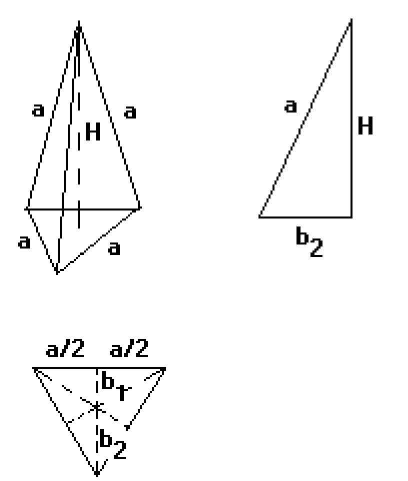
$$\gamma = \left(I\sin^2\theta_\circ + I_s\cos^2\theta_\circ\right)\dot\phi + I_s\dot\psi\cos\theta_\circ - I_s\dot\phi\cos^2\theta_\circ - I_s\dot\psi\cos\theta_\circ$$

$= I\dot\phi\sin^2\theta_\circ \approx \dfrac{mglI\sin^2\theta_\circ}{L_{z'}} = \dfrac{mglI\sin^2\theta_\circ}{I_s\left(\dot\psi + \dot\phi\cos\theta_\circ\right)}$ and since $\dot\psi >> 0$, we can ignore the $\dot\phi$ term in the denominator and we have …

$$\dot\phi \approx \frac{mgl}{I_s\dot\psi}$$

Hence, if $\dot\psi$ large, $\left.\dot\theta\right|_{\theta=\theta_1} = 0$ and $\left.\dot\phi\right|_{\theta=\theta_1} = \dfrac{mgl}{I_s\dot\psi}$ the top will precess without nutation at $\theta_1 = \theta_\circ$ the place where $V(\theta) = \min$.

9.25



$$b_1 + b_2 = \sqrt{3}\left(\frac{a}{2}\right)$$

$$\frac{b_1}{b_2} = \sin 30^\circ = \frac{1}{2}$$

$$b_1 = \frac{a}{2\sqrt{3}} \qquad b_2 = \frac{a}{\sqrt{3}}$$

$$a^2 = H^2 + b_2^2$$

$$H^2 = a^2 - b_2^2 = a^2 - \frac{a^2}{3} = \frac{2a^2}{3}$$

$H = \sqrt{\frac{2}{3}}a$

Thus, the coordinates of the 4 atoms are:

Oxygen: $(0, 0, H)$

Hydrogen: $\left(-b_1, \frac{a}{2}, 0\right); \left(-b_1, -\frac{a}{2}, 0\right)$

Carbon: $(b_2, 0, 0)$

(a) The axes 1, 2, 3 are principal axes if the products of inertia are zero.

$$-I_{xy} = \sum m_i x_i y_i = 0 + \left(-b_1 \frac{a}{2}\right) + \left(-b_1\left(\frac{-a}{2}\right)\right) + 0 \equiv 0$$

$$-I_{yz} = 0 + 0 + 0 + 0 \equiv 0$$

$$-I_{xz} = 0 + 0 + 0 + 0 \equiv 0$$

The 1,2,3 axes are principal axes.

(b) Find principal moments

$$I_1 = I_{xx} = \sum m_i \left(y_i^2 + z_i^2\right) = 16H^2 + 1\left(\frac{a}{2}\right)^2 + 1\left(\frac{a}{2}\right)^2 = \frac{67}{6}a^2$$

$$I_2 = I_{yy} = \sum m_i \left(x_i^2 + z_i^2\right) = 16H^2 + 1\left(b_1^2\right) + 1\left(b_1^2\right) + 12b_2^2 = \frac{89}{6}a^2$$

$$I_3 = I_{zz} = \sum m_i \left(x_i^2 + y_i^2\right) = 0 + \left(b_1^2 + \left(\frac{a}{2}\right)^2\right) + \left(b_1^2 + \left(\frac{a}{2}\right)^2\right) + b_2^2 = \frac{14}{3}a^2$$

$$I = \begin{pmatrix} \frac{67}{6} & 0 & 0 \\ 0 & \frac{89}{6} & 0 \\ 0 & 0 & \frac{14}{3} \end{pmatrix} a^2$$

(c) $I_3 < I_1 < I_2$ therefore rotation about the1-axis is unstable (see discussion Sec. 9.4)
