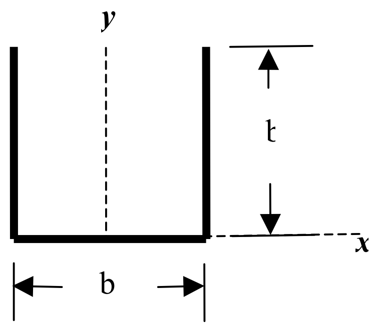
CHAPTER 8
MECHANICS OF RIGID BODIES: PLANAR MOTION

8.1 (a) For each portion of the wire having a mass $\frac{m}{3}$ and centered at

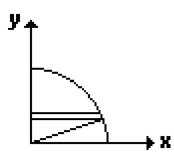


$$\left(-\frac{b}{2},\frac{b}{2}\right),\ (0,0),\ \text{and}\ \left(\frac{b}{2},\frac{b}{2}\right)\ \ldots$$

$$x_{cm}=\frac{1}{m}\left[-\left(\frac{b}{2}\right)\left(\frac{m}{3}\right)+0+\left(\frac{b}{2}\right)\left(\frac{m}{3}\right)\right]=0$$

$$y_{cm}=\frac{1}{m}\left[\left(\frac{b}{2}\right)\left(\frac{m}{3}\right)+0+\left(\frac{b}{2}\right)\left(\frac{m}{3}\right)\right]=\frac{b}{3}$$

(b)



$$ds = xdy = \left(b^2-y^2\right)^{\frac{1}{2}}dy$$

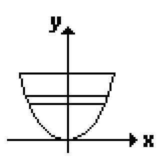
$$y_{cm}=\frac{1}{m}\int_0^b \rho y\left(b^2-y^2\right)^{\frac{1}{2}}dy$$

$$y_{cm}=\frac{-\frac{\rho}{2}\int_{y=0}^{y=b}\left(b^2-y^2\right)^{\frac{1}{2}}d\left(b^2-y^2\right)}{\frac{1}{4}\pi b^2\rho}$$

$$y_{cm}=\frac{4b}{3\pi}$$

From symmetry, $x_{cm}=\frac{4b}{3\pi}$

(c)



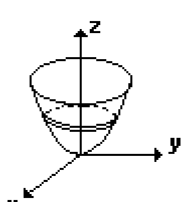
The center of mass is on the y-axis.

$$ds = 2xdy = 2\left(by\right)^{\frac{1}{2}}dy$$

$$y_{cm}=\frac{\int_0^b 2\rho y\left(by\right)^{\frac{1}{2}}dy}{\int_0^b 2\rho\left(by\right)^{\frac{1}{2}}dy}=\frac{\int_0^b y^{\frac{3}{2}}dy}{\int_0^b y^{\frac{1}{2}}dy}$$

$$y_{cm}=\frac{3b}{5}$$

(d)



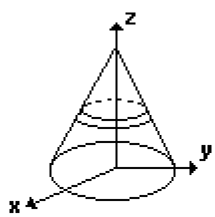
The center of mass is on the z-axis.

$$dv=\pi r^2dz=\pi\left(x^2+y^2\right)dz=\pi bzdz$$

$$z_{cm} = \frac{\int_0^b \rho z \pi b z dz}{\int_0^b \rho \pi b z dz} = \frac{\int_0^b z^2 dz}{\int_0^b z dz}$$

$$z_{cm} = \frac{2}{3} b$$

(e)



The center of mass is on the z-axis.
α is the half-angle of the apex of the cone. $r_\circ$ is the radius of the base at $z = 0$ and r is radius of a circle at some arbitrary z in a plane parallel to the base.

$$\tan\alpha = \frac{r_\circ}{b} = \frac{r}{b-z}\text{, a constant}$$

$$dv = \pi r^2 dz = \pi (b-z)^2 \tan^2\alpha \, dz$$

$$m = \rho \frac{1}{3} \pi r_\circ^2 b = \frac{1}{3} \pi \rho b^3 \tan^2\alpha$$

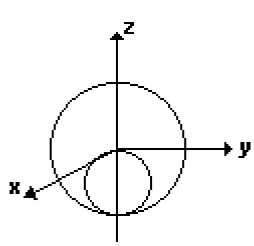
$$z_{cm} = \frac{\int_0^b \rho z \pi (b-z)^2 \tan^2\alpha \, dz}{\frac{1}{3} \pi \rho b^3 \tan^2\alpha} = \frac{3}{b^3} \int_0^b \left(b^2 z - 2bz^2 + z^3\right) dz$$

$$z_{cm} = \frac{b}{4}$$

8.2 $$x_{cm} = \frac{\int \rho x dx}{\int \rho dx} = \frac{\int_0^b c x^2 dx}{\int_0^b c x dx}$$

$$x_{cm} = \frac{2b}{3}$$

8.3 The center of mass is on the z-axis. Consider the sphere with the cavity to be made



of a (i) solid sphere of radius a and mass M_s, with its center of mass at $z = 0$, and (ii) a solid sphere the size of the cavity, with mass $-M_c$ and center of mass at $z = -\frac{a}{2}$. The actual sphere with the cavity has a mass $m = M_s - M_c$ and center of mass z_{cm}.

$$0 = \frac{1}{M_s}\left[M_c \left(-\frac{a}{2}\right) + m z_{cm} \right]$$

$$M_s = \frac{4}{3} \pi a^3 \rho \text{, } M_c = \frac{4}{3} \pi \left(\frac{a}{2}\right)^3 \rho$$

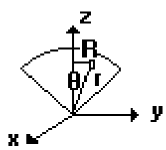
$$0 = \frac{1}{a^3} \left\{ \left(\frac{a}{2}\right)^3 \left(-\frac{a}{2}\right) + \left[a^3 - \left(\frac{a}{2}\right)^3 \right] z_{cm} \right\}$$

$$z_{cm} = \frac{a}{14}$$

8.4 (a) $I_z = \sum_i m_i R_i^2 = \frac{m}{3}\left[\left(\frac{b}{2}\right)^2 + 0 + \left(\frac{b}{2}\right)^2\right]$

$$I_z = \frac{mb^2}{6}$$

(b)



$$ds = r d\theta dr \ , \ R = r\sin\theta$$

$$I_z = \int R^2 \rho ds$$

$$I_z = \rho \int_{r=0}^{r=b} r^2 \, r dr \int_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} \sin^2\theta \, d\theta$$

$$I_z = \frac{\rho b^4}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2\theta d\theta$$

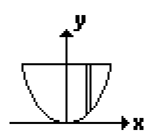
$$\left[\int \sin^2\theta d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]$$

$$I_z = \frac{\rho b^4}{4}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$m = \frac{1}{4}\rho\pi b^2$$

$$I_z = \frac{mb^2}{4\pi}(\pi - 2)$$

(c)



$$ds = hdx = \left(b - \frac{x^2}{b}\right)dx$$

Where the parabola intersects the line $y = b$,

$$x = (by)^{\frac{1}{2}} = \pm b$$

$$I_y = \int_{-b}^{b} x^2 \rho\left(b - \frac{x^2}{b}\right)dx = \rho\int_{-b}^{b}\left(bx^2 - \frac{x^4}{b}\right)dx$$

$$I_y = \frac{4}{15}\rho b^4$$

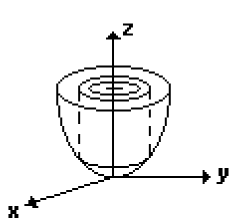
$$m = \int_{-b}^{b}\rho\left(b - \frac{x^2}{b}\right)dx = \frac{4}{3}\rho b^2$$

$$I_y = \frac{1}{5}mb^2$$

(d)

$$dv = 2\pi R h dR$$

$$h = b - z$$



$$R = \left(x^2 + y^2\right)^{\frac{1}{2}} = (bz)^{\frac{1}{2}}$$

$$dR = \frac{1}{2}\left(\frac{b}{z}\right)^{\frac{1}{2}} dz$$

$$I_z = \int R^2 \rho dv = \int_0^b bz\rho 2\pi (bz)^{\frac{1}{2}} (b-z)\frac{1}{2}\left(\frac{b}{z}\right)^{\frac{1}{2}} dz$$

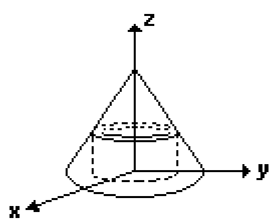
$$I_z = \pi\rho b^2 \int_0^b \left(bz - z^2\right) dz = \frac{1}{6}\pi\rho b^5$$

$$m = \int \rho dv = \int_0^b \rho 2\pi (bz)^{\frac{1}{2}} (b-z)\frac{1}{2}\left(\frac{b}{z}\right)^{\frac{1}{2}} dz$$

$$m = \pi\rho b\int_0^b (b-z)\, dz = \frac{1}{2}\pi\rho b^3$$

$$I_z = \frac{1}{3} mb^2$$

(e)



α is the half-angle of the apex of the cone. $r_\circ$ is the radius of the base at $z = 0$ and r is radius of a circle at some arbitrary z in a plane parallel to the base.

$$\tan\alpha = \frac{R_\circ}{b} = \frac{R}{b-z}, \text{ a constant}$$

$$dv = 2\pi R h dR = 2\pi \frac{(b-z)R_\circ}{b} z dR$$

Since $R = \frac{(\mathrm{b-z})R_\circ}{b}$, $dR = -\frac{R_\circ}{b} dz$, and the limits of integration for $R = 0 \rightarrow R_\circ$ correspond to $z = b \rightarrow 0$

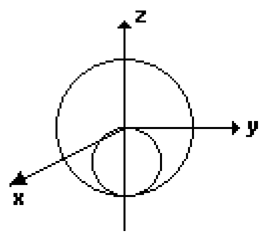
$$I_z = \int R^2 \rho dv = \int_b^0 \frac{(b-z)^2 R_\circ^2}{b^2} \rho 2\pi \frac{(b-z)R_\circ}{b} z\left(-\frac{R_\circ}{b}\right) dz$$

$$I_z = +2\pi\rho \frac{R_\circ^4}{b^4}\int_0^b \left(b^3 z - 3b^2 z^2 + 3bz^3 - z^4\right) dz = \frac{1}{10}\pi\rho R_\circ^4 b$$

$$m = \rho\frac{1}{3}\pi R_\circ^2 b$$

$$I_z = \frac{3}{10} m R_\circ^2$$

8.5 Consider the sphere with the cavity to be made of a (i) solid sphere of radius a and and mass M_s, with its center of mass at $z=0$, and (ii) a solid sphere the size of the cavity, with mass $-M_c$ and center of mass at $z=-\frac{a}{2}$. The actual sphere with the cavity has a mass $m=M_s-M_c$ and center of mass z_{cm}.



$$m=M_s-M_c=\frac{4}{3}\pi a^3\rho-\frac{4}{3}\pi\left(\frac{a}{2}\right)^3\rho=\frac{7}{8}\frac{4}{3}\pi a^3\rho$$

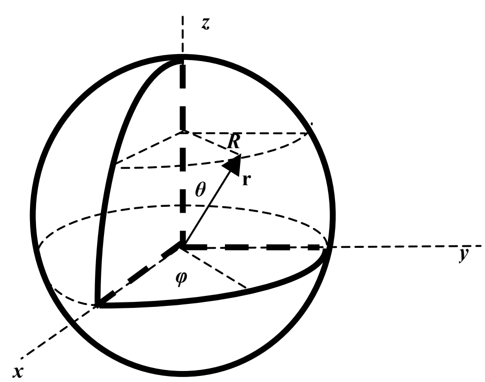
$$M_s=\frac{8}{7}m \text{ and } M_c=\frac{1}{7}m$$

From eqn. 8.3.2, $\quad I_s=I_c+I$

$$\frac{2}{5}M_s a^2=\frac{2}{5}M_c\left(\frac{a}{2}\right)^2+I$$

$$I=\frac{2}{5}\left(\frac{8}{7}m\right)a^2-\frac{2}{5}\left(\frac{1}{7}m\right)\frac{a^2}{4}=\frac{31}{70}ma^2$$

8.6 The moment of inertia about one of the straight edges is $I_z=\int R^2\rho dv$ where $R^2=x^2+y^2$. From Appendix F …



$$dv=r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$R^2=x^2+y^2=r^2\sin^2\theta$$

Let a = radius of sphere

$$I_z=\int_{r=0}^{r=a}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{\phi=0}^{\phi=\frac{\pi}{2}} r^2\sin^2\theta\rho r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$I_z=\rho\frac{\pi}{2}\int_{r=0}^{r=a}\int_{\theta=0}^{\theta=\frac{\pi}{2}} r^4\sin^3\theta dr d\theta$$

$$I_z=\frac{1}{10}\rho\pi a^5\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta$$

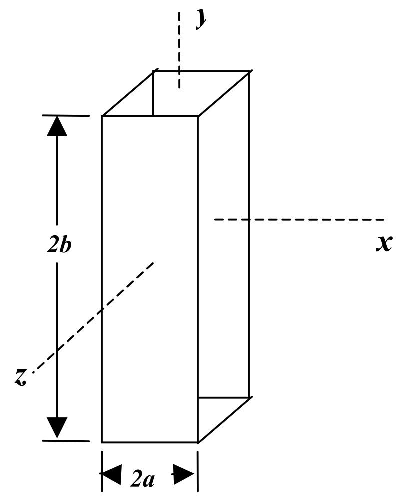
$$\left[\int\sin^3\theta d\theta=\frac{\cos^3\theta}{3}-\cos\theta\right]$$

$$I_z=\frac{2}{30}\rho\pi a^5$$

$$m=\frac{1}{8}\frac{4}{3}\pi a^3\rho=\frac{1}{6}\pi a^3\rho$$

$$I_z=\frac{2}{5}ma^2$$

8.7 For a rectangular parallelepiped:



$dv = hdxdy$ h is the length of the box in the z-direction

$$R = \left(x^2 + y^2\right)^{\frac{1}{2}}$$

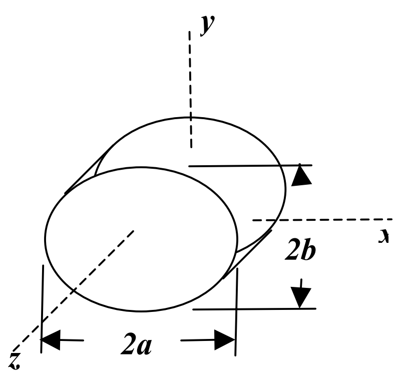
$$I_z = \int R^2 \rho dv = \int_{x=-a}^{x=a} \int_{y=-b}^{y=b} \left(x^2 + y^2\right) \rho h dxdy$$

$$I_z = \rho h \int_{-a}^{a} \left(2bx^2 + \frac{2b^3}{3}\right) dx = \frac{4}{3} \rho hab\left(a^2 + b^2\right)$$

$$m = \rho(2a)(2b)h = 4\rho abh$$

$$I_z = \frac{m}{3}\left(a^2 + b^2\right)$$

For an elliptic cylinder:



Again $dv = hdxdy$, and $R = \left(x^2 + y^2\right)^{\frac{1}{2}}$

On the surface, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$x = \pm a\left(1 - \frac{y^2}{b^2}\right)^{\frac{1}{2}}$$

$$I_z = \int R^2 dv = \int_{y=-b}^{y=b} \int_{x=-a\left(1-\frac{y^2}{b^2}\right)^{\frac{1}{2}}}^{x=a\left(1-\frac{y^2}{b^2}\right)^{\frac{1}{2}}} \left(x^2 + y^2\right) \rho h dxdy$$

$$I_z = \rho h \int_{-b}^{b} \left[\frac{2}{3} a^3 \left(1 - \frac{y^2}{b^2}\right)^{\frac{3}{2}} + 2a\left(1 - \frac{y^2}{b^2}\right)^{\frac{1}{2}} y^2\right] dy$$

$$= 2\rho h \frac{a}{b}\left[\frac{a^2}{3b^2}\int_{-b}^{b}\left(b^2 - y^2\right)^{\frac{3}{2}} dy + \int_{-b}^{b}\left(b^2 - y^2\right)^{\frac{1}{2}} y^2 dy\right]$$

From a table of integrals:

$$\int\left(b^2 - y^2\right)^{\frac{3}{2}} dy = \frac{y}{4}\left(b^2 - y^2\right)^{\frac{3}{2}} + \frac{3}{8}b^2 y\left(b^2 - y^2\right)^{\frac{1}{2}} + \frac{3}{8}b^4 \sin^{-1}\frac{y}{b}$$

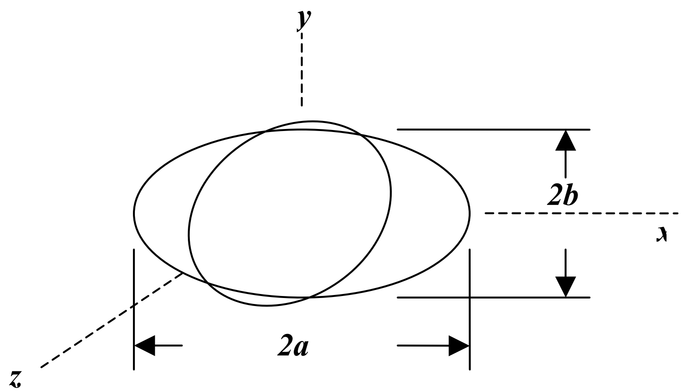
$$\int\left(b^2 - y^2\right)^{\frac{1}{2}} y^2 dy = -\frac{y}{4}\left(b^2 - y^2\right)^{\frac{3}{2}} + \frac{b^2 y}{8}\left(b^2 - y^2\right)^{\frac{1}{2}} + \frac{b^4}{8}\sin^{-1}\frac{y}{b}$$

$$I_z = 2\rho h \frac{a}{b}\left[\frac{a^2}{3b^2}\left(\frac{3}{8}b^4 \pi\right) + \frac{b^4}{8}\pi\right] = \frac{1}{4}\rho h \pi ab\left(a^2 + b^2\right)$$

$$m = \rho h(\pi ab)$$

$$I_z = \frac{m}{4}(a^2 + b^2)$$

For an ellipsoid: $dv = hdxdy$, $R^2 = x^2 + y^2$ and on the surface, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



$$z = \pm c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}}$$

$$h = 2|z| = 2c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}}$$

In the xy plane $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$x = \pm a\left(1 - \frac{y^2}{b^2}\right)^{\frac{1}{2}}$$

$$I_z = R^2 \rho dv = \int_{y=-b}^{y=b} \int_{x=-a\left(1-\frac{y^2}{b^2}\right)^{\frac{1}{2}}}^{x=a\left(1-\frac{y^2}{b^2}\right)^{\frac{1}{2}}} (x^2 + y^2)\rho 2c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}} dxdy$$

$$I_z = \frac{2\rho c}{a}\int_{y=-b}^{y=b}\left\{\int\left[\left(a^2 - \frac{a^2y^2}{b^2}\right) - x^2\right]^{\frac{1}{2}} x^2 dx + y^2 \int\left[\left(a^2 - \frac{a^2y^2}{b^2}\right) - x^2\right]^{\frac{1}{2}} dx\right\}$$

From a table of integrals:

$$\int (k^2 - x^2)^{\frac{1}{2}} x^2 dx = -\frac{x}{4}(k^2 - x^2)^{\frac{3}{2}} + \frac{k^2 x}{8}(k^2 - x^2)^{\frac{1}{2}} + \frac{k^4}{8}\sin^{-1}\frac{x}{k}$$

$$\int (k^2 - x^2)^{\frac{1}{2}} dx = \frac{x}{2}(k^2 - x^2)^{\frac{1}{2}} + \frac{k^2}{2}\sin^{-1}\frac{x}{k}$$

$$I_z = \frac{2\rho c}{a}\int_{-b}^{b}\left[\frac{a^4}{8}\left(1 - \frac{y^2}{b^2}\right)^2 \pi + y^2 \frac{a^2}{2}\left(1 - \frac{y^2}{b^2}\right)\pi\right] dy$$

$$I_z = \rho \pi ac \int_{-b}^{b}\left[\frac{a^2}{4}\left(1 - \frac{2y^2}{b^2} + \frac{y^4}{b^4}\right) + y^2 - \frac{y^4}{b^2}\right] dy$$

$$I_z = \frac{4}{15}\rho \pi abc(a^2 + b^2)$$

For an ellipsoid, $m = \rho \frac{4}{3}\pi abc$, so $I_z = \frac{m}{5}(a^2 + b^2)$

8.8 (See Figure 8.4.1) Note that $l'+l$ is the distance from $0'$ to 0, defined as d

From eqn. 8.4.13, $k_{cm}^2 = ll'$

$$k_{cm}^2 + l^2 = ll' + l^2 = l\left(l'+l\right)$$

$$k_{cm}^2 + l^2 = ld$$

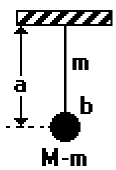
From eqn. 8.4.9b, $k^2 = k_{cm}^2 + l^2$

$$k^2 = ld$$

From eqn. 8.4.6, $T_{\circ} = 2\pi\sqrt{\dfrac{k^2}{gl}}$

$$T_{\circ} = 2\pi\sqrt{\frac{d}{g}}$$

8.9



Period of a simple pendulum: $T = 2\pi\sqrt{\dfrac{a}{M}}$

Period of real pendulum: $T_{\circ} = 2\pi\sqrt{\dfrac{I}{Mgl}}$ (eqn. 8.4.5)

Where $I =$ moment of inertia
$l =$ distance to CM of physical pendulum
$a =$ distance to CM of *bob*
$b =$ radius of *bob*

location of CM of physical pendulum:

$$l = \frac{m\dfrac{(a-b)}{2} + (M-m)a}{m + (M-m)} = a - \frac{m}{M}\left(\frac{a}{2}+\frac{b}{2}\right) = a - \frac{m}{2M}(a+b)$$

$$l = a\left[1 - \frac{m}{2M} - \frac{m}{2M}\frac{b}{a}\right]$$

Moment of inertia:

$$I_{bob} = (M-m)a^2 + \frac{2}{5}(M-m)b^2 = (M-m)\left(a^2 + \frac{2}{5}b^2\right)$$

$$= Ma^2\left(1-\frac{m}{M}\right)\left(1+\frac{2}{5}\frac{b^2}{a^2}\right)$$

$$I_{rod} = \frac{1}{3}m(a-b)^2 = \frac{1}{3}Ma^2\left[\frac{m}{M}\left(1-\frac{b}{a}\right)^2\right]$$

$$\therefore I = I_{bob} + I_{rod} = Ma^2\left[\left(1-\frac{m}{M}\right)\left(1+\frac{2}{5}\frac{b^2}{a^2}\right) + \frac{1}{3}\frac{m}{M}\left(1-\frac{b}{a}\right)^2\right]$$

letting $\alpha = \dfrac{m}{M}$ and $\beta = \dfrac{b}{a}$

$$T_\circ = 2\pi \left\{ \frac{Ma^2\left[(1-\alpha)\left(1+\frac{2}{5}\beta^2\right)+\frac{1}{3}\alpha(1-\beta)^2\right]}{Mga\left[1-\frac{\alpha}{2}-\frac{\alpha\beta}{2}\right]} \right\}^{\frac{1}{2}}$$

(a) $$\frac{T_\circ}{T} = \sqrt{\frac{(1-\alpha)\left(1+\frac{2}{5}\beta^2\right)+\frac{1}{3}\alpha(1-\beta)^2}{\left(1-\frac{\alpha}{2}-\frac{\alpha\beta}{2}\right)}} \approx 1-\frac{1}{12}\alpha$$ to 1^{st} order in α

(b) $m = 10g$ $\quad M = 1kg$ $\quad a = 1.27m$ $\quad b = 5cm$

$\alpha = 0.01$ $\quad \beta = 0.0394$

$\frac{T_\circ}{T} \approx 1-\frac{1}{12}\alpha = 0.9992$ (actually 0.9994 using complete expression)

8.10 The period of the "seconds" pendulum is

$$T_2 = 2\pi\sqrt{\frac{I}{Mgl}} = 2\,s$$

The period of the modified pendulum is

$$T' = 2\pi\sqrt{\frac{I'}{M'gl'}} = 2\left(\frac{n}{n-20}\right)$$

where I', M', l' refer to parameters of pendulum *with m attached* and $n\ (= 24\times 60\times 60)$ is the number of seconds in a day.

$$I' = I + ml_m^2 \qquad l' = \frac{Ml+ml_m}{M}$$

where l_m is the distance of the attached mass m from the pivot point.

So $$\left(1-\frac{20}{n}\right)^{-2} = \frac{\pi^2 I'}{M'gl'} = \frac{\pi^2 I + \pi^2 ml_m^2}{(Ml+ml_m)g}$$

$$= \frac{Mgl + \pi^2 ml_m^2}{(Ml+ml_m)g}$$

Thus $$\left(1-\frac{20}{n}\right)^{2} = \frac{(Ml+ml_m)g}{(Mgl+\pi^2 ml_m^2)}$$

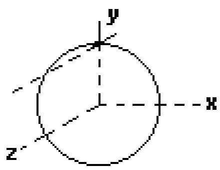
Or $$1-\frac{40}{n} \approx \frac{\left(1+\frac{\alpha l_m}{l}\right)}{\left(1+\frac{\pi^2\alpha l_m^2}{gl}\right)} \qquad \alpha = \frac{m}{M}$$

Solving for α gives the approximate result

$$\alpha = \frac{m}{M} \cong \frac{\frac{40}{n}\frac{l}{l_m}}{\left(\frac{\pi^2 l_m}{g} - 1\right)}$$

Letting $l_m = 1.3m$; $l = 1.0m$; we obtain $\alpha \cong 1.15 \cdot 10^{-3}$

8.11 (a) $I_{\perp cm} = ma^2$ (all mass in rim)



$$I_{\perp rim} = ma^2 + ma^2 = 2ma^2$$

$$\therefore T = 2\pi\sqrt{\frac{I_{\perp rim}}{mga}} = 2\pi\sqrt{\frac{2a}{g}}$$

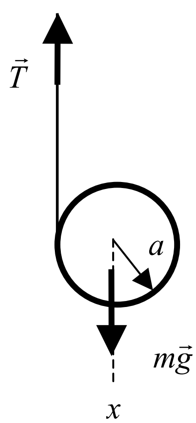
(b) $I_z = I_x + I_y = 2I_{\| cm} = ma^2 \qquad \left(= I_{\perp cm}\right)$

$$\therefore I_{\| cm} = \frac{ma^2}{2}$$

hence $I_{\| rim} = \frac{ma^2}{2} + ma^2 = \frac{3}{2}ma^2$

$$\therefore T = 2\pi\sqrt{\frac{I_{\| rim}}{mga}} = 2\pi\sqrt{\frac{3a}{2g}}$$

8.12



$$m\ddot{x}_{cm} = mg - T$$

$$I_{cm}\dot{\omega} = aT$$

$$\ddot{x}_{cm} = a\dot{\omega}$$

$$I_{cm} = \frac{2}{5}ma^2$$

$$m\ddot{x}_{cm} = mg - \frac{I_{cm}\dot{\omega}}{a} = mg - \frac{1}{a}\left(\frac{2}{5}ma^2\frac{\ddot{x}_{cm}}{a}\right) = mg - \frac{2}{5}m\ddot{x}_{cm}$$

$$\ddot{x}_{cm} = \frac{5}{7}g$$

8.13 When two men hold the plank, each supports $\frac{mg}{2}$.

When one man lets go: $mg - R = m\ddot{x}_{cm}$ and $R\frac{l}{2} = I_{cm}\dot{\omega}$

From table 8.3.1, $I_{cm} = \frac{ml^2}{12}$

$$\dot{\omega} = \frac{Rl}{2}\cdot\frac{12}{ml^2} = \frac{6R}{ml}$$

$$\ddot{x}_{cm} = \frac{l}{2}\dot{\omega} = \frac{3R}{m}$$

$$mg - R = m\left(\frac{3R}{m}\right) = 3R$$

$$R = \frac{mg}{4}$$

$$\ddot{x}_{end} = l\dot{\omega} = l\frac{6R}{ml} = \frac{6}{m}\left(\frac{mg}{4}\right)$$

$$\ddot{x}_{end} = \frac{3}{2}g$$

8.14 For a solid sphere:

$$M_s = \frac{4}{3}\pi a^3\rho \text{ and } I_s = \frac{2}{5}M_s a^2$$

$$\left(k_{cm}^2\right)_s = \frac{2}{5}a^2$$

For subscript c representing a solid sphere the size of the cavity, from eqn. 8.3.2:

$$I_s = I + I_c$$

$$I = \frac{2}{5}\left(\frac{4}{3}\pi a^3\rho\right)a^2 - \frac{2}{5}\left[\frac{4}{3}\pi\left(\frac{a}{2}\right)^3\rho\right]\left(\frac{a}{2}\right)^2 = \frac{2}{5}\cdot\frac{4}{3}\cdot\frac{31}{32}\pi a^5\rho$$

$$m = \frac{4}{3}\pi a^3\rho - \frac{4}{3}\pi\left(\frac{a}{2}\right)^3\rho = \frac{4}{3}\cdot\frac{7}{8}\pi a^3\rho$$

$$k_{cm}^2 = \frac{I}{m} = \frac{2}{5}\cdot\frac{31}{32}\cdot\frac{8}{7}a^2 = \frac{31}{70}a^2$$

From eqn. 8.6.11, for a sphere rolling down a rough inclined plane:

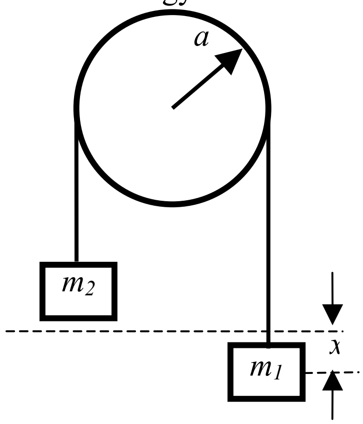
$$\ddot{x}_{cm} = \frac{g\sin\theta}{1+\left(\frac{k_{cm}^2}{a^2}\right)}$$

$$\frac{\ddot{x}}{\ddot{x}_s} = \frac{1+\frac{\left(k_{cm}^2\right)_s}{a^2}}{1+\frac{k_{cm}^2}{a^2}}$$

$$= \frac{1+\frac{2}{5}}{1+\frac{31}{70}} = \frac{\frac{7}{5}}{\frac{101}{70}}$$

$$\frac{\ddot{x}}{\ddot{x}_s} = \frac{98}{101}$$

8.15 Energy is conserved:

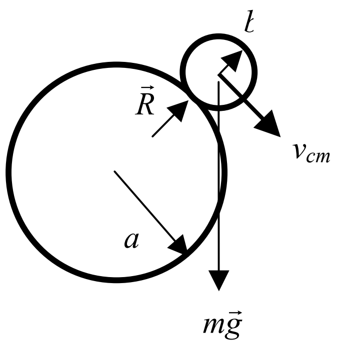


$$\frac{1}{2}m_1\dot{x}^2+\frac{1}{2}m_2\dot{x}^2+\frac{1}{2}I\left(\frac{\dot{x}}{2}\right)^2+m_2gx-m_1gx=E$$

$$m_1\dot{x}\ddot{x}+m_2\dot{x}\ddot{x}+\frac{I}{a^2}\dot{x}\ddot{x}+m_2g\dot{x}-m_1g\dot{x}=0$$

$$\ddot{x}=\frac{(m_1-m_2)g}{m_1+m_2+\frac{I}{a^2}}$$

8.16 While the cylinders are in contact:



$$f_r=\frac{mv_{cm}^2}{r}=mg\cos\theta-R$$

$r=a+b$, so $\frac{mv_{cm}^2}{a+b}=mg\cos\theta-R$

From conservation of energy:

$$mg(a+b)=\frac{1}{2}mv_{cm}^2+\frac{1}{2}I\omega^2+mg(a+b)\cos\theta$$

From table 8.3.1, $I=\frac{1}{2}ma^2$

$$\omega=\frac{v_{cm}}{a}$$

$$\frac{1}{2}mv_{cm}^2+\frac{1}{2}\left(\frac{1}{2}ma^2\right)\left(\frac{v_{cm}^2}{a^2}\right)=mg(a+b)(1-\cos\theta)$$

$$\frac{mv_{cm}^2}{a+b}=\frac{4}{3}mg(1-\cos\theta)$$

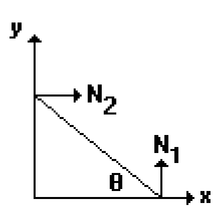
When the rolling cylinder leaves, $R=0$:

$$mg\cos\theta=\frac{4}{3}mg(1-\cos\theta)$$

$$\frac{7}{3}\cos\theta=\frac{4}{3}$$

$$\theta=\cos^{-1}\frac{4}{7}$$

8.17



$$m\ddot{x}=N_2$$

$$m\ddot{y}=N_1-mg$$

$$\frac{1}{2}mv_{cm}^2+\frac{1}{2}I\omega^2+mgy=mgy_\circ$$

$$x=\frac{l}{2}\cos\theta,\qquad \dot{x}=-\frac{l}{2}\dot{\theta}\sin\theta,\qquad \ddot{x}=\frac{l}{2}\left(\dot{\theta}^2\cos\theta+\ddot{\theta}\sin\theta\right)$$

$$y=\frac{l}{2}\sin\theta,\qquad \dot{y}=\frac{l}{2}\dot{\theta}\cos\theta,\qquad \ddot{y}=\frac{l}{2}\left(-\dot{\theta}^2\sin\theta+\ddot{\theta}\cos\theta\right)$$

$$v_{cm}^2=\dot{x}^2+\dot{y}^2=\left(-\frac{l}{2}\dot{\theta}\sin\theta\right)^2+\left(\frac{l}{2}\dot{\theta}\cos\theta\right)^2=\frac{l^2\dot{\theta}^2}{4}$$

$I=\dfrac{ml^2}{12}$, and $\omega=\dot{\theta}$

$$\frac{1}{2}m\frac{l^2\dot{\theta}^2}{4}+\frac{1}{2}\frac{ml^2}{12}\dot{\theta}^2+mg\frac{l}{2}\sin\theta=mg\frac{l}{2}\sin\theta_\circ$$

$$\frac{l}{3}\dot{\theta}^2=g\left(\sin\theta_\circ-\sin\theta\right)$$

$$\dot{\theta}=\left[\frac{3g}{l}\left(\sin\theta_\circ-\sin\theta\right)\right]^{\frac{1}{2}}$$

$$\ddot{\theta}=\frac{1}{2}\left[\frac{3g}{l}\left(\sin\theta_\circ-\sin\theta\right)\right]^{-\frac{1}{2}}\left(\frac{3g}{l}\right)\left(-\cos\theta\right)\dot{\theta}=-\frac{3g}{2l}\cos\theta$$

$$N_2=m\ddot{x}=-\frac{ml}{2}\left[\cos\theta\left(\frac{3g}{l}\right)\left(\sin\theta_\circ-\sin\theta\right)+\sin\theta\left(-\frac{3g}{2l}\right)\cos\theta\right]$$

Separation occurs when $N_2=0$:

$$\sin\theta_\circ-\sin\theta-\frac{1}{2}\sin\theta=0,\qquad \theta=\sin^{-1}\left(\frac{2}{3}\sin\theta_\circ\right)$$

8.18 $R_x=m\ddot{x}$

$$R_y-mg=m\ddot{y}$$

$$x=\frac{l}{2}\sin\theta,\qquad \dot{x}=\frac{l}{2}\dot{\theta}\cos\theta,\quad \ddot{x}=\frac{l}{2}\left(-\dot{\theta}^2\sin\theta+\ddot{\theta}\cos\theta\right)$$

$$y=\frac{l}{2}\cos\theta,\qquad \dot{y}=-\frac{l}{2}\dot{\theta}\sin\theta,\qquad \ddot{y}=-\frac{l}{2}\left(\dot{\theta}^2\cos\theta+\ddot{\theta}\sin\theta\right)$$

$$\frac{1}{2}mv_{cm}^2+\frac{1}{2}I\dot{\theta}^2+mg\frac{l}{2}\cos\theta=mg\frac{l}{2}$$

$$v_{cm}=\frac{l}{2}\dot{\theta},\qquad I=\frac{ml^2}{12}$$

$$\frac{m}{2}\frac{l^2\dot{\theta}^2}{4}+\frac{1}{2}\frac{ml^2}{12}\dot{\theta}^2=mg\frac{l}{2}(1-\cos\theta)$$
$$\frac{l\dot{\theta}^2}{3}=g(1-\cos\theta)$$
$$\dot{\theta}=\left[\frac{3g}{l}(1-\cos\theta)\right]^{\frac{1}{2}}$$
$$\ddot{\theta}=\frac{1}{2}\left[\frac{3g}{l}(1-\cos\theta)\right]^{-\frac{1}{2}}\left(\frac{3g}{l}\right)\sin\theta\dot{\theta}=\frac{3g}{2l}\sin\theta$$
$$R_x=\frac{ml}{2}\left[(-\sin\theta)\left(\frac{3g}{l}\right)(1-\cos\theta)+\cos\theta\left(\frac{3g}{2l}\sin\theta\right)\right]$$
$$R_x=\frac{3mg}{4}\sin\theta(3\cos\theta-2)$$
$$R_y=mg-\frac{ml}{2}\left[\cos\theta\left(\frac{3g}{l}\right)(1-\cos\theta)+\sin\theta\left(\frac{3g}{2l}\sin\theta\right)\right]$$
$$R_y=mg-\frac{3mg}{2}\left(\cos\theta-\cos^2\theta+\frac{\sin^2\theta}{2}\right)$$
$$R_y=\frac{mg}{4}(3\cos\theta-1)^2$$

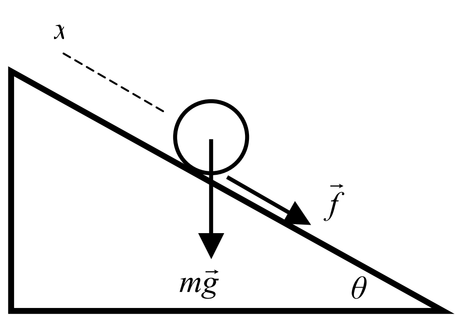
The reaction force constrains the tail of the rocket from sliding backward for $R_x>0$:

$$3\cos\theta-2>0$$
$$\theta<\cos^{-1}\frac{2}{3}$$

The rocket is constrained from sliding forward for $R_x<0$:

$$\theta>\cos^{-1}\frac{2}{3}$$

8.19 $m\ddot{x}=-mg\sin\theta-\mu mg\cos\theta$



$$\ddot{x}=-g(\sin\theta+\mu\cos\theta)$$

Since acceleration is constant, $x=\dot{x}_\circ t+\frac{1}{2}\ddot{x}t^2$:

$$x=v_\circ t-\frac{gt^2}{2}(\sin\theta+\mu\cos\theta)$$

Meanwhile $(\mu mg\cos\theta)a=I\dot{\omega}=\frac{2}{5}ma^2\dot{\omega}$

$$\dot{\omega}=\frac{5}{2}\frac{\mu g\cos\theta}{a}$$

$$\omega=\frac{5}{2}\frac{\mu g\cos\theta}{a}t$$

The ball begins pure rolling when $v = a\omega$ …

$$v = v_\circ + \ddot{x}t = v_\circ - g\left(\sin\theta + \mu\cos\theta\right)t = a\frac{5}{2}\frac{\mu g\cos\theta}{a}t$$

$$t = \frac{2v_\circ}{g\left(2\sin\theta + 7\mu\cos\theta\right)}$$

At that time:

$$x = \frac{2v_\circ^2}{g\left(2\sin\theta + 7\mu\cos\theta\right)} - \frac{g}{2}\frac{4v_\circ^2\left(\sin\theta + \mu\cos\theta\right)}{g^2\left(2\sin\theta + 7\mu\cos\theta\right)^2}$$

$$x = \frac{2v_\circ^2}{g}\frac{\left(\sin\theta + 6\mu\cos\theta\right)}{\left(2\sin\theta + 7\mu\cos\theta\right)^2}$$

8.20 $m\ddot{x} = \mu mg$

$$\ddot{x} = \mu g$$

$\dot{x} = \mu gt$, and $x = \frac{1}{2}\mu gt^2$

$$I\dot{\omega} = \frac{2}{5}ma^2\dot{\omega} = -\mu mga$$

$$\dot{\omega} = -\frac{5}{2}\frac{\mu g}{a}$$

$$\omega = \omega_\circ - \frac{5}{2}\frac{\mu g}{a}t$$

Slipping ceases to occur when $v = a\omega$ …

$$\mu gt = a\omega_\circ - \frac{5}{2}\mu gt$$

$$t = \frac{2}{7}\frac{a\omega_\circ}{g}$$

$$x = \frac{1}{2}\mu g\left(\frac{2}{7}\frac{a\omega_\circ}{\mu g}\right)^2$$

$$x = \frac{2}{49}\frac{a^2\omega_\circ^2}{\mu g}$$

8.21 Let the moments of inertia of A and B be $I_a\left(=\frac{1}{2}M_a a^2\right)$ and $I_b\left(=\frac{1}{2}M_b b^2\right)$.

The angular velocity of A is $\dot{\alpha}$ while that of B is $\dot{\beta} - \dot{\alpha} + \dot{\phi}$ (remember that in two dimensions, angular velocity is the rate of change of an angle between an line or direction fixed to the body and one fixed in space). For rolling contact, lengths traveled along the perimeters of the disks A and B must be equal to the arc length traveled along the track C.

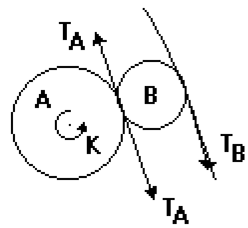
$$a\phi = b\beta = \left(a + 2b\right)\left(\alpha - \phi\right)$$

so that $\phi = \dfrac{(a+2b)\alpha}{2(a+b)}$ and $\beta = \dfrac{a(a+2b)\alpha}{2b(a+b)}$

After some algebra … the angular velocity of B is found to be …

$$\omega_B = \dot{\beta} - \dot{\alpha} + \dot{\phi} = \frac{a\dot{\alpha}}{2b}$$

For A, we take moments about O and for B we take moments about its center. Call T_A and T_B the components of the reaction forces tangent to A and B (the "upward-going" T_A acts on disk B. The "downward-going" T_A acts on disk A)



Thus $K - T_A a = I_A \ddot{\alpha}$ (Torque on Disk A)

$$-T_A b - T_B b = -I_B\left(\ddot{\beta} - \ddot{\alpha} + \ddot{\phi}\right) = -I_B a \frac{\ddot{\alpha}}{2b} \qquad \text{(Torque on Disk B)}$$

$$T_A - T_B = M_B(a+b)\left(\ddot{\alpha} - \ddot{\phi}\right) = \frac{1}{2} M_B a \ddot{\alpha} \qquad \text{(Force on Disk B)}$$

Eliminate T_A and T_B

$$K = \frac{\ddot{\alpha}}{4b^2}\left(4b^2 I_A + M_B a^2 b^2 + a^2 I_B\right)$$

Integrating this equation gives :

$$Kt = \frac{\dot{\alpha}}{4b^2}\left(4b^2 I_A + M_B a^2 b^2 + a^2 I_B\right)$$

Putting $\omega_A = \dot{\alpha}$ at $t = t_\circ$ gives

$$\omega_A = \frac{4b^2 K t_\circ}{\left(4b^2 I_A + M_B a^2 b^2 + a^2 I_B\right)}$$

Putting in values for I_A and I_B gives

$$\omega_A = \frac{4Kt_\circ}{a^2\left(2M_A + \frac{3}{2}M_B\right)}$$

Since the angular velocity of B is $\omega_B = \dot{\beta} - \dot{\alpha} + \dot{\phi} = \dfrac{a\dot{\alpha}}{2b}$, we have

$$\omega_B = \frac{a}{2b}\omega_A = \frac{2Kt_\circ}{ab\left(2M_A + \frac{3}{2}M_B\right)}$$

8.22 From section 8.7 (see Figure 8.7.1), the instantaneous center of rotation is the point O. If x is the distance from the center of mass to O and $\frac{l}{2}$ is the distance from the center of mass to the center of percussion O', then from eqn. 8.7.10 …

$$x\left(\frac{l}{2}\right)=\frac{I_{cm}}{M}=\frac{Ml^2}{12}\left(\frac{1}{M}\right)=\frac{l^2}{12}$$

$$x=\frac{l}{6}$$

8.23 In order that no reaction occurs between the table surface and the ball, the ball must approach and recede from its collision with the cushion by rolling without slipping. Using a prime to denote velocity and rotational velocity after the collision:

$$v'_{cm}=v_{cm}-\frac{\hat{P}}{m}$$

$$\omega'_{cm}=\omega_{cm}-\frac{\hat{P}}{I}(h-a)$$

The conditions for no reaction are $v_{cm}=a\omega_{cm}$ and $v'_{cm}=a\omega'_{cm}$.

$$a\omega_{cm}-\frac{\hat{P}}{m}=a\left[\omega_{cm}-\frac{\hat{P}}{I}(h-a)\right]$$

$$\frac{1}{m}=\frac{1}{I}a(h-a)$$

$$I=\frac{2}{5}ma^2$$

$$h=\frac{2ma^2}{5ma}+a=\frac{7}{5}a$$

$$a=\frac{d}{2}$$

$$h=\frac{7}{10}d$$

8.24 During the collision, angular momentum about the point 0 is conserved:

$$m'v_\circ l'=I\dot{\theta}$$

$$\dot{\theta}=\frac{m'v_\circ l'}{I}$$

After the collision, energy is conserved:

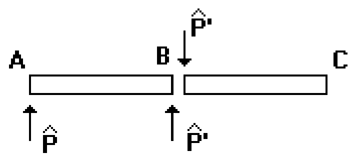
$$\frac{1}{2}I\dot{\theta}^2-mg\frac{l}{2}-m'gl'=-mg\frac{l}{2}\cos\theta_\circ-m'gl'\cos\theta_\circ$$

$$\frac{1}{2}\frac{m'^2v_\circ^2l'^2}{I}=g(1-\cos\theta_\circ)\left(\frac{ml}{2}+m'l'\right)$$

$$I = \frac{ml^2}{3} + m'l'^2$$

$$v_\circ = \frac{1}{m'l'}\left[2g\left(1-\cos\theta_\circ\right)\left(\frac{ml}{2}+m'l'\right)\left(\frac{ml^2}{3}+m'l'^2\right)\right]^{\frac{1}{2}}$$

8.25



The effect of rod BC acting on rod AB is impulse $+\hat{P}'$.
The effect of AB on BC is $-\hat{P}'$.

$$mv_1 = \hat{P} + \hat{P}' \qquad\qquad mv_2 = -\hat{P}'$$

$$I\omega_1 = \frac{l}{2}\left(\hat{P} - \hat{P}'\right) \qquad\qquad I\omega_2 = -\frac{l}{2}\hat{P}'$$

$$I = \frac{ml^2}{12}$$

$$\omega_1 = \frac{6}{ml}\left(\hat{P} - \hat{P}'\right) \qquad\qquad \omega_2 = -\frac{6}{ml}\hat{P}'$$

$$v_B = v_1 - \frac{l}{2}\omega_1 \qquad\qquad v_B = v_2 + \frac{l}{2}\omega_2$$

$$v_B = \frac{\hat{P}+\hat{P}'}{m} - \frac{l}{2}\left(\frac{6}{ml}\right)\left(\hat{P}-\hat{P}'\right)$$

$$v_B = -\frac{\hat{P}'}{m} + \frac{l}{2}\left(-\frac{6}{ml}\hat{P}'\right)$$

$$\hat{P} + \hat{P}' - 3\left(\hat{P} - \hat{P}'\right) = -\hat{P}' - 3\hat{P}'$$

$$8\hat{P}' = 2\hat{P}$$

$$\hat{P}' = \frac{\hat{P}}{4}$$

$$v_1 = \frac{5\hat{P}}{4m} \qquad\qquad v_2 = -\frac{\hat{P}}{4m}$$

$$\omega_1 = \frac{9\hat{P}}{2ml} \qquad\qquad \omega_2 = -\frac{3\hat{P}}{2ml}$$

$$v_B = -\frac{\hat{P}}{m}$$
