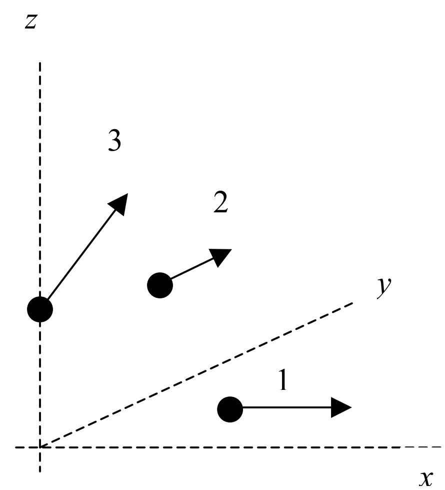
Chapter 7
Dynamics of Systems of Particles



7.1 From eqn. 7.1.1, $\vec{r}_{cm} = \frac{1}{m}\sum_i m_i \vec{r}_i$

$$\vec{r}_{cm} = \frac{1}{3}\left(\vec{r}_1 + \vec{r}_2 + \vec{r}_3\right) = \frac{1}{3}\left(\hat{i} + \hat{j} + \hat{j} + \hat{k} + \hat{k}\right)$$

$$\vec{r}_{cm} = \frac{1}{3}\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

$$\vec{v}_{cm} = \frac{d}{dt}\vec{r}_{cm} = \frac{1}{3}\left(\vec{v}_1 + \vec{v}_2 + \vec{v}_3\right) = \frac{1}{3}\left(2\hat{i} + \hat{j} + \hat{i} + \hat{j} + \hat{k}\right)$$

$$\vec{v}_{cm} = \frac{1}{3}\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$$

From eqn 7.1.3, $\vec{p} = \sum_i m_i \vec{v}_i = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$

$$\vec{p} = 3\hat{i} + 2\hat{j} + \hat{k}$$

7.2 (a) From eqn. 7.2.15, $T = \sum_i \frac{1}{2} m_i v_i^2$

$$T = \frac{1}{2}\left[2^2 + 1^2 + \left(1^2 + 1^2 + 1^2\right)\right] = 4$$

(b) From Prob. 7.1, $\vec{v}_{cm} = \frac{1}{3}\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$

$$\frac{1}{2} m v_{cm}^2 = \frac{1}{2} \times 3 \times \frac{1}{9}\left(3^2 + 2^2 + 1^2\right) = 2\frac{1}{3}$$

(c) From eqn.7.2.8, $\vec{L} = \sum_i \vec{r}_i \times m\vec{v}_i$

$$\vec{L} = \left[\left(\hat{i} + \hat{j}\right) \times 2\hat{i}\right] + \left[\left(\hat{j} + \hat{k}\right) \times \hat{j}\right] + \left[\hat{k} \times \left(\hat{i} + \hat{j} + \hat{k}\right)\right]$$

$$\vec{L} = \left(-2\hat{k}\right) + \left(-\hat{i}\right) + \left(\hat{j} - \hat{i}\right) = -2\hat{i} + \hat{j} - 2\hat{k}$$

7.3 $\vec{v}_\circ = \vec{v}_b - \vec{v}_g$

Since momentum is conserved and the bullet and gun were initially at rest:

$$0 = m\vec{v}_b + M\vec{v}_g$$

$$\vec{v}_g = -\gamma \vec{v}_b, \qquad \gamma = \frac{m}{M}$$

$$\vec{v}_\circ = \left(1 + \gamma\right)\vec{v}_b$$

$$\vec{v}_b = \frac{\vec{v}_\circ}{1 + \gamma}$$

$$\vec{v}_g = -\frac{\gamma \vec{v}_\circ}{1+\gamma}$$

7.4 Momentum is conserved: $mv_\circ = m\left(\frac{v_\circ}{2}\right) + Mv_{blk}$

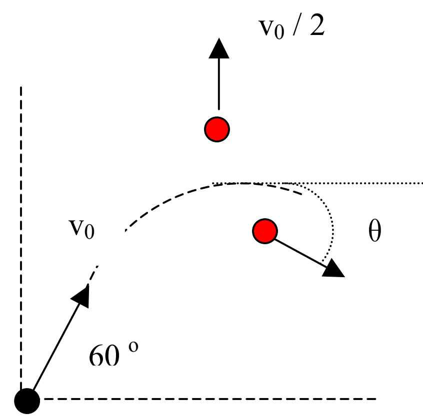
$$v_{blk} = \frac{1}{2}\gamma v_\circ \qquad \gamma = \frac{m}{M}$$

$$T_i - T_f = \frac{1}{2}mv_\circ^2 - \left[\frac{1}{2}m\left(\frac{v_\circ}{2}\right)^2 + \frac{1}{2}M\left(\frac{\gamma v_\circ}{2}\right)^2\right]$$

$$= \frac{1}{2}mv_\circ^2\left(1 - \frac{1}{4} - \frac{1}{m}M\frac{\gamma^2}{4}\right)$$

$$\frac{T_i - T_f}{T_i} = \frac{3}{4} - \frac{\gamma}{4}$$

7.5 At the top of the trajectory:



$$\vec{v} = \hat{i}v_\circ \cos 60^\circ = \hat{i}\frac{v_\circ}{2}$$

Momentum is conserved:

$$\hat{i}m\frac{v_\circ}{2} = \hat{j}\left(\frac{m}{2}\right)\left(\frac{v_\circ}{2}\right) + \frac{m}{2}\vec{v}_2$$

$$\vec{v}_2 = \hat{i}v_\circ - \hat{j}\frac{v_\circ}{2}$$

Direction: $\theta = \tan^{-1}\left(\frac{-\frac{v_\circ}{2}}{v_\circ}\right) = 26.6^\circ$ below the horizontal.

Speed: $v_2 = \left[v_\circ^2 + \left(\frac{v_\circ}{2}\right)^2\right]^{\frac{1}{2}} = 1.118\, v_\circ$

7.6 When a ball reaches the floor, $\frac{1}{2}mv^2 = mgh$.

As a result of the bounce, $\frac{v'}{v} = \varepsilon$.

The height of the first bounce: $mgh' = \frac{1}{2}mv'^2$

$$h' = \frac{v'^2}{2g} = \frac{\varepsilon^2 v^2}{2g} = \varepsilon^2 h$$

Similarly, the height of the second bounce, $h'' = \varepsilon^2 h' = \varepsilon^4 h$

$$\text{Total distance} = h + 2\varepsilon^2 h + 2\varepsilon^4 h + \ldots = h\left(-1 + \sum_{n=0}^{\infty} 2\varepsilon^{2n}\right)$$

Now $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$, $|r| < 1$.

$$\left(-1 + \sum_{n=0}^{\infty} 2\varepsilon^{2n}\right) = -1 + \frac{2}{1-\varepsilon^2} = \frac{1+\varepsilon^2}{1-\varepsilon^2}$$

$$\text{total distance} = h\left(\frac{1+\varepsilon^2}{1-\varepsilon^2}\right)$$

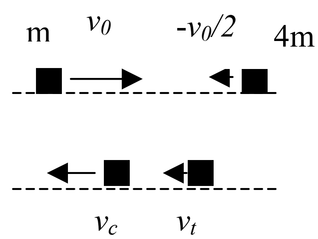
For the first fall, $\frac{1}{2} g t_\circ^2 = h$, so $t_\circ = \sqrt{\dfrac{2h}{g}}$

For the fall from height h': $t_1 = \sqrt{\dfrac{2h'}{g}} = \varepsilon\sqrt{\dfrac{2h}{g}}$

Accounting for equal rise and fall times:

$$\text{Total time} = \sqrt{\frac{2h}{g}}\left(1 + 2\varepsilon + 2\varepsilon^2 + \ldots\right) = \sqrt{\frac{2h}{g}}\left(-1 + \sum_{n=0}^{\infty} 2\varepsilon^n\right)$$

$$\text{Total time} = \sqrt{\frac{2h}{g}}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)$$



7.7 From eqn. 7.5.5:

$$\dot{x}_1' = \frac{(m_1 - \varepsilon m_2)\dot{x}_1 + (m_2 + \varepsilon m_2)\dot{x}_2}{m_1 + m_2}$$

$$\dot{x}_2' = \frac{(m_1 + \varepsilon m_1)\dot{x}_1 + (m_2 - \varepsilon m_1)\dot{x}_2}{m_1 + m_2}$$

$$v_c = \frac{\left(m - \frac{1}{4}4m\right)v_\circ + \left(4m + \frac{1}{4}4m\right)\left(-\frac{v_\circ}{2}\right)}{m + 4m} = \frac{0 + 5m\left(-\frac{v_\circ}{2}\right)}{5m} = -\frac{v_\circ}{2}$$

$$v_t = \frac{\left(m + \frac{1}{4}m\right)v_\circ + \left(4m - \frac{1}{4}m\right)\left(-\frac{v_\circ}{2}\right)}{m + 4m}$$

$$= \frac{\frac{5}{4}mv_\circ + \frac{15}{4}m\left(-\frac{v_\circ}{2}\right)}{5m} = -\frac{v_\circ}{8}$$

Both car and truck are traveling in the initial direction of the truck with speeds $\dfrac{v_\circ}{2}$ and $\dfrac{v_\circ}{8}$, respectively.

7.8 From eqn. 7.2.15, $T=\sum_i \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$

Meanwhile:

$$\frac{1}{2}mv_{cm}^2 + \frac{1}{2}\mu v^2 = \frac{1}{2}m\left(\frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m}\right)^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\left(\vec{v}_1 - \vec{v}_2\right)^2$$

$$= \frac{1}{2m}\left[m_1^2 v_1^2 + m_2^2 v_2^2 + 2m_1 m_2 \vec{v}_1\cdot\vec{v}_2 + m_1 m_2\left(v_1^2 + v_2^2 - 2\vec{v}_1\cdot\vec{v}_2\right)\right]$$

$$= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

Therefore, $T = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}\mu v^2$

7.9 From Prob. 7.8, $T = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}\mu v^2$

$Q = T - T'$ and since $v_{cm} = v'_{cm}$:

$$Q = \frac{1}{2}\mu v^2 - \frac{1}{2}\mu v'^2$$

From eqn. 7.5.4, $\varepsilon = \frac{v'}{v}$

$$Q = \frac{1}{2}\mu v^2\left(1-\varepsilon^2\right)$$

7.10 Conservation of momentum:

$$m_1 v_1 = m_1 v'_1 + m_2 v'_2$$

$$v'_1 = v_1 - \frac{m_2}{m_1}v'_2$$

$$v_1'^2 = v_1^2 - \frac{2m_2}{m_1}v_1 v'_2 + \frac{m_2^2}{m_1^2}v_2'^2$$

Conservation of energy:

$$\frac{1}{2}m_1 v_1^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2$$

$$\frac{1}{2}m_1 v_1^2 = \frac{1}{2}m_1 v_1^2 - m_2 v_1 v'_2 + \frac{m_2^2}{2m_1}v_2'^2 + \frac{1}{2}m_2 v_2'^2$$

$$\frac{m_2}{2}\left(\frac{m_2}{m_1}+1\right)v_2'^2 - m_2 v_1 v'_2 = 0$$

$$v'_2 = \frac{2m_1 v_1}{m_1+m_2}$$

$$T_1 - T'_1 = \frac{1}{2}m_1 v_1^2 - \frac{1}{2}m_1 v_1'^2 = \frac{1}{2}m_2 v_2'^2 = \frac{2m_1^2 m_2}{\left(m_1+m_2\right)^2}v_1^2$$

$$\frac{T_1 - T_1'}{T_1} = \frac{\frac{2m_1\mu}{m}v_1^2}{\frac{1}{2}m_1 v_1^2} = \frac{4\mu}{m}$$

7.11 From eqn. 7.2.14, $\vec{L} = \vec{r}_{cm} \times m\vec{v}_{cm} + \sum_i \vec{\bar{r}}_i \times m_i \vec{\bar{v}}_i$

$$\sum_i \vec{\bar{r}}_i \times m_i \vec{\bar{v}}_i = \vec{\bar{r}}_1 \times m_1 \vec{\bar{v}}_1 + \vec{\bar{r}}_2 \times m_2 \vec{\bar{v}}_2$$

From eqn. 7.3.2, $\vec{R} = \vec{\bar{r}}_1\left(1 + \frac{m_1}{m_2}\right) = \vec{\bar{r}}_1\left(\frac{m_1 + m_2}{m_2}\right) = \frac{m_1}{\mu}\vec{\bar{r}}_1$

Since from eqn. 7.3.1, $\vec{\bar{r}}_1 = -\frac{m_2}{m_1}\vec{\bar{r}}_2$

$$\vec{R} = -\frac{m_2}{\mu}\vec{\bar{r}}_2$$

$$\sum_i \vec{\bar{r}}_i \times m_i \vec{\bar{v}}_i = \frac{\mu}{m_1}\vec{R} \times m_1 \vec{\bar{v}}_1 + \left(-\frac{\mu}{m_2}\right)\vec{R} \times m_2 \vec{\bar{v}}_2$$

$$= \mu\vec{R} \times \left(\vec{\bar{v}}_1 - \vec{\bar{v}}_2\right) = \vec{R} \times \mu\vec{v}$$

$$\vec{L} = \vec{r}_{cm} \times m\vec{v}_{cm} + \vec{R} \times \mu\vec{v}$$

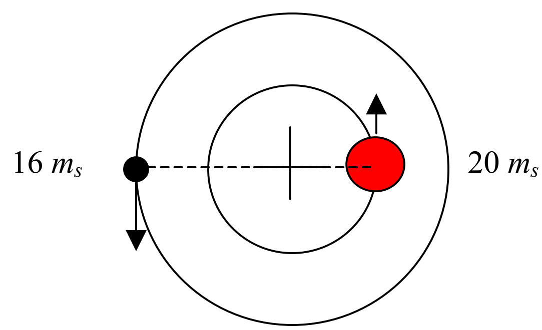
7.12 Let m_s = mass of Sun and a_e = semi-major axis of Earth's orbit then from eqn. 7.3.9c,

$$\tau = \left(\frac{m_1}{m_s} + \frac{m_2}{m_s}\right)^{-\frac{1}{2}}\left(\frac{a}{a_e}\right)^{\frac{3}{2}} yr$$

$$\frac{a}{a_e} = \tau^{\frac{2}{3}}\left(\frac{m_1}{m_s} + \frac{m_2}{m_s}\right)^{\frac{1}{3}}$$

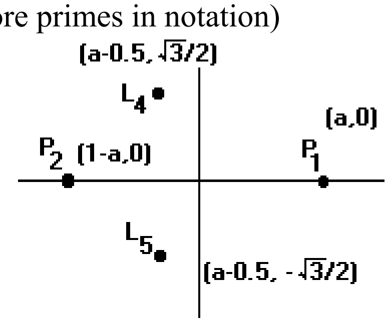
$$= \left(5.6\,da \times \frac{1yr}{365\,da}\right)^{\frac{2}{3}}\left(20 + 16\right)^{\frac{1}{3}} yr^{-\frac{2}{3}}$$

$$a = 0.20\,a_e = \frac{1}{5}a_e$$



7.13 (Ignore primes in notation)

a)



The coordinates of the two primaries, P1 and P2, are shown at left – along with the coordinates of L_4 and L_5.

b) $V(x,y) = -\dfrac{(1-\alpha)}{\left[(x-\alpha)^2 + y^2\right]^{\frac{1}{2}}} - \dfrac{\alpha}{\left[(x+1-\alpha)^2 + y^2\right]^{\frac{1}{2}}} - \dfrac{x^2+y^2}{2}$ (7.4.13)

$$\frac{\partial V}{\partial x} = \frac{(1-\alpha)(x-\alpha)}{[\ ]^{\frac{3}{2}}} + \frac{\alpha(x+1-\alpha)}{[\ ]^{\frac{3}{2}}} - x$$

Now $x = \alpha - 0.5$ at L_4 and L_5

also, each bracket term in the denominator equals 1 at L_4, L_5

$$\frac{\partial V}{\partial x} = (1-\alpha)(\alpha - 0.5 - \alpha) + \alpha(\alpha - 0.5 + 1 - \alpha) - (\alpha - 0.5)$$

$$= -0.5 + 0.5\alpha + 0.5\alpha - \alpha + 0.5 \equiv 0$$

$$\frac{\partial V}{\partial y} = \frac{(1-\alpha)y}{[\ ]^{\frac{3}{2}}} + \frac{\alpha y}{[\ ]^{\frac{3}{2}}} - y$$

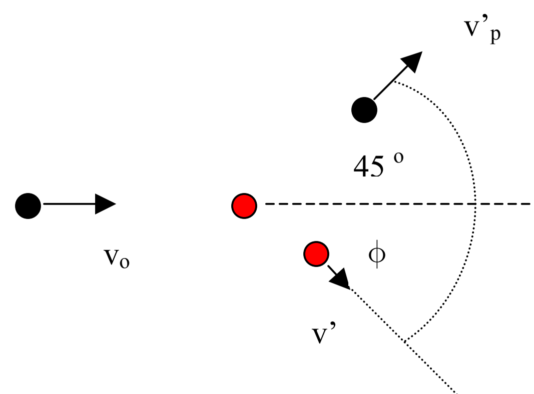
Again, the denominator in brackets equals 1 @ L_4, L_5

So, $\dfrac{\partial V}{\partial y} = (1-\alpha)\left(\pm\dfrac{\sqrt{3}}{2}\right) + \alpha\left(\pm\dfrac{\sqrt{3}}{2}\right) - \left(\pm\dfrac{\sqrt{3}}{2}\right)$

$$= \pm\frac{\sqrt{3}}{2} \mp \alpha\frac{\sqrt{3}}{2} \pm \alpha\frac{\sqrt{3}}{2} \mp \frac{\sqrt{3}}{2} \equiv 0$$

Thus $\bar{\nabla} V(x,y) = \hat{i}\dfrac{\partial V}{\partial x} + \hat{j}\dfrac{\partial V}{\partial y} \equiv 0$ at L_4, L_5.

7.14 Conservation of momentum:



$$m_p\vec{v}_\circ = m_p\vec{v}'_p + 4m_p\vec{v}'_\alpha$$

$$v_\circ = v'_p\cos 45^\circ + 4v'_\alpha\cos\phi$$

$$4v'_\alpha\cos\phi = v_\circ - \frac{v'_p}{\sqrt{2}}$$

$$0 = v'_p\sin 45^\circ - 4v'_\alpha\sin\phi$$

$$4v'_\alpha\sin\phi = \frac{v'_p}{\sqrt{2}}$$

$$16v'^2_\alpha = v^2_\circ - \sqrt{2}\,v_\circ v'_p + v'^2_p$$

Conservation of energy:

$$\frac{1}{2}m_p v^2_\circ = \frac{1}{2}m_p v'^2_p + \frac{1}{2}4m_p v'^2_\alpha$$

$$16v'^2_\alpha = 4v^2_\circ - 4v'_p$$

Subtracting: $0 = -3v^2_\circ - \sqrt{2}\,v_\circ v'_p + 5v'^2_p$

$$v'_p = \frac{\sqrt{2}v_\circ \pm \sqrt{2v_\circ^2 + 60v_\circ^2}}{10} = \frac{v_\circ}{10}\left(\sqrt{2} \pm \sqrt{62}\right)$$

$v'_p > 0$, so the positive square root is used.

$$v'_p = 0.9288\,v_\circ$$

$$v'_{px} = v'_{py} = \frac{v'_p}{\sqrt{2}} = 0.657\,v_\circ$$

$$v'_\alpha = \frac{1}{2}\left(v_\circ^2 - v'^2_p\right)^{\frac{1}{2}} = \frac{v_\circ}{2}\left(1 - .9288^2\right)^{\frac{1}{2}}$$

$$v'_\alpha = 0.1853 v_\circ$$

$$\tan\phi = \frac{\frac{v'_p}{\sqrt{2}}}{v_\circ - \frac{v'_p}{\sqrt{2}}} = \frac{v'_p}{\sqrt{2}v_\circ - v'_p} = \frac{.9288}{\sqrt{2} - .9288}$$

$$\phi = \tan^{-1} 1.9134 = 62.41^\circ$$

$$v'_{\alpha x} = v'_\alpha \cos\phi = 0.086\,v_\circ$$

$$v'_{\alpha y} = -v'_\alpha \sin\phi = -0.164\,v_\circ$$

7.15 Conservation of energy:

$$\frac{1}{2}m_p v_\circ^2 = \frac{1}{2}m_p v'^2_p + \frac{1}{2}4m_p v'^2_\alpha + \frac{1}{4}\left(\frac{1}{2}m_p v_\circ^2\right)$$

$$16v'^2_\alpha = 3v_\circ^2 - 4v'^2_p$$

From the conservation of momentum eqn of Prob. 7.14:

$$16v'^2_\alpha = v_\circ^2 - \sqrt{2}v_\circ v'_p + v'^2_p$$

Subtracting: $0 = -2v_\circ^2 - \sqrt{2}v_\circ v'_p + 5v'^2_p$

$$v'_p = \frac{\sqrt{2}v_\circ \pm \sqrt{2v_\circ^2 + 40v_\circ^2}}{10} = \frac{v_\circ}{10}\left(\sqrt{2} \pm \sqrt{42}\right)$$

Using the positive square root, since $v'_p > 0$:

$$v'_p = 0.7895\,v_\circ$$

$$v'_{px} = v'_{py} = \frac{v'_p}{\sqrt{2}} = 0.558\,v_\circ$$

$$v'_\alpha = \left(\frac{3}{16}v_\circ^2 - \frac{1}{4}v'^2_p\right)^{\frac{1}{2}} = \frac{v_\circ}{2}\left(.75 - .7895^2\right)^{\frac{1}{2}}$$

$$v'_\alpha = 0.1780 v_\circ$$

From the conservation of momentum eqns of Prob. 7.14:

$$\tan\phi = \frac{v'_p}{\sqrt{2}v_\circ - v'_p} = \frac{.7895}{\sqrt{2} - .7895}$$

$$\phi = \tan^{-1} 1.2638 = 51.65^\circ$$
$$v'_{\alpha x} = v'_\alpha \cos\phi = 0.110\, v_\circ$$
$$v'_{\alpha y} = -v'_\alpha \sin\phi = -0.140\, v_\circ$$

7.16 From eqn. 7.6.14, $\tan\phi_1 = \dfrac{\sin\theta}{\gamma + \cos\theta}$ ϕ_1 and θ are the scattering angles in the Lab and C.M. frames respectively.

From eqn. 7.6.16, for Q = 0, $\gamma = \dfrac{m_1}{m_2}$

$$\tan 45^\circ = 1 = \frac{\sin\theta}{\frac{1}{4} + \cos\theta}$$

$$\frac{1}{4} + \cos\theta = \sin\theta \qquad \text{and squaring …}$$

$$\frac{1}{16} + \frac{1}{2}\cos\theta + \cos^2\theta = 1 - \cos^2\theta$$

$$2\cos^2\theta + \frac{1}{2}\cos\theta - \frac{15}{16} = 0$$

$$\cos\theta = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{15}{2}}}{4} = -.125 \pm .696$$

Since $0 < \theta < \dfrac{\pi}{2}$, $\theta = \cos^{-1} .571 \approx 55.2^\circ$

7.17 From eqn. 7.6.14, $\tan\phi_1 = \dfrac{\sin\theta}{\gamma + \cos\theta}$

From eqn. 7.6.18, $\gamma = \dfrac{m_1}{m_2}\left[1 - \dfrac{Q}{T}\left(1 + \dfrac{m_1}{m_2}\right)\right]^{-\frac{1}{2}}$

$$\gamma = \frac{1}{4}\left[1 - \frac{1}{4}\left(1 + \frac{1}{4}\right)\right]^{-\frac{1}{2}} = 0.3015$$

$$\tan 45^\circ = \frac{\sin\theta}{.3015 + \cos\theta}$$

$.3015 + \cos\theta = \sin\theta$ (since $\sin\theta > \cos\theta$, $\theta > 45^\circ$)

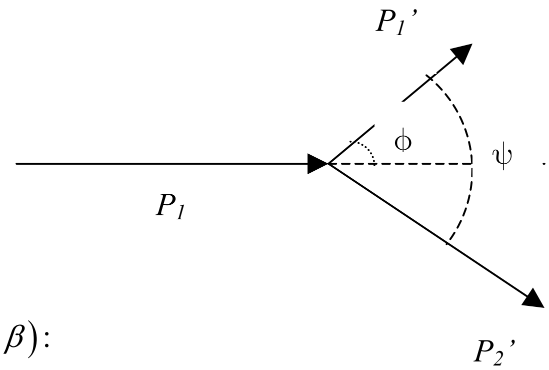
$$.3015^2 = \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta$$

Using the identity $2\sin\theta\cos\theta = \sin 2\theta$

$$\sin 2\theta = 1 - .3015^2 = 0.9091$$

Since $\theta > 45^\circ$, $2\theta > 90^\circ$: $2\theta = \sin^{-1} .9091 = 114.62^\circ$

$$\theta = 57.3^\circ$$



7.18 Conservation of momentum:

$$P_1 = P_1' \cos\phi + P_2' \cos(\psi - \phi)$$

$$0 = P_1' \sin\phi - P_2' \sin(\psi - \phi)$$

From Appendix B for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$:

$$P_1 = P_1' \cos\phi + P_2' (\cos\psi\cos\phi + \sin\psi\sin\phi)$$

$$0 = P_1' \sin\phi - P_2' (\sin\psi\cos\phi - \cos\psi\sin\phi)$$

$$P_1^2 = P_1'^2 \cos^2\phi + P_2'^2 \left(\cos^2\psi\cos^2\phi + 2\cos\psi\cos\phi\sin\phi\sin\psi + \sin^2\psi\sin^2\phi\right)$$
$$+2P_1'P_2'\left(\cos^2\phi\cos\psi + \cos\phi\sin\psi\sin\phi\right)$$

$$0 = P_1'^2 \sin^2\phi + P_2'^2 \left(\sin^2\psi\cos^2\phi - 2\sin\psi\cos\phi\cos\psi\sin\phi + \cos^2\psi\sin^2\phi\right)$$
$$-2P_1'P_2'\left(\sin\phi\sin\psi\cos\phi - \cos\psi\sin^2\phi\right)$$

Adding: $P_1^2 = P_1'^2 + P_2'^2 + 2P_1'P_2'\cos\psi$

Conservation of energy:

$$\frac{P_1^2}{2m} = \frac{P_1'^2}{2m} + \frac{P_2'^2}{2m} + Q$$

$$Q = \frac{1}{2m}\left(P_1^2 - P_1'^2 - P_2'^2\right) = \frac{1}{2m}\left(2P_1'P_2'\cos\psi\right)$$

$$Q = \frac{P_1'P_2'\cos\psi}{m}$$

7.19 $T_1 = \frac{1}{2}m_1v_1^2$ $\qquad$ $T_1' = \frac{1}{2}m_1v_1'^2$

let $r = \frac{T_1'}{T_1} = \frac{v_1'^2}{v_1^2}$ … ratio of scattered particle to incident particle energy

Looking at Figure 7.6.2 …

$$\bar{v}_1' \cdot \bar{v}_1' = (v_1' - v_{cm}) \cdot (v_1' - v_{cm})$$

$$\bar{v}_1'^2 = v_1'^2 + v_{cm}^2 - 2v_1'v_{cm}\cos\phi_1$$

hence $v_1'^2 = \bar{v}_1'^2 - v_{cm}^2 + 2v_1'v_{cm}\gamma$ $\qquad$ where $\gamma = \cos\phi_1$

$$\therefore r = \frac{\bar{v}_1'^2}{v_1^2} - \frac{v_{cm}^2}{v_1^2} + \frac{2v_1'v_{cm}\gamma}{v_1^2}$$

but $\bar{v}_1' = \bar{v}_1$ …the center of mass speeds of the incident and scattered particle are the same.

$$\frac{\bar{v}_1'}{v_1} = \frac{m_2}{m_2 + m_1} = \frac{\alpha}{1+\alpha}$$ …from equation 7.6.12 where $\alpha = \frac{m_2}{m_1}$

$$\frac{v_{cm}}{v_1}=\frac{m_1}{m_2+m_1}=\frac{1}{1+\alpha} \qquad \text{Equation 7.6.11}$$

Thus

$$r=\frac{\alpha^2}{(1+\alpha)^2}-\frac{1}{(1+\alpha)^2}+\frac{2\gamma}{(1+\alpha)} \qquad \frac{v_1'}{v_1}=\frac{\alpha^2}{(1+\alpha)^2}-\frac{1}{(1+\alpha)^2}+\frac{2\gamma}{(1+\alpha)}r^{\frac{1}{2}}$$

Simplifying

$$r-\frac{2\gamma}{1+\alpha}r^{\frac{1}{2}}+\left(\frac{1-\alpha}{1+\alpha}\right)=0$$

Let $x^2=r$ and solving the resulting quadratic for x

$$x=\frac{\gamma}{1+\alpha}+\frac{1}{1+\alpha}\left[\gamma^2-\left(1-\alpha^2\right)\right]^{\frac{1}{2}}$$

Squaring

$$r=x^2=\frac{1}{(1+\alpha)^2}\left[2\gamma^2+\alpha^2-1+2\gamma\left(\gamma^2+\alpha^2-1\right)^{\frac{1}{2}}\right]$$

Now $\frac{\Delta T_1}{T_1}=1-r=1-\frac{1}{(1+\alpha)^2}\left[2\gamma^2+\alpha^2-1+2\gamma\left(\gamma^2+\alpha^2-1\right)^{\frac{1}{2}}\right]$

And, after a little algebra, we get the desired solution

$$\frac{\Delta T_1}{T_1}=\frac{2}{1+\alpha}-\frac{2\gamma}{(1+\alpha)^2}\left[\gamma+\sqrt{\gamma^2+\alpha^2-1}\right]$$

7.20 From Equation 7.6.15 … $\gamma=\frac{m_1 v_1}{\bar{v}_1'(m_1+m_2)}=\frac{m_1}{m_2(1+m_1/m_2)}\frac{v_1}{\bar{v}_1'}$

Now we solve for $\frac{v_1}{\bar{v}_1'}$ …

$v_1=\left(\frac{2T}{m_1}\right)^{\frac{1}{2}}$ and now solving for $\bar{v}_1'$ starting with Equation 7.6.9 …

$\frac{1}{2}\mu\bar{v}_1'^2=\frac{1}{2}\mu\bar{v}_1^{\,2}-Q$ and using $\bar{v}_1'=\frac{m_2}{m_1+m_2}v_1$ we get …

$$=\frac{1}{2}m_1 v_1^{\,2}\frac{1}{(1+m_1/m_2)}-Q=\frac{T}{(1+m_1/m_2)}-Q$$

$$\bar{v}_1'^2=\frac{2}{m_1(1+m_1/m_2)}\left[\frac{T}{(1+m_1/m_2)}-Q\right].$$

Thus, solving for γ …

$$\gamma = \frac{m_1}{m_2}\frac{\left(2T/m_1\right)^{\frac{1}{2}}}{\left(2/m_1\right)^{\frac{1}{2}}\left(1+m_1/m_2\right)^{\frac{1}{2}}\left[\frac{T}{\left(1+m_1/m_2\right)}-Q\right]^{\frac{1}{2}}}$$

Finally…

$$\gamma = \frac{m_1}{m_2}\frac{1}{\left[1-\frac{Q\left(1+m_1/m_2\right)}{T}\right]^{\frac{1}{2}}}$$

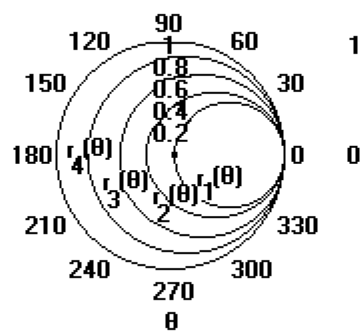
7.21 The time of flight, $\tau = \text{constant}$—so $\tau = \frac{r}{v_1'}$ but from problem 7.19 above

$$r = v_1'\tau = \frac{v_1\tau}{1+\alpha}\left[\gamma + \sqrt{\gamma^2+\alpha^2-1}\right]$$

As an example, let $v_1\tau = 1$ and we have

$r_1 = \gamma$	$\alpha = 1$	pp scattering
$r_2 = \frac{1}{3}\left[\gamma+\sqrt{\gamma^2+3}\right]$	$\alpha = 2$	p – D
$r_3 = \frac{1}{5}\left[\gamma+\sqrt{\gamma^2+15}\right]$	$\alpha = 4$	p – He
$r_4 = \frac{1}{13}\left[\gamma+\sqrt{\gamma^2+143}\right]$	$\alpha = 12$	p – C

Below is a polar plot of these four curves.



7.22 From eqn. 7.7.6, $F_u - F_g = m\dot{v} + v\dot{m}$

since $v = \text{constant}$, $\dot{v} = 0$

$\dot{m} = \lambda\dot{z} = \lambda v$, $\quad\lambda = \text{mass per unit length}$

$F_g = \left(\lambda z\right)g$

$$F_u = \lambda zg + \left(\lambda v\right)v = g\lambda\left(z+\frac{v^2}{g}\right)$$

F_u is equal to the weight of a length $z+\frac{v^2}{g}$ of chain.

7.23 $m = \frac{4}{3}\pi r^3 \rho$

$\dot{m} = 4\pi r^2 \rho \dot{r} \quad \propto \quad \pi r^2 \dot{z}$ where $v = \dot{z}$

$\dot{r} = k\dot{z}$ k a constant of proportionality

$r = r_\circ + kz$

From eqn. 7.7.6, $mg = m\dot{v} + v\dot{m}$

$$\frac{4}{3}\pi r^3 \rho g = \frac{4}{3}\pi r^3 \rho \ddot{z} + 4\pi r^2 \rho (k\dot{z})\dot{z}$$

$$g = \ddot{z} + \frac{3k\dot{z}^2}{r}$$

$$\ddot{z} = g - \frac{3\dot{z}^2}{z + \frac{r_\circ}{k}}$$

For $r_\circ = 0$, $\ddot{z} = g - \frac{3\dot{z}^2}{z}$

A series solution is used for this differential equation:

$$\dot{z}^2 = \sum_{n=0}^{\infty} a_n z^n$$

$$\ddot{z} = \frac{d\dot{z}}{dt} = \frac{d\dot{z}}{dz}\cdot\frac{dz}{dt} = \dot{z}\frac{d\dot{z}}{dz} = \frac{1}{2}\frac{d(\dot{z}^2)}{dz}$$

$$\frac{d(\dot{z}^2)}{dz} = \sum_n a_n n z^{n-1}$$

$$\frac{\dot{z}^2}{z} = \sum_n a_n z^{n-1}$$

$$\therefore \quad \ddot{z} = \frac{1}{2}\sum_n a_n n z^{n-1} = g - 3\sum_n a_n z^{n-1}$$

For $n = 1$: $\frac{1}{2}a_1 = g - 3a_1$

$$a_1 = \frac{2}{7}g$$

For $n \neq 1$: $\frac{1}{2}na_n = -3a_n$

Since n is an integer, $a_n = 0$ for $n \neq 1$

$$\dot{z}^2 = \frac{2}{7}g\,z$$

$$\ddot{z} = g - \frac{3}{z}\left(\frac{2}{7}g\,z\right) = \frac{g}{7}$$

7.24 From eqn. 7.7.6, $mg = m\dot{v} + v\dot{m}$, where m and v refer to the portion of the chain hanging over the edge of the table.

$m = \lambda z$ and $v = \dot{z}$ where λ is the mass per unit length of chain

$\dot{m} = \lambda \dot{z}$ and $\dot{v} = \ddot{z}$

$$\ddot{z} = \frac{d\dot{z}}{dt} = \frac{d\dot{z}}{dz}\cdot\frac{dz}{dt} = \dot{z}\frac{d\dot{z}}{dz} = \frac{1}{2}\frac{d(\dot{z}^2)}{dz}$$

$$\lambda z g = \lambda z\left(\frac{1}{2}\frac{d(\dot{z}^2)}{dz}\right) + \dot{z}\left(\lambda \dot{z}\right)$$

$$\ddot{z} = \frac{1}{2}\frac{d(\dot{z}^2)}{dz} = g - \frac{\dot{z}^2}{z}$$

Because of the initial condition $z_\circ = b \neq 0$, a normal power series solution to this differential equation (…as in Prob. 7.22) does not work. Instead, we use the Method of Frobenius …

$$\dot{z}^2 = \sum_{n=0}^{\infty} a_n z^{n+s}$$

$$\frac{d(\dot{z}^2)}{dz} = \sum_n a_n \left(n+s\right) z^{n+s-1}$$

$$\frac{\dot{z}^2}{z} = \sum_n a_n z^{n+s-1}$$

$$\ddot{z} = \frac{1}{2}\sum_n a_n \left(n+s\right) z^{n+s-1} = g - \sum_n a_n z^{n+s-1}$$

Equality can be attained for $a_n \neq 0$ at $n = 0$ and $n = 3$

…otherwise $a_n = 0$ $\quad n \neq 0, 3$

For $n = 0$, $\frac{1}{2}a_\circ s = -a_\circ$

$$s = -2$$

$$\ddot{z} = \frac{1}{2}\sum_n a_n \left(n-2\right) z^{n-3} = g - \sum_n a_n z^{n-3}$$

For $n = 3$, $\frac{1}{2}a_3 = g - a_3$

$$a_3 = \frac{2}{3}g$$

For all $n \neq 0$, 3: $\frac{1}{2}a_n\left(n-2\right) = -a_n$

$$a_n = 0, \qquad n \neq 0, 3.$$

$$\dot{z}^2 = a_\circ z^{-2} + \frac{2}{3}gz$$

At $t=0$, $\dot{z}=0$, and $z=b$

$$0=\frac{a_\circ}{b^2}+\frac{2gb}{3}$$

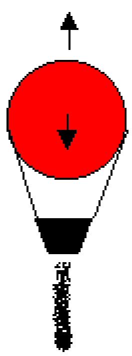
$$a_\circ=-\frac{2}{3}gb^3$$

$$\dot{z}^2=-\frac{2}{3}g\frac{b^3}{z^2}+\frac{2}{3}gz$$

At $z=a$, $\dot{z}^2=\frac{2}{3}g\left(a-\frac{b^3}{a^2}\right)=\frac{2g}{3a^2}\left(a^3-b^3\right)$

$$\dot{z}=\left[\frac{2g}{3a^2}\left(a^3-b^3\right)\right]^{\frac{1}{2}}$$

7.25 Initially, the upward buoyancy force balances the weight of the balloon and sand.



$$F_B-\left(M+m_\circ\right)g=0 \qquad (1)$$

Let $m=m(t)-$the mass of sand at time t where $0\le t\le t_\circ$

$$m=m_\circ\left(1-\frac{t}{t_\circ}\right) \qquad (2)$$

The velocity of sand relative to the balloon is zero upon release so $\bar{V}=0$ in equation 7.7.5 … there is no upward "rocket-thrust."

As sand is released, the net upward force is the difference between the initial buoyancy force, F_B, and the weight of the balloon and remaining sand. Let y be the subsequent displacement of the balloon, so equation 7.7.5 reduces to $F = ma$

$$F_B-\left(M+m\right)g=\left(M+m\right)\frac{dv}{dt}$$

and using (1) and (2) above we get

$$\frac{dv}{dt}=\frac{m_\circ gt}{\left(M+m_\circ\right)t_\circ-m_\circ t}=-g+\frac{\left(M+m_\circ\right)gt_\circ}{\left(M+m_\circ\right)t_\circ-m_\circ t}$$

whose solution is:

$$v=\frac{dy}{dt}=-gt-\frac{\left(M+m_\circ\right)gt_\circ}{m_\circ}\ln\left(1-\frac{m_\circ t}{\left(M+m_\circ\right)t_\circ}\right)$$

$$y=C-\int\left(gt+\frac{g}{k}\ln\left(1-kt\right)\right)dt\,, \qquad k=\frac{m_\circ}{t_\circ\left(M+m_\circ\right)}$$

$$=C-\frac{1}{2}gt^2-\frac{gt}{k}\ln\left(1-kt\right)-g\int\frac{tdt}{1-kt}$$

Integrating by parts

$$= C - \frac{1}{2}gt^2 - \frac{gt}{k}\ln(1-kt) - \frac{g}{k}\int\left(-1+\frac{1}{1-kt}\right)dt$$

$$= C - \frac{1}{2}gt^2 - \frac{gt}{k}\ln(1-kt) + \frac{gt}{k} + \frac{g}{k^2}\ln(1-kt)$$

$$= C - \frac{1}{2}gt^2 + \frac{g}{k^2}(1-kt)\ln(1-kt) + \frac{gt}{k}$$

but $y = 0$ at $t = 0$ so $C = 0$

$$y = \frac{gt}{k} - \frac{1}{2}gt^2 + \frac{g}{k^2}(1-kt)\ln(1-kt)$$

and at $t = t_\circ$

(a) $$H = \frac{gt_\circ^2}{2m_\circ^2}\left[(2M+m_\circ)m_\circ + 2M(M+m_\circ)\ln\left(\frac{M}{M+m_\circ}\right)\right]$$

(b) $$v = \frac{gt_\circ}{m_\circ}\left[(M+m_\circ)\ln\frac{(M+m_\circ)}{M} - m_\circ\right]$$

(c) letting $\varepsilon = \frac{m_\circ}{M} << 1$ we have

$$H = \frac{gt_\circ^2}{2\varepsilon^2}\left[(2+\varepsilon)\varepsilon - 2(1+\varepsilon)\ln(1+\varepsilon)\right]$$

$$= \frac{gt_\circ^2}{2\varepsilon^2}\left[2\varepsilon + \varepsilon^2 - 2(1+\varepsilon)\left(\varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \ldots\right)\right]$$

$$H \cong \frac{gt_\circ^2}{6}\varepsilon$$

Similarly:

$$v = \frac{gt_\circ}{\varepsilon}\left[(1+\varepsilon)\ln(1+\varepsilon) - \varepsilon\right]$$

$$= \frac{gt_\circ}{\varepsilon}\left[(1+\varepsilon)\left(\varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \ldots\right) - \varepsilon\right]$$

$$\approx \frac{1}{2}gt_\circ\varepsilon$$

(d) $H = 327m$; $v = 9.8ms^{-1}$

7.26 $\dot{m} = -k$ or $m = m_\circ - kt$

Burn-out occurs at time $T = \frac{\varepsilon m_\circ}{k}$

So – the rocket equation (7.7.7) becomes

$$m\frac{dv}{dt} = -V\dot{m}$$ (-) since V is oppositely directed to $\dot{v}$

$$\frac{dv}{dt} = +\frac{Vk}{m_\circ - kt}$$

Thus

$$v = Vk\int \frac{dt}{m_\circ - kt} = -V\ln(m_\circ - kt) + C_1 \qquad \text{where } C_1 \text{ is a constant.}$$

Now, $v = 0 @ t = 0$ so $C_1 = V\ln m_\circ$

Hence $v = -V\ln\left[\frac{(m_\circ - kt)}{m_\circ}\right], \qquad 0 \le t \le \frac{\varepsilon m_\circ}{k}$

Let y be the displacement at the time t so

$$y = -V\int \ln\left[\frac{(m_\circ - kt)}{m_\circ}\right]dt + C_2$$

Integrating the above expression by parts

$$y = -Vt\ln\left[\frac{(m_\circ - kt)}{m_\circ}\right] - Vk\int \frac{t\,dt}{m_\circ - kt} + C_2$$

$$= -Vt\ln\left[\frac{(m_\circ - kt)}{m_\circ}\right] + V\int\left(1 - \frac{m_\circ}{m_\circ - kt}\right)dt + C_2$$

$$= -Vt\ln\left[\frac{(m_\circ - kt)}{m_\circ}\right] + Vt + \frac{Vm_\circ}{k}\ln(m_\circ - kt) + C_2$$

since $y = 0$ at $t = 0$, $C_2 = \frac{-Vm_\circ \ln m_\circ}{k}$ and we have

$$y = Vt + \frac{V}{k}(m_\circ - kt)\ln\left[\frac{(m_\circ - kt)}{m_\circ}\right]$$

At burn-out $t = T = \frac{\varepsilon m_\circ}{k}$ so

(a) $\quad y(\varepsilon) = D = \frac{m_\circ V}{k}\left[\varepsilon + (1-\varepsilon)\ln(1-\varepsilon)\right]$

(b) $\quad \varepsilon$ cannot exceed 1.0 although it can approach 1.0 for small payloads

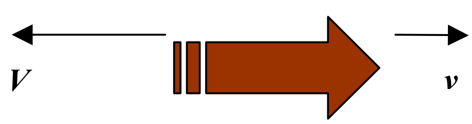
Thus $y_{\max} = \lim_{\varepsilon \to 1} y(\varepsilon) = \frac{m_\circ V}{k}$

7.27 From eqn. 7.7.5, $-k\vec{v} = m\dot{\vec{v}} - \vec{V}\dot{m}$

Since $\vec{V}$ is opposite in direction from $\vec{v}$

$$-kv = m\dot{v} + V\dot{m}$$

$$v = -\frac{m}{k}\dot{v} - \frac{\dot{m}}{k}V$$



$\alpha = \left|\frac{\dot{m}}{k}\right|$ and since $\dot{m} < 0$, $\alpha = -\frac{\dot{m}}{k}$

$$v-\alpha V=\alpha\frac{m}{\dot{m}}\dot{v}=\alpha\frac{m}{\dot{m}}\frac{dv}{dt}=\alpha\frac{m}{\dot{m}}\frac{dv}{dm}\frac{dm}{dt}=\alpha m\frac{dv}{dm}$$

$$\frac{dm}{\alpha m}=\frac{dv}{v-V\alpha}$$

$$\frac{1}{\alpha}\int_{m_\circ}^{m}\frac{dm}{m}=\int_0^v\frac{dv}{v-V\alpha}$$

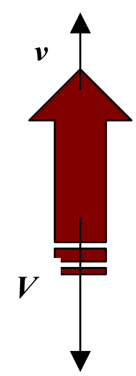
$$\frac{1}{\alpha}\ln\left(\frac{m}{m_\circ}\right)=\ln\left(\frac{v-V\alpha}{-V\alpha}\right)$$

$$\left(\frac{m}{m_\circ}\right)^{\frac{1}{\alpha}}=-\frac{v}{V\alpha}+1$$

$$v=V\alpha\left[1-\left(\frac{m}{m_\circ}\right)^{\frac{1}{\alpha}}\right]$$

7.28 From eqn. 7.7.5, $-m\vec{g}=m\dot{\vec{v}}-\vec{V}\dot{m}$

since $\vec{V}$ is opposite in direction from $\vec{v}$,



$$-mg=m\dot{v}+V\dot{m}$$

$$-mgdt=mdv+Vdm$$

$$\dot{m}=\frac{dm}{dt} \qquad \text{so } dt=\frac{dm}{\dot{m}}$$

$$-mg\frac{dm}{\dot{m}}=mdv+Vdm$$

$$dv=-dm\left(\frac{g}{\dot{m}}+\frac{V}{m}\right)$$

$$\int_0^{v_e}dv=-\int_{m_\circ}^{m_p}dm\left(\frac{g}{\dot{m}}+\frac{V}{m}\right) \qquad m_p=\text{payload mass}$$

$$v_e=\frac{g}{\dot{m}}\left(m_\circ-m_p\right)+V\ln\frac{m_\circ}{m_p}$$

$$m_f=m_\circ-m_p\approx m_\circ \qquad m_f=\text{fuel mass}$$

$$v_e=\frac{g}{\dot{m}}m_f+V\ln\left(1+\frac{m_f}{m_p}\right)$$

$$\ln\left(1+\frac{m_f}{m_p}\right)=\frac{v_e}{V}-\frac{g}{V}\frac{m_\circ}{\dot{m}}$$

$$\frac{m_f}{m_p}=\exp\left(\frac{v_e}{V}-\frac{g}{V}\frac{m_\circ}{\dot{m}}\right)-1$$

For $V = kv_e$, $\frac{m_f}{m_p} = \exp\left(\frac{1}{k} - \frac{g}{kv_e}\frac{m_\circ}{\dot{m}}\right) - 1$

From chap. 2, Section 2.3 … $v_e \approx 11\frac{km}{s}$

For $|\dot{m}| = 0.01 m_\circ s^{-1}$ and $k = \frac{1}{4}$:

$$\frac{m_f}{m_p} = \exp\left[4 - \frac{9.8}{\frac{1}{4}(11{,}000)(-.01)}\right] - 1$$

$$\frac{m_f}{m_p} = 77$$

7.29 We can use Equation 7.7.9 to calculate the final velocity attained by the ion rocket during the 100 hour burn. Assuming the rocket starts from rest (even if the ion rocket is turned on while in Earth orbit, the initial rocket speed $v_0 \approx 10^{-4} c \approx 0$. Thus …

$v \approx V \ln\frac{m_0}{m_p}$ and $m_0 = m_F + m_p = 2m_p + m_p = 3m_p$

$v \approx V \ln 3 = 0.1099 c$. The final rocket velocity is a little more than 10% *c*.

$$T = \frac{L}{0.1099 c} = \frac{4 LY}{0.1099 LY/yr} = 36.4 yr$$

7.30 We again use Equation 7.7.9 …

$v \approx V_{ion} \ln\frac{m_0}{m_p} = V_{ion} \ln\frac{m_F + m_p}{m_p} = V_{ion} \ln 2$ for the ion rocket. For the chemical rocket …

$v \approx V_{chem} \ln\frac{m_F + m_p}{m_p}$. Setting these two equations equal …

$V_{chem} \ln\frac{m_F + m_p}{m_p} = V_{ion} \ln 2$. Solving for m_F …

$\frac{m_F + m_p}{m_p} = 2^{(V_{ion}/V_{chem})} = 2^{\left(0.1c/10^{-5} c\right)} = 2^{10^4} \approx 10^{500}$ which demonstrates the virtue of ejecting mass at high velocity!
