CHAPTER 6
GRAVITATIONAL AND CENTRAL FORCES

6.1 $m = \rho V = \rho \frac{4}{3}\pi r_s^3$

$$r_s = \left(\frac{3m}{4\pi\rho}\right)^{\frac{1}{3}}$$

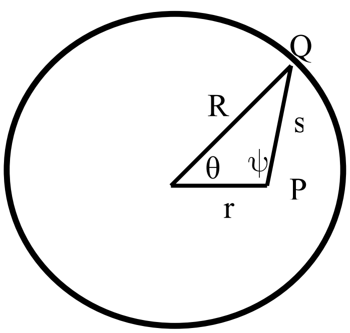
$$F = \frac{Gmm}{(2r_s)^2} = \frac{Gm^2}{4}\left(\frac{4\pi\rho}{3m}\right)^{\frac{2}{3}} = \frac{G}{4}\left(\frac{4\pi\rho}{3}\right)^{\frac{2}{3}} m^{\frac{4}{3}}$$

$$\frac{F}{W} = \frac{F}{mg} = \frac{Gm^2}{4g}\left(\frac{4\pi\rho}{3}\right)^{\frac{2}{3}} m^{\frac{1}{3}}$$

$$\frac{F}{W} = \frac{6.672\times10^{-11} N\cdot m^2\cdot kg^{-2}}{4\times9.8\, m\cdot s^{-2}}\left(\frac{4\pi\times11.35\, g\cdot cm^{-3}}{3}\times\frac{1kg}{10^3 g}\times\frac{10^6 cm^3}{1m^3}\right)^{\frac{2}{3}}\times(1kg)^{\frac{1}{3}}$$

$$\frac{F}{W} = 2.23\times10^{-9}$$

6.2 (a) The derivation of the force is identical to that in Section 6.2 except here r < R. This means that in the last integral equation, (6.2.7), the limits on u are R – r to R + r.



$$F = \frac{GmM}{4Rr^2}\int_{R-r}^{R+r}\left(1+\frac{r^2-R^2}{s^2}\right)ds$$

$$= \frac{GmM}{4Rr^2}\left[R+r-(R-r)+\frac{R^2-r^2}{R+r}-\frac{R^2-r^2}{R-r}\right]$$

$$F = \frac{GmM}{4Rr^2}\left[2r+R-r-(R+r)\right] = 0$$

(b) Again the derivation of the gravitational potential energy is identical to that in Example 6.7.1, except that the limits of integration on s are $(R-r)\rightarrow(R+r)$.

$$\phi = -G\frac{2\pi\rho R^2}{rR}\int_{R-r}^{R+r} ds$$

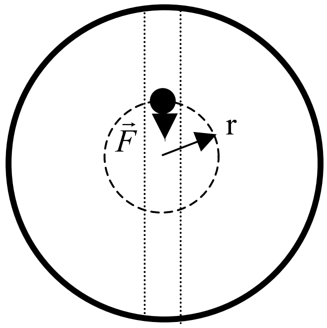
$$= -G\frac{2\pi\rho R^2}{rR}\left[R+r-(R-r)\right]$$

$$\phi = -G\frac{4\pi R^2\rho}{R} = -G\frac{M}{R}$$

For $r < R$, ϕ is independent of r. It is constant inside the spherical shell.

6.3 $\vec{F} = -\frac{GMm}{r^2}\hat{e}_r$

The gravitational force on the particle is due only to the mass of the earth that is inside the particle's instantaneous displacement from the center of the earth, r. The net effect of the mass of the earth outside r is zero (See Problem 6.2).



$$M = \frac{4}{3}\pi r^3 \rho$$

$$\vec{F} = -\frac{4}{3}G\pi\rho m r\hat{e}_r = -kr\hat{e}_r$$

The force is a linear restoring force and induces simple harmonic motion.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{3}{4G\pi\rho}}$$

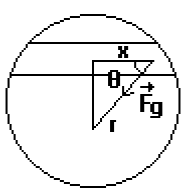
The period depends on the earth's density but is independent of its size.
At the surface of the earth,

$$mg = \frac{GMm}{R_e^2} = \frac{Gm}{R_e^2}\cdot\frac{4}{3}\pi R_e^3\rho$$

$$\frac{4G\pi\rho}{3} = \frac{g}{R_e}$$

$$T = 2\pi\sqrt{\frac{R_e}{g}} = 2\pi\sqrt{\frac{6.38\times10^6 m}{9.8 m\cdot s^{-2}}}\times\frac{1 hr}{3600 s} \approx 1.4 hr$$

6.4



$$\vec{F}_g = -\frac{GMm}{r^2}\hat{e}_r\text{, where } M = \frac{4}{3}\pi r^3\rho$$

The component of the gravitational force perpendicular to the tube is balanced by the normal force arising from the side of the tube. The component of force along the tube is

$$F_x = F_g\cos\theta$$

The net force on the particle is …

$$\vec{F} = -\hat{i}\frac{4}{3}G\pi\rho m r\cos\theta$$

$$r\cos\theta = x$$

$$\vec{F} = -\hat{i}\frac{4}{3}G\pi\rho m x = -\hat{i}kx$$

As in problem 6.3, the motion is simple harmonic with a period of 1.4 hours.

6.5 $$\frac{GMm}{r^2}=\frac{mv^2}{r} \quad \text{so} \quad v^2=\frac{GM}{r}$$

for a circular orbit r, v is constant.

$$T=\frac{2\pi r}{v}$$

$$T^2=\frac{4\pi^2 r^2}{v^2}=\frac{4\pi^2}{GM}r^3 \propto r^3$$

6.6 (a) $T=\dfrac{2\pi r}{v}$

From Example 6.5.3, the speed of a satellite in circular orbit is …

$$v=\left(\frac{gR_e^2}{r}\right)^{\frac{1}{2}}$$

$$T=\frac{2\pi r^{\frac{3}{2}}}{g^{\frac{1}{2}}R_e}$$

$$r=\left(\frac{T^2 gR_e^2}{4\pi^2}\right)^{\frac{1}{3}}$$

$$\frac{r}{R_e}=\left(\frac{T^2 g}{4\pi^2 R_e}\right)^{\frac{1}{3}}=\left(\frac{24^2\, hr^2\times 3600^2\, s^2\cdot hr^{-2}\times 9.8\, m\cdot s^{-2}}{4\pi^2 6.38\times 10^6\, m}\right)^{\frac{1}{3}}$$

$$\frac{r}{R_e}=6.62\approx 7$$

(b) $$T=\frac{2\pi r^{\frac{3}{2}}}{g^{\frac{1}{2}}R_e}=\frac{2\pi\left(60R_e\right)^{\frac{3}{2}}}{g^{\frac{1}{2}}R_e}=2\pi\sqrt{\frac{60^3 R_e}{g}}$$

$$=2\pi\left(\frac{60^3\times 6.38\times 10^6\, m}{9.8\, m\cdot s^{-2}\times 3600^2\, s^2\cdot hr^{-2}\times 24^2\, hr^2\cdot day^{-2}}\right)^{\frac{1}{2}}$$

$$T=27.27\, day\approx 27\, day$$

6.7 From Example 6.5.3, the speed of a satellite in a circular orbit just above the earth's surface is …

$$v=\sqrt{gR_e}$$

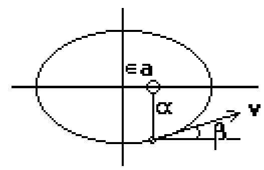
$$T=\frac{2\pi R_e}{v}=2\pi\sqrt{\frac{R_e}{g}}$$

This is the same expression as derived in Problem 6.3 for a particle dropped into a hole drilled through the earth. $T \approx 1.4$ hours.

6.8 The Earth's orbit about the Sun is counter-clockwise as seen from, say, the north star. It's coordinates on approach at the latus rectum are $(x,y)=(\varepsilon a,-\alpha)$.

The easiest way to solve this problem is to note that $\varepsilon=\frac{1}{60}$ is small. The orbit is almost circular!

$$\therefore \frac{GM_S m}{r^2}=\frac{mv^2}{r} \text{ and } v^2=\frac{GM_S}{r}$$



with $r=\alpha\approx a\approx b$ when $\varepsilon\approx 0$

$$v\approx\left(\frac{GM_S}{\alpha}\right)=3\cdot 10^4\frac{m}{s}$$

More exactly

$$|\vec{r}\times\vec{v}|=\alpha v\cos\beta=l\text{, but }\alpha=\frac{ml^2}{k} \qquad \text{(equation 6.5.19)}$$

Since $k=GM_S m$ $\qquad \alpha=\frac{l^2}{GM_S}$, $\qquad$ hence $l=\alpha v\cos\beta=(\alpha GM_S)^{\frac{1}{2}}$

Or $\qquad v=\left(\frac{GM_S}{\alpha}\right)^{\frac{1}{2}}\frac{1}{\cos\beta}$

The angle β can be calculated as follows:

$$\frac{y^2}{b^2}+\frac{x^2}{a^2}=1 \qquad \text{(see appendix C)}$$

$$\therefore\frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y} \qquad \text{and at } (x,y)=(\varepsilon a,-\alpha)$$

so $\qquad \frac{dy}{dx}=\frac{b^2}{a^2}\frac{\varepsilon a}{\alpha}=\frac{b^2}{a^2}\frac{\varepsilon}{(1-\varepsilon^2)}=\varepsilon$ since $1-\varepsilon^2=\frac{b^2}{a^2}$

here $\qquad \frac{dy}{dx}=\tan\beta=\varepsilon \qquad$ or $\beta\approx\varepsilon$ (small ε)

and $\qquad v=\left(\frac{GM_S}{\alpha}\right)^{\frac{1}{2}}\frac{1}{\cos\varepsilon}\approx\left(\frac{GM_S}{\alpha}\right)^{\frac{1}{2}}$ as before.

6.9 $\quad F(r)=F_s+F_d$

$$F_s=-\frac{GMm}{r^2}$$

$$F_d=-\frac{GM_d m}{r^2}$$

The net effect of the dust outside the planet's radius is zero (Problem 6.2). The mass of the dust inside the planet's radius is:

$$M_d = \frac{4}{3}\pi r^3 \rho$$

$$F(r) = -\frac{GMm}{r^2} - \frac{4}{3}\pi\rho m G r$$

6.10 $u = \frac{1}{r} = \frac{1}{r_\circ} e^{-k\theta}$

$$\frac{du}{d\theta} = -\frac{k}{r_\circ} e^{-k\theta}$$

$$\frac{d^2u}{d\theta^2} = \frac{k^2}{r_\circ} e^{-k\theta} = k^2 u$$

From equation 6.5.10 …

$$\frac{d^2u}{du^2} + u = k^2 u + u = -\frac{1}{ml^2u^2} f\left(u^{-1}\right)$$

$$f\left(u^{-1}\right) = -ml^2\left(k^2+1\right)u^3$$

$$f(r) = -\frac{ml^2\left(k^2+1\right)}{r^3}$$

The force varies as the inverse cube of r.

From equation 6.5.4, $r^2\dot{\theta} = l$

$$\frac{d\theta}{dt} = \frac{l}{r_\circ^2} e^{-2k\theta}$$

$$e^{2k\theta} d\theta = \frac{l}{r_\circ^2} dt$$

$$\frac{1}{2k} e^{2k\theta} = \frac{lt}{r_\circ^2} + C$$

$$\theta = \frac{1}{2k} \ln\left(\frac{2klt}{r_\circ^2} + C'\right)$$

θ varies logarithmically with t.

6.11 $f(r) = \frac{k}{r^3} = ku^3$

From equation 6.5.10 …

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2} \cdot ku^3 = -\frac{ku}{ml^2}$$

$$\frac{d^2u}{d\theta^2} + \left(1 + \frac{k}{ml^2}\right)u = 0$$

If $\left(1+\frac{k}{ml^2}\right)<0$, $\frac{d^2u}{d\theta^2}-cu=0$, $c>0$, for which $u=ae^{b\theta}$ is a solution.

If $\left(1+\frac{k}{ml^2}\right)=0$, $\frac{d^2u}{d\theta^2}=0$

$$\frac{du}{d\theta}=C_1$$

$$u=c_1\,\theta+c_2$$

$$r=\frac{1}{c_1\,\theta+c_2}$$

If $\left(1+\frac{k}{ml^2}\right)>0$, $\frac{d^2u}{d\theta^2}+cu=0$, $c>0$

$$u=A\cos\left(\sqrt{c}\theta+\delta\right)$$

$$r=\left[A\cos\left(\sqrt{1+\frac{k}{ml^2}}\theta+\delta\right)\right]^{-1}$$

6.12 $u=\frac{1}{r}=\frac{1}{r_\circ\cos\theta}$

$$\frac{du}{d\theta}=\frac{\sin\theta}{r_\circ\cos^2\theta}$$

$$\frac{d^2u}{d\theta^2}=\frac{1}{r_\circ}\left(\frac{1}{\cos\theta}+\frac{2\sin^2\theta}{\cos^3\theta}\right)=\frac{1}{r_\circ\cos\theta}\left(1+\frac{2-2\cos^2\theta}{\cos^2\theta}\right)=\frac{1}{r_\circ\cos\theta}\left(\frac{2}{\cos^2\theta}-1\right)$$

$$\frac{d^2u}{d\theta^2}=u\left(2r_\circ^2u^2-1\right)=2r_\circ^2u^3-u$$

Substituting into equation 6.5.10 …

$$2r_\circ^2u^3-u+u=-\frac{1}{ml^2u^2}f\left(u^{-1}\right)$$

$$f\left(u^{-1}\right)=-2r_\circ^2ml^2u^5$$

$$f\left(r\right)=-\frac{2r_\circ^2ml^2}{r^5}$$

6.13 From Chapter 1, the transverse component of the acceleration is … $a_\theta=r\ddot{\theta}+2\dot{r}\dot{\theta}$

If this term is nonzero, then there must be a transverse force given by …

$$f(\theta)=m(r\ddot{\theta}+2\dot{r}\dot{\theta})$$

For $r=a\theta$, and $\theta=bt$

$$f\left(\theta\right)=2mab^2\neq 0$$

Since $f\left(\theta\right)\neq 0$, the force is not a central field.

For $r = a\theta$, and the force to be central, try $\theta = bt^n$

$$f(\theta) = m\left[2ab^2n^2t^{2n-2} + ab^2n(n-1)t^{2n-2}\right]$$

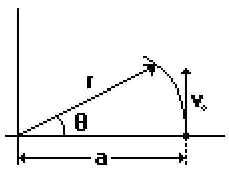
For a central field … $f(\theta) = 0$

$$2n + (n-1) = 0$$

$$n = \frac{1}{3}$$

$$\theta = bt^{\frac{1}{3}}$$

6.14



(a)

Calculating the potential energy

$$-\frac{dv}{dr} = f(r) = -k\left(\frac{4}{r^3} + \frac{a^2}{r^5}\right)$$

Thus, $V = -k\left(\frac{2}{r^2} + \frac{a^2}{4r^4}\right)$

The total energy is …

$$E = T_\circ + V_\circ = \frac{1}{2}v_\circ^2 - k\left(\frac{2}{a^2} + \frac{1}{4a^2}\right) = \frac{1}{2}\left(\frac{9k}{2a^2}\right) - \frac{9k}{4a^2} = 0$$

Its angular momentum is …

$$l^2 = a^2v_\circ^2 = \frac{9k}{2} = constant = r^4\dot{\theta}^2$$

Its KE is …

$$T = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) = \frac{1}{2}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\dot{\theta}^2 = \frac{1}{2}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\frac{l^2}{r^4}$$

The energy equation of the orbit is …

$$T + V = 0 = \frac{1}{2}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\frac{l^2}{r^4} - k\left(\frac{2}{r^2} + \frac{a^2}{4r^4}\right)$$

$$= \left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\frac{9k}{4r^4} - k\left(\frac{2}{r^2} + \frac{a^2}{4r^4}\right)$$

or $\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{9}\left(a^2 - r^2\right)$

Letting $r = a\cos\phi$ then $\frac{dr}{d\theta} = -a\sin\phi\frac{d\phi}{d\theta}$

So $\left(\frac{d\phi}{d\theta}\right)^2 = \frac{1}{9}$ $\therefore \phi = \frac{1}{3}\theta$

Thus $r = a\cos\frac{1}{3}\theta$ $\left(r = a @ \theta = 0^\circ\right)$

(b) at $\theta = \frac{3\pi}{2}$ $r \to 0$ the origin of the force. To find how long it takes …

$$\dot{\theta} = \frac{l}{r^2} = \frac{a v_\circ}{a^2\cos^2\frac{1}{3}\theta} = \frac{v_\circ}{a\cos^2\frac{1}{3}\theta}$$

$$dt = \frac{a}{v_\circ}\cos^2\frac{1}{3}\theta\, d\theta$$

$$T = \int_0^{\frac{3\pi}{2}} \frac{a}{v_\circ}\cos^2\frac{1}{3}\theta\, d\theta = \frac{3a}{v_\circ}\int_0^{\frac{\pi}{2}}\cos^2\phi\, d\phi = \frac{3\pi a}{4v_\circ}$$

Since $v_\circ = \left(\frac{9k}{2a^2}\right)^{\frac{1}{2}}$

$$T = \frac{3}{4}\pi a^2\left(\frac{2}{9k}\right)^{\frac{1}{2}} = \frac{\pi a^2}{4}\left(\frac{2}{k}\right)^{\frac{1}{2}}$$

(c) Since the particle falls into the center of the force

$v \to \infty$ (since $l = v r_\perp = const$)

6.15 From Example 6.5.4 … $\frac{v_\circ}{v_c} = \left(\frac{2r_1}{r_1 + r_\circ}\right)^{\frac{1}{2}}$

Letting $V = \frac{v_\circ}{v_c}$ we have $V = \left(\frac{2}{1 + \frac{r_\circ}{r_1}}\right)^{\frac{1}{2}}$

So: $$\frac{dV}{dr_1} = \frac{1}{2V}\left[-2\left(1+\frac{r_\circ}{r_1}\right)^{-2}\left(-\frac{r_\circ}{r_1^2}\right)\right]$$

Thus $$\frac{\frac{dV}{V}}{\left(\frac{dr_1}{r_1}\right)} = \frac{1}{\left(\frac{2}{1+\frac{r_\circ}{r_1}}\right)}\frac{1}{\left(1+\frac{r_\circ}{r_1}\right)^2}\frac{r_\circ}{r_1} = \frac{1}{2}\frac{1}{\left(1+\frac{r_\circ}{r_1}\right)}\frac{r_\circ}{r_1}$$

(a) $$\frac{\left(\frac{dV}{V}\right)}{\left(\frac{dr_1}{r_1}\right)} \approx \frac{1}{2}\frac{r_\circ}{r_1}$$ (b) $$\left(\frac{dr_1}{r_1}\right) = \frac{2r_1}{r_\circ}\left(\frac{dV}{V}\right) = 2(60)1\% = 120\%!$$

The approximation of a differential has broken down – a correct result can be obtained by calculating finite differences, but the implication is clear – a 1% error in boost causes rocket to miss the moon by a huge factor --- $\sim$ 2!

6.16 From section 6.5, $\varepsilon = 0.967$ and $r_\circ = 55\times10^6$ mi.

From equations 6.5.21a&b, $r_1 = r_\circ \frac{1+\varepsilon}{1-\varepsilon}$

$$a = \frac{1}{2}(r_\circ + r_1) = \frac{r_\circ}{1-\varepsilon} = \frac{55\times10^6\, mi}{1-0.967}\times\frac{1\, AU}{93\times10^6\, mi}$$

$$a = 17.92\, AU$$

From equation 6.6.5, $\tau = ca^{\frac{3}{2}}$

$$\tau = 1yr\cdot AU^{-\frac{3}{2}}\times 17.92^{\frac{3}{2}}\, AU^{\frac{3}{2}}$$

$$\tau = 75.9\, yr$$

From equation 6.5.21a and 6.5.19 …

$$\varepsilon = \frac{\alpha}{r_0} - 1 = \frac{ml^2}{kr_0} - 1$$

$$\varepsilon = \frac{mr_\circ v_\circ^2}{k} - 1 \quad \text{and} \qquad k = GMm$$

$$v_\circ = \left[\frac{GM}{r_\circ}(\varepsilon+1)\right]^{\frac{1}{2}}$$

From Example 6.5.3 we can translate the factor GM into the more convenient

$GM = a_e v_e^2$ … with a_e the radius of a circular orbit and v_e the orbital speed …

$$v_\circ = \left[\frac{a_e v_e^2}{r_\circ}(\varepsilon+1)\right]^{\frac{1}{2}} = \left[\frac{93\times10^6\, mi}{55\times10^6\, mi}(1.967)\right]^{\frac{1}{2}} v_e$$

$$v_0 = 1.824\, v_e$$

Since l is constant … $r_1 v_1 = r_\circ v_\circ$

$$v_1 = \frac{r_\circ}{r_1}v_\circ = \frac{1-\varepsilon}{1+\varepsilon}v_\circ = \frac{1-.967}{1.967}\times 1.824\, v_e = 0.0306\, v_e$$

$$v_e \approx \frac{2\pi a_e}{\tau} = \frac{2\pi\times 93\times10^6\ mi}{1\, yr\times 365\, day\cdot yr^{-1}\times 24\, hr\cdot day^{-1}} = 66{,}705\ mph$$

$$v_\circ = 1.22\times10^5\, mph \qquad \text{and} \qquad v_1 = 2.04\times10^3\, mph$$

6.17 From Example 6.10.1 …

$$\varepsilon = \left[1 + \left(q^2 - \frac{2}{d}\right)(qd\sin\phi)^2\right]^{\frac{1}{2}} \qquad \text{where } q = \frac{v}{v_e} \text{ and } d = \frac{r}{a_e}$$

are dimensionless ratios of the comet's speed and distance from the Sun in terms of the Earth's orbital speed and radius, respectively (q and d are the same as the factors V and R in Example 6.10.1). ϕ is the angle between the comet's orbital velocity and direction vector towards the Sun (see Figure 6.10.1).

The orbit is hyperbolic, parabolic, or elliptic as ε is > , = , or < 1 …

i.e., as $\left(q^2 - \frac{2}{R}\right)$ is > , = , or < 0.

$\left(q^2 - \frac{2}{R}\right)$ is > , = , or < 0 as $q^2 d$ is > , = , or < 2.

6.18 Since l is constant, v_{max} occurs at $r_\circ$ and v_{min} occurs at r_1, i.e. $v_{max} = v_\circ$ and $v_{min} = v_1$ and form the constancy of l … $v_1 r_1 = v_\circ r_\circ$

$$v_{min} v_{max} = v_1 v_\circ = \frac{r_\circ}{r_1} v_\circ^2$$

$$r_\circ v_\circ^2 = \frac{k}{m}(\varepsilon + 1) \qquad \text{(See Example 6.5.4)}$$

From equation 6.6.5 … $\frac{k}{m} = GM_\odot = \left(\frac{2\pi a}{\tau}\right)^2 a$

$$v_{min} v_{max} = \left(\frac{2\pi a}{\tau}\right)^2 \cdot \frac{a(\varepsilon+1)}{r_1}$$

From equation 6.5.21a&b … $r_1 = r_\circ \frac{1+\varepsilon}{1-\varepsilon}$. With $2a = r_\circ + r_1$:

$$\frac{a(\varepsilon+1)}{r_1} = \frac{1}{2}\frac{(r_\circ + r_1)(\varepsilon+1)}{r_1} = \frac{1}{2}\left(\frac{r_\circ}{r_1} + 1\right)(\varepsilon+1) = \frac{1}{2}\left[\left(\frac{1-\varepsilon}{1+\varepsilon}\right) + 1\right](\varepsilon+1) = 1$$

$$v_{min} v_{max} = \left(\frac{2\pi a}{\tau}\right)^2$$

6.19 As a result of the impulse, the speed of the planet <u>instantaneously</u> changes; its orbital radius does not, so there is no change in its potential energy V. The instantaneous change in its total orbital energy E is due to the change in its kinetic energy, T, only, so

$$\delta E = \delta T = \delta\left(\frac{1}{2}mv^2\right) = mv\delta v = mv^2 \frac{\delta v}{v} = 2T\frac{\delta v}{v}$$

$$\frac{\delta E}{T} = 2\frac{\delta v}{v}$$

But the total orbital energy is

$$E=-\frac{k}{2a} \qquad \text{So} \qquad \delta E=\frac{k}{2a^2}\delta a$$

Since planetary orbits are nearly circular

$$V\sim-\frac{k}{a} \qquad \text{and} \qquad T\sim\frac{k}{2a}$$

Thus, $\delta E\cong T\frac{\delta a}{a}$ and $\frac{\delta E}{T}=\frac{\delta a}{a}$

We obtain $\frac{\delta a}{a}=2\frac{\delta v}{v}$

6.20 (a) $\bar{V}=\frac{1}{\tau}\int_0^\tau V dt$

$$V(r)=-\frac{k}{r}$$

From equation 6.5.4, $l=r^2\dot{\theta}$

$$\frac{d\theta}{dt}=\frac{l}{r^2} \text{ or } dt=\frac{r^2 d\theta}{l}$$

$$\int_0^\tau V dt=-\int_0^{2\pi}\frac{kr}{l}d\theta$$

From equation 6.5.18a …

$$r=\frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}$$

$$\int_0^\tau V dt=-\frac{ka\left(1-\varepsilon^2\right)}{l}\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta}$$

From equation 6.6.4 … $\tau=\frac{2\pi a^2}{l}\sqrt{1-\varepsilon^2}$

$$\bar{V}=-\frac{k\sqrt{1-\varepsilon^2}}{2\pi a}\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta}$$

$$\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta}=\frac{2\pi}{\sqrt{1-\varepsilon^2}}\ , \ \varepsilon^2<1 \ \therefore \bar{V}=-\frac{k}{a}$$

(b) This problem is an example of the *virial theorem* which, for a bounded, periodic system, relates the time average of the quantity $\int_0^\tau \sum_i \bar{p}\cdot\bar{r}_i$ to its kinetic energy T. We will derive it for planetary motion as follows:

$$\frac{1}{\tau}\int_0^\tau \bar{p}\cdot\bar{r}dt=\frac{1}{\tau}\int_0^\tau m\ddot{\bar{r}}\cdot\bar{r}dt=\frac{1}{\tau}\int_0^\tau \bar{F}\cdot\bar{r}dt$$

Integrate LHS by parts

$$\frac{1}{\tau}\left[m\dot{\bar{r}}\cdot\bar{r}\right]\Big|_0^\tau-\frac{1}{\tau}\int_0^\tau m\dot{\bar{r}}^2dt=\frac{1}{\tau}\int_0^\tau \bar{F}\cdot\bar{r}dt$$

The first term is zero – since the quantity has the same value at 0 and τ.

Thus $2\langle T\rangle=-\langle\bar{F}\cdot\bar{r}\rangle$ where $\langle\ \rangle$ denote time average of the quantity within brackets.

but $-\langle\bar{r}\cdot\bar{F}\rangle=\langle r\cdot\bar{\nabla}V\rangle=\left\langle r\frac{dV}{dr}\right\rangle=\left\langle\frac{k}{r}\right\rangle=-\langle V\rangle$

hence $2\langle T\rangle=-\langle V\rangle$

but $\langle E\rangle=\langle T\rangle+\langle V\rangle=-\frac{\langle V\rangle}{2}+\langle V\rangle=\frac{\langle V\rangle}{2}$

hence $\langle V\rangle=2\langle E\rangle$ but $E=-\frac{k}{2a}=constant$

and $\langle E\rangle=\frac{1}{\tau}\int_0^\tau Edt=E=-\frac{k}{2a}$ so $2E=-\frac{k}{a}$

Thus: $\langle V\rangle=-\frac{k}{a}$ as before and therefore $\langle T\rangle=-\frac{1}{2}\langle V\rangle=\frac{k}{2a}$

6.21 The energy of the initial orbit is

$$\frac{1}{2}mv^2-\frac{k}{r}=E=-\frac{k}{2a}$$

(1) $$v^2=\frac{k}{m}\left(\frac{2}{r}-\frac{1}{a}\right)$$

Since $r_a=a(1+\varepsilon)$ at apogee, the speed v_1, at apogee is

$$v_1^2=\frac{k}{m}\left(\frac{2}{a(1+\varepsilon)}-\frac{1}{a}\right)=\frac{k}{ma}\frac{(1-\varepsilon)}{(1+\varepsilon)}$$

To place satellite in circular orbit, we need to boost its speed to v_c such that

$$\frac{1}{2}mv_c^2-\frac{k}{r_a}=-\frac{k}{2r_a}$$ since the radius of the orbit is r_a

$$v_c^2=\frac{k}{mr_a}=\frac{k}{ma(1+\varepsilon)}$$

Thus, the boost in speed $\Delta v_1=v_c-v_1$

(2) $$\Delta v_1=\left[\frac{k}{ma(1+\varepsilon)}\right]^{\frac{1}{2}}\left[1-(1-\varepsilon)^{\frac{1}{2}}\right]$$

Now we solve for the semi-major axis a and the eccentricity ε of the first orbit. From (1) above, at launch $v=v_\circ$ at $r=R_E$, so

$$v_\circ^2 = \frac{k}{m}\left(\frac{2}{R_E} - \frac{1}{a}\right)$$

and solving for a

$$a = \frac{R_E}{2 - mv_\circ^2 \dfrac{R_E}{k}} \quad \text{noting that } \ldots$$

(3) $$\frac{k}{mR_E} = \frac{GM_E}{R_E} = gR_E$$

$$a = \frac{R_E}{\left(2 - \dfrac{v_\circ^2}{gR_E}\right)} = \frac{R_E}{1.426} = \underline{4.49 \cdot 10^3}\, km$$

The eccentricity ε can be found from the angular momentum per unit mass, l, equation 6.5.19, and the data on ellipses defined in figure 6.5.1 …

$$l = r^2\dot{\theta} = v_\circ\left(R_E \sin\theta_\circ\right) = \left[\frac{k\alpha}{m}\right]^{\frac{1}{2}} = \left[\frac{ka\left(1-\varepsilon^2\right)}{m}\right]^{\frac{1}{2}}$$

where $v_\circ$, $\theta_\circ$ are the launch velocity, angle

Solving for ε (using (3) above)

$$\varepsilon^2 = 1 - \frac{v_\circ^2}{gR_E}\left(2 - \frac{v_\circ^2}{gR_E}\right)\sin^2\theta_\circ = 0.795$$

$$\therefore \underline{\varepsilon = 0.892}$$

Inserting these values for a, ε into (2) and using (3) gives

(a) $$\Delta v_1 = \left[gR_E\left(\frac{R_E/a}{1+\varepsilon}\right)\right]^{\frac{1}{2}}\left[1-\left(1-\varepsilon\right)^{\frac{1}{2}}\right] = \underline{4.61 \cdot 10^3}\ km \cdot s^{-1}$$

(b) $h = a\left(1+\varepsilon\right) - R_E = \underline{2.09 \cdot 10^3}\, km$ {altitude above the Earth … at perigee}

6.22 $$f'\left(r\right) = -k\left(\frac{-be^{-br}}{r^2} - 2\frac{e^{-br}}{r^3}\right) = k\frac{e^{-br}}{r^2}\left(b + \frac{2}{r}\right)$$

$$\frac{f'\left(a\right)}{f\left(a\right)} = -\left(b + \frac{2}{a}\right)$$

From equation 6.14.3, $\psi = \pi\left[3 + a\dfrac{f'\left(a\right)}{f\left(a\right)}\right]^{-\frac{1}{2}} = \pi\left[3 - \left(ab+2\right)\right]^{-\frac{1}{2}}$

$$\psi = \frac{\pi}{\sqrt{1-ab}}$$

6.23 From Problem 6.9, $f(r) = -\frac{GMm}{r^2} - \frac{4}{3}\pi\rho mGr$

$$f'(r) = \frac{2GMm}{r^3} - \frac{4}{3}\pi\rho mG$$

$$\frac{f'(a)}{f(a)} = \frac{2GMma^{-3} - \frac{4}{3}\pi\rho mG}{-GMma^{-2} - \frac{4}{3}\pi\rho mGa} = \frac{-2 + \frac{4\pi\rho a^3}{3M}}{a\left(1 + \frac{4\pi\rho a^3}{3M}\right)}$$

From equation 6.14.3, $\psi = \pi\left[3 + a\frac{f'(a)}{f(a)}\right]^{-\frac{1}{2}}$

$$\psi = \pi\left[3 + \frac{-2 + \frac{4\pi\rho a^3}{3M}}{1 + \frac{4\pi\rho a^3}{3M}}\right]^{-\frac{1}{2}} = \pi\left[\frac{1 + 4\left(\frac{4\pi\rho a^3}{3M}\right)}{1 + \frac{4\pi\rho a^3}{3M}}\right]^{-\frac{1}{2}}$$

$$\psi = \pi\left(\frac{1+c}{1+4c}\right)^{\frac{1}{2}}, \qquad c = \frac{4\pi\rho a^3}{3M}$$

6.24 We differentiate equation 6.11.1b to obtain $m\ddot{r} = -\frac{dU(r)}{dr}$

For a circular orbit at $r = a$, $\ddot{r} = 0$ so

$$\frac{dU}{dr}\Big|_{r=a} = 0$$

For small displacements x from $r = a$,

$$r = x + a \qquad \text{and} \qquad \ddot{r} = \ddot{x}$$

From Appendix D …

$$f(x+a) = f(a) + xf'(a) + \frac{x^2}{2}f''(a) + \ldots$$

Taking $f(r)$ to be $\frac{dU}{dr}$, $f'(r) = \frac{d^2U}{dr^2}$

Near $r = a$ …

$$\frac{dU}{dr} = \frac{dU}{dr}\Big|_{r=a} + x\frac{d^2U}{dr^2}\Big|_{r=a} + \ldots$$

$$m\ddot{x} = -x\frac{d^2U}{dr^2}\Big|_{r=a}$$

This represents a "restoring force," i.e., stable motion, so long as $\frac{d^2U}{dr^2} > 0$ at $r = a$.

6.25 $f'(r) = \frac{2k}{r^3} + \frac{4\varepsilon}{r^5}$

From equation 6.13.7, the condition for stability is $f(a) + \frac{a}{3} f'(a) < 0$

$$-\frac{k}{a^2} - \frac{\varepsilon}{a^4} + \frac{a}{3}\left(\frac{2k}{a^3} + \frac{4\varepsilon}{a^5}\right) < 0$$

$$-\frac{k}{3a^2} + \frac{\varepsilon}{3a^4} < 0$$

$$\frac{\varepsilon}{a^2} < k$$

$$a > \left(\frac{\varepsilon}{k}\right)^{\frac{1}{2}}$$

6.26 (a) $f(r) = -k\frac{e^{-br}}{r^2}$

$$f'(r) = -ke^{-br}\left(-\frac{b}{r^2} - \frac{2}{r^3}\right) = k\frac{e^{-br}}{r^2}\left(b + \frac{2}{r}\right)$$

From equation 6.13.7, the condition for stability is $f(a) + \frac{a}{3} f'(a) < 0$

$$-k\frac{e^{-ba}}{a^2} + \frac{a}{3}k\frac{e^{-ba}}{a^2}\left(b + \frac{2}{a}\right) < 0$$

$$-k\frac{e^{-ba}}{3a^2} + k\frac{be^{-ba}}{3a} < 0$$

$$b < \frac{1}{a} \qquad a < \frac{1}{b}$$

(b) $f(r) = -\frac{k}{r^3}$

$$f'(r) = \frac{3k}{r^4}$$

$$f(a) + \frac{a}{3} f'(a) = -\frac{k}{a^3} + \frac{a}{3}\left(\frac{3k}{a^4}\right) = 0$$

Since $f(a) + \frac{a}{3} f'(a)$ is not less than zero, the orbit is not stable.

6.27 (See Figure 6.10.1) From equation 6.5.18a $r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}$

and the data on ellipses in Figure 6.5.1 $p = a(1-\varepsilon)$ so

$$r = p\frac{(1+\varepsilon)}{1+\varepsilon\cos\theta}$$

For a parabolic orbit, $\varepsilon = 1$

The comet intersects earth's orbit at $r = a$.

$$a = \frac{2p}{1+\cos\theta}$$

$$\cos\theta = -1 + \frac{2p}{a}$$

6.28 (See Figure 6.10.1) $T = \int dt$ along the comet's trajectory inside earth's orbit

From equation 6.5.4, $r^2\dot{\theta} = r^2\frac{d\theta}{dt} = l$ so $dt = \frac{r^2 d\theta}{l}$

$$T = \int \frac{r^2 d\theta}{l}$$

From equation 6.5.18a $r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}$

and the data on ellipses in Figure 6.5.1 $p = a(1-\varepsilon)$ so

$$r = p\frac{(1+\varepsilon)}{1+\varepsilon\cos\theta}$$

From equation 6.5.18b, with $\varepsilon = 1$ for a parabolic orbit:

$$r = \frac{2p}{1+\cos\theta}$$

At $\theta = \frac{\pi}{2}$ the distance to the comet is $r = \alpha = \frac{2p}{1+\cos\frac{\pi}{2}} = 2p$

From equation 6.5.19, $\alpha = \frac{ml^2}{k}$, where $k = GMm$, so $p = \frac{l^2}{2GM}$

As shown in Example 6.5.3, $GM = av_e^2$

For a circular orbit, $v_e = \frac{2\pi a}{1\ yr}$

$$l = (2GMp)^{\frac{1}{2}} = (2ap)^{\frac{1}{2}} v_e = (2a)^{\frac{3}{2}} p^{\frac{1}{2}} \pi\ yr^{-1}$$

$$T = \int_{-\theta_\circ}^{+\theta_\circ} \frac{r^2 d\theta}{l} = \int_{-\theta_\circ}^{+\theta_\circ} \frac{4p^2}{(1+\cos\theta)^2} \cdot (2a)^{-\frac{3}{2}} p^{-\frac{1}{2}} \pi^{-1} d\theta\ yr$$

where $\theta_\circ = \cos^{-1}\left(-1 + \frac{2p}{a}\right)$ from Problem 6.27

$$T = \frac{\sqrt{2}p^{\frac{3}{2}}}{\pi a^{\frac{3}{2}}}\int_{-\theta_\circ}^{+\theta_\circ}\frac{d\theta}{(1+\cos\theta)^2}\ \ yr$$

From a table of integrals, $\int\frac{dx}{(1+\cos x)^2} = \frac{1}{2}\tan\frac{x}{2}+\frac{1}{6}\tan^3\frac{x}{2}$

$$T = \frac{\sqrt{2}}{\pi}\left(\frac{p}{a}\right)^{\frac{3}{2}}\left[\tan\frac{\theta_\circ}{2}+\frac{1}{3}\tan^3\frac{\theta_\circ}{2}\right]yr$$

$$\tan\frac{x}{2} = \left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}}$$

$$\tan\frac{\theta_\circ}{2} = \left(\frac{2-\frac{2p}{a}}{\frac{2p}{a}}\right)^{\frac{1}{2}} = \left(\frac{a-p}{p}\right)^{\frac{1}{2}}$$

$$T = \frac{\sqrt{2}}{\pi}\left(\frac{p}{a}\right)^{\frac{3}{2}}\left[\left(\frac{a-p}{p}\right)^{\frac{1}{2}}+\frac{1}{3}\left(\frac{a-p}{p}\right)^{\frac{3}{2}}\right]yr$$

$$= \frac{\sqrt{2}}{\pi}\left(\frac{p}{a}\right)^{\frac{3}{2}}\left(\frac{a-p}{p}\right)^{\frac{1}{2}}\left(1+\frac{a-p}{3p}\right)yr$$

$$T = \frac{\sqrt{2}}{3\pi}\left(\frac{2p}{a}+1\right)\left(1-\frac{p}{a}\right)^{\frac{1}{2}}\ yr$$

T is a maximum when $(2p+a)(a-p)^{\frac{1}{2}}$ is a maximum.

$$\frac{d}{dp}\left[(2p+a)^2(a-p)\right] = 2(2p+a)(2)(a-p)+(2p+a)^2(-1)$$

$$= (2p+a)(3a-6p)$$

T is a maximum when $p = \frac{a}{2}$.

$$T = \frac{\sqrt{2}}{3\pi}(2)\left(\frac{1}{2}\right)^{\frac{1}{2}} = \frac{2}{3\pi}\ yr = 77.5\ day$$

When $p = 0.6a$

$$T = \frac{\sqrt{2}}{3\pi}(2.2)\sqrt{.04} = 0.2088\,yr = 76.2\,day$$

6.29 $V(r) = -\frac{k}{r}-\frac{k\varepsilon}{r^3}$

$$f(r) = -\frac{dV}{dr} = \frac{k}{r^2} + \frac{3k\varepsilon}{r^4} = \frac{k}{r^4}\left(r^2 + 3\varepsilon\right)$$

$$f'(r) = -\frac{2k}{r^3} - \frac{12k\varepsilon}{r^5} = -\frac{2k}{r^5}\left(r^2 + 6\varepsilon\right)$$

$$\frac{f'(a)}{f(a)} = -\frac{2}{a}\left(\frac{r^2 + 6\varepsilon}{r^2 + 3\varepsilon}\right)$$

From equation 6.14.3, $\psi = \pi\left[3 + a\frac{f'(a)}{f(a)}\right]^{-\frac{1}{2}}$

$$\psi = \pi\left[3 - 2\left(\frac{r^2 + 6\varepsilon}{r^2 + 3\varepsilon}\right)\right]^{-\frac{1}{2}} = \pi\sqrt{\frac{r^2 + 3\varepsilon}{r^2 - 3\varepsilon}}$$

For $\varepsilon = \frac{2}{5}R\Delta R$, $R = 4000\,mi$, $\Delta R = 13\,mi$

$$\varepsilon = \frac{2}{5}(4000)(13) = 2.08\times 10^4\,mi^2$$

For $r \approx R$, $r^2 = 1.6\times 10^7\,mi^2$

$$\psi = 1.0039\pi = 180.7^\circ$$

6.30 $V_{rel} = -\frac{k}{r} - \frac{1}{2m_\circ c^2}\left(E + \frac{k}{r}\right)^2$

$$f(r) = -\frac{dV}{dr} = -\frac{k}{r^2} + \frac{2}{2m_\circ c^2}\left(E + \frac{k}{r}\right)\left(-\frac{k}{r^2}\right)$$

$$f(r) = -\frac{k}{r^2}\left[1 + \frac{1}{m_\circ c^2}\left(E + \frac{k}{r}\right)\right]$$

$$f'(r) = \frac{2k}{r^3}\left[1 + \frac{1}{m_\circ c^2}\left(E + \frac{k}{r}\right)\right] - \frac{k}{r^2}\left(\frac{1}{m_\circ c^2}\right)\left(-\frac{k}{r^2}\right)$$

$$f'(r) = \frac{k}{r^3}\left[2 + \frac{1}{m_\circ c^2}\left(2E + \frac{3k}{r}\right)\right]$$

$$\frac{f'(a)}{f(a)} = -\frac{1}{a}\left[\frac{2 + \frac{1}{m_\circ c^2}\left(2E + \frac{3k}{a}\right)}{1 + \frac{1}{m_\circ c^2}\left(E + \frac{k}{a}\right)}\right]$$

$$3 + a\frac{f'(a)}{f(a)} = \frac{\left[3 + \frac{3}{m_\circ c^2}\left(E + \frac{k}{a}\right) - 2 - \frac{1}{m_\circ c^2}\left(2E + \frac{3k}{a}\right)\right]}{\left[1 + \frac{1}{m_\circ c^2}\left(E + \frac{k}{a}\right)\right]}$$

$$= \frac{\left[1 + \frac{E}{m_\circ c^2}\right]}{\left[1 + \frac{1}{m_\circ c^2}\left(E + \frac{k}{a}\right)\right]}$$

$$3 + a\frac{f'(a)}{f(a)} = \left[\frac{m_\circ c^2 + E}{m_\circ c^2 + E + \frac{k}{a}}\right]$$

$$\psi = \pi\left[3 + a\frac{f'(a)}{f(a)}\right]^{-\frac{1}{2}}$$

$$\psi = \pi\left[1 + \frac{\frac{k}{a}}{m_\circ c^2 + E}\right]^{\frac{1}{2}}$$

6.31 From equation 6.5.18a $r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}$ … (Here θ is the polar angle of conic section trajectories as illustrated by the coordinates in Figure 6.5.1)
… and the data on ellipses in Figure 6.5.1 $r_0 = a(1-\varepsilon)$ so

$$r_{com} = r_\circ \frac{1+\varepsilon}{1+\varepsilon\cos\theta}$$

From equation 6.5.18b $r = \frac{\alpha}{1+\varepsilon\cos\theta}$ and at $\theta = 0^\circ$ $r_0 = \frac{\alpha}{1+\varepsilon}$

And from equation 6.5.19 $\alpha = \frac{ml^2}{k}$ so $r_0 = \frac{ml^2}{k(1+\varepsilon)}$

$$\frac{m}{k} = \frac{m}{GMm} = \frac{1}{GM}$$

From Example 6.5.3, $GM = a_e v_e^2$ and $l^2 = r_{com}^2 v_{com}^2 \sin^2\phi$

$$r_{com} = \frac{r_{com}^2 v_{com}^2 \sin^2\phi(1+\varepsilon)}{a_e v_e^2(1+\varepsilon)(1+\varepsilon\cos\theta)}$$

$$1 = RV^2\sin^2\phi\frac{1}{1+\varepsilon\cos\theta}$$

$$\cos\theta = \frac{1}{\varepsilon}\left(RV^2\sin^2\phi - 1\right)$$

$$\sin\theta = \left(1-\cos^2\theta\right)^{\frac{1}{2}} = \left[1 - \frac{1}{\varepsilon^2}\left(RV^2\sin^2\phi - 1\right)^2\right]^{\frac{1}{2}}$$

$$\sin\theta = \frac{1}{\varepsilon}\left[\varepsilon^2 - \left(RV^2\sin^2\phi\right)^2 + 2\left(RV^2\sin^2\phi\right) - 1\right]^{\frac{1}{2}}$$

Again from Example 6.5.3 …

$$\varepsilon = \left[1 + \left(V^2 - \frac{2}{R}\right)\left(RV\sin\phi\right)^2\right]^{\frac{1}{2}}$$

$$\varepsilon^2 = 1 + \left(RV^2\sin\phi\right)^2 - 2RV^2\sin^2\phi$$

$$\sin\theta = \frac{1}{\varepsilon}\left[\left(RV^2\sin\phi\right)^2 - \left(RV^2\sin^2\phi\right)^2\right]^{\frac{1}{2}}$$

$$\sin\theta = \frac{1}{\varepsilon}RV^2\sin\phi\,\cos\phi$$

$$\frac{\cos\theta}{\sin\theta} = \frac{RV^2\sin^2\phi - 1}{RV^2\sin\phi\cos\phi}$$

$$\cot\theta = \tan\phi - \frac{2}{RV^2\sin 2\phi}$$

$$\theta = \cot^{-1}\left(\tan\phi - \frac{2}{RV^2}\csc 2\phi\right)$$

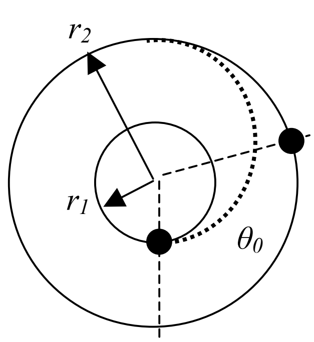
For $V = 0.5$, $R = 4$, $\phi = 30^\circ$:

$$\theta = \cot^{-1}\left(\tan 30^\circ - \frac{2}{4(.5)^2}\csc 60^\circ\right)$$

$$= \cot^{-1}\left(\frac{1}{\sqrt{3}} - \frac{2}{\frac{\sqrt{3}}{2}}\right) = \cot^{-1}\left(-\sqrt{3}\right)$$

$$\theta = -30^\circ$$

6.32



The picture at left shows the orbital transfer and the position of the two satellites at the moment the transfer is initiated. Satellite B is "ahead" of satellite A by the angle θ_0

$a = \frac{r_1 + r_2}{2}$ is the semi-major axis of the elliptical transfer orbit.

From Kepler's 3rd law (Equation 6.6.5) applied to objects in orbit about Earth …

$$\tau^2 = \frac{4\pi^2}{GM_E{}^2}a^3$$

The time to intercept is …

$$T_t = \frac{\tau}{2} = \pi\frac{1}{\sqrt{GM_E}}a^{\frac{3}{2}} = \frac{\pi}{R_E\sqrt{g}}a^{\frac{3}{2}} \qquad \text{since } \frac{GM_E}{R_E{}^2} = g$$

Letting $r_1 = R_E + h_1$ and $r_2 = R_E + h_2$ where h_1 and h_2 are the heights of the 2 satellites above the ground. Inserting these into the above gives …

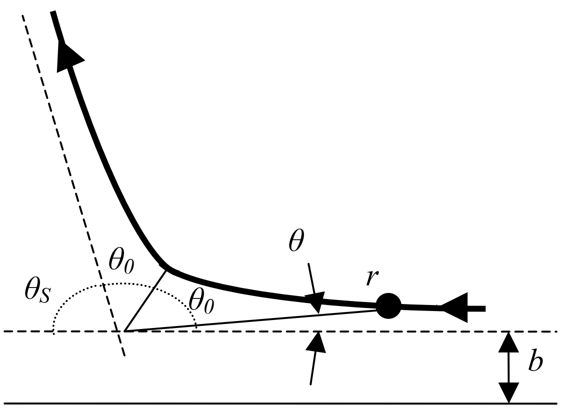
$$T_t = \frac{\pi}{R_E\sqrt{g}} R_E^{\frac{3}{2}} \left(2 + \frac{h_1 + h_2}{R_E}\right)^{\frac{3}{2}} = \frac{\pi}{\sqrt{R_E g}} \left(2 + \frac{h_1 + h_2}{R_E}\right)^{\frac{3}{2}}$$

From Example 6.6.2, R_E = 6371 km, h_1 = 200 mi = 324 km and $h_2 = r_2 - R_E$ = 42,400 km - 36,029 km. Putting in the numbers …

T_t = 4.79 hr

(b) Thus, $\theta_0 = 180^0\left(1 - \frac{T_t}{12}\right) = 108^0$

6.33



The potential for the inverse-cube force law is … $V(r) = \frac{k}{2r^2}$

Letting $u = r^{-1}$, we have (Equation 6.9.3)

$$\frac{1}{2} m l^2 \left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + V\left(u^{-1}\right) = E$$

$$\frac{du}{d\theta} = \sqrt{\frac{2(E - V)}{ml^2} - u^2}$$

$$d\theta = \frac{du}{\sqrt{\frac{2(E-V)}{ml^2} - u^2}} = \frac{ml}{\sqrt{2m(E-V) - m^2 l^2 u^2}} du$$

Now, integrating from $r = \infty\ (u = 0)$ up to $r = r_{\min}\ (u = u_{\max})$ …

$$\theta_0 = \int_0^{u_{\max}} \frac{ml}{\sqrt{2m(E-V) - m^2 l^2 u^2}} du$$

But $E = \frac{1}{2} m v_0^2$, $l = b v_0$, so …

$$\theta_0 = \int_0^{u_{\max}} \frac{m b v_0}{\sqrt{2m\left(\frac{1}{2} m v_0^2 - \frac{1}{2} k u^2\right) - m^2 b^2 v_0^2 u^2}} du$$

$$\theta_0 = \int_0^{u_{\max}} \frac{m b v_0}{\sqrt{2m\left(\frac{1}{2} m v_0^2 - \frac{1}{2} k u^2\right) - m^2 b^2 v_0^2 u^2}} du$$

Before evaluating this integral, we need to find $u_{\max}\left(=r_{\min}{}^{-1}\right)$, in other words, the distance of closest approach to the scattering center.

$$E=T\left(r_{\min}\right)+V\left(r_{\min}\right)=\frac{1}{2}mv^2+\frac{1}{2}\frac{k}{r_{\min}{}^2}=\frac{1}{2}mv_0{}^2$$

But, the angular momentum per unit mass l is …

$l=bv_0=r_{\min}v$ and substituting for v into the above gives …

$$\frac{ml^2}{r_{\min}{}^2}+\frac{k}{r_{\min}{}^2}=mv_0{}^2 \text{ so … } \frac{1}{r_{\min}{}^2}=\frac{mv_0{}^2}{ml^2+k}=u_{\max}{}^2$$

Solving for $u_{\max}$ …

$$u_{\max}=\frac{1}{\sqrt{b^2+\dfrac{k}{mv_0{}^2}}}$$

Now we evaluate the integral for θ_0 …

$$\theta_0=\int_0^{u_{\max}}\frac{b}{\sqrt{1-\left(b^2+\dfrac{k}{mv_0{}^2}\right)u^2}}du=\frac{b}{\left(b^2+\dfrac{k}{mv_0{}^2}\right)}\sin^{-1}\frac{u}{u_{\max}}\Bigg|_0^{u_{\max}}=\frac{b}{\left(b^2+\dfrac{k}{mv_0{}^2}\right)}\frac{\pi}{2}$$

Solving for b …

$$b\left(\theta_0\right)=\frac{k}{mv_0{}^2}\frac{2\theta_0}{\sqrt{\pi^2-4\theta_0{}^2}}$$

But $\theta_0=\frac{1}{2}\left(\pi-\theta_S\right)$. Thus, we have …

$$b\left(\theta_S\right)=\frac{k}{mv_0{}^2}\frac{\pi-\theta_S}{\sqrt{\theta_S\left(2\pi-\theta_S\right)}}$$

We can now compute the differential cross section …

$$\sigma\left(\theta_S\right)=\frac{b}{\sin\theta_S}\left|\frac{db}{d\theta_S}\right|=\frac{k\pi^2\left(\pi-\theta_S\right)}{mv_0{}^2\theta_S{}^2\left(2\pi-\theta_S\right)^2\sin\theta_S}$$

Since $d\Omega=2\pi\sin\theta_s\,d\theta_s$ we get …

$$\sigma\left(\theta_S\right)d\Omega=2\pi\left|bdb\right|=\frac{k\pi^3}{E}\left[\frac{\left(\pi-\theta_S\right)}{\left(2\pi-\theta_S\right)^2\theta_S{}^2}\right]d\theta_S$$
