CHAPTER 4
GENERAL MOTION OF A PARTICLE IN THREE DIMENSIONS

Note to instructors … there is a typo in equation 4.3.14. The range of the projectile is …

$$R = x = \frac{v_0^{\ 2}\sin 2\alpha}{g} \quad \text{… NOT …} \quad \frac{v_0^{\ 2}\sin^2 2\alpha}{g}$$

4.1 (a) $\vec{F} = -\vec{\nabla}V = -\hat{i}\,\frac{\partial V}{\partial x} - \hat{j}\,\frac{\partial V}{\partial y} - \hat{k}\,\frac{\partial V}{\partial z}$

$$\vec{F} = -c\left(\hat{i}yz + \hat{j}xz + \hat{k}xy\right)$$

(b) $\vec{F} = -\vec{\nabla}V = -\hat{i}\,2\alpha x - \hat{j}\,2\beta y - \hat{k}\,2\gamma z$

(c) $\vec{F} = -\vec{\nabla}V = ce^{-(\alpha x+\beta y+\gamma z)}\left(\hat{i}\alpha + \hat{j}\beta + \hat{k}\gamma\right)$

(d) $\vec{F} = -\vec{\nabla}V = -\hat{e}_r\,\frac{\partial V}{\partial r} - \hat{e}_\theta\,\frac{1}{r}\frac{\partial V}{\partial \theta} - \hat{e}_\phi\,\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$

$$\vec{F} = -\hat{e}_r cnr^{n-1}$$

4.2 (a)

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0 \qquad \text{conservative}$$

(b)

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z^2 \end{vmatrix} = \hat{k}\left(-1-1\right) \neq 0$$

~~conservative~~

(c)

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & z^3 \end{vmatrix} = \hat{k}\left(1-1\right) = 0 \qquad \text{conservative}$$

(d)

$$\vec{\nabla}\times\vec{F}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{e}_r & \hat{e}_\theta r & \hat{e}_\phi r\sin\theta\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ -kr^{-n} & 0 & 0\end{vmatrix}=0 \quad \text{conservative}$$

4.3 (a)

$$\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & cx^2 & z^3\end{vmatrix}=k\left(2cx-x\right)$$

$$2cx-x=0$$

$$c=\frac{1}{2}$$

(b)

$$\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{z}{y} & \frac{cxz}{y^2} & \frac{x}{y}\end{vmatrix}$$

$$=\hat{i}\left(-\frac{x}{y^2}-\frac{cx}{y^2}\right)+\hat{j}\left(\frac{1}{y}-\frac{1}{y}\right)+\hat{k}\left(\frac{cz}{y^2}+\frac{z}{y^2}\right)$$

$$-\frac{x}{y^2}-\frac{cx}{y^2}=0 \qquad c=-1$$

also $\frac{cz}{y^2}+\frac{z}{y^2}=0$ implies that $c=-1$ as it must

4.4 (a) $E=\text{constant}=V\left(x,y,z\right)+\frac{1}{2}mv^2$

at the origin $E=0+\frac{1}{2}mv_\circ^2$

at $\left(1,1,1\right)$ $E=\alpha+\beta+\gamma+\frac{1}{2}mv^2=\frac{1}{2}mv_\circ^2$

$$v^2 = v_\circ^2 - \frac{2}{m}(\alpha+\beta+\gamma)$$

$$v = \left[v_\circ^2 - \frac{2}{m}(\alpha+\beta+\gamma)\right]^{\frac{1}{2}}$$

(b) $$v_\circ^2 - \frac{2}{m}(\alpha+\beta+\gamma) = 0$$

$$v_\circ = \left[\frac{2}{m}(\alpha+\beta+\gamma)\right]^{\frac{1}{2}}$$

(c) $$m\ddot{x} = F_x = -\frac{\partial V}{\partial x}$$

$$m\ddot{x} = -\alpha$$

$$m\ddot{y} = -\frac{\partial V}{\partial y} = -2\beta y$$

$$m\ddot{z} = -\frac{\partial V}{\partial z} = -3\gamma z^2$$

4.5 (a) $\vec{F} = \hat{i}x + \hat{j}y$

on the path $x = y$: $d\vec{r} = \hat{i}dx + \hat{j}dy$

$$\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = \int_0^1 F_x dx + \int_0^1 F_y dy = \int_0^1 x dx + \int_0^1 y dy = 1$$

on the path along the x-axis: $d\vec{r} = \hat{i}dx$

and on the line $x = 1$: $d\vec{r} = \hat{j}dy$

$$\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = \int_0^1 F_x dx + \int_0^1 F_y dy = 1$$

$\vec{F}$ is conservative.

(b) $\vec{F} = \hat{i}y - \hat{j}x$

on the path $x = y$:

$$\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = \int_0^1 F_x dx + \int_0^1 F_y dy = \int_0^1 y dx - \int_0^1 x dy$$

and, with $x = y$ $$\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = \int_0^1 x dx - \int_0^1 y dy = 0$$

on the x-axis: $$\int_{(0,0)}^{(1,0)} \vec{F}\cdot d\vec{r} = \int_0^1 F_x dx = \int_0^1 y dx$$

and, with $y = 0$ on the x-axis $$\int_{(0,0)}^{(1,0)} \vec{F}\cdot d\vec{r} = 0$$

on the line $x = 1$: $$\int_{(1,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = \int_0^1 F_y dy = \int_0^1 x dy$$

and, with $x = 1$ $$\int_{(1,0)}^{(1,1)} \vec{F} \cdot d\vec{r} = \int_0^1 dy = 1$$

on this path $$\int_{(0,0)}^{(1,1)} \vec{F} \cdot d\vec{r} = 0 + 1 = 1$$

$\vec{F}$ is *not* conservative.

4.6 From Example 2.3.2, $V(z) = -mg\dfrac{r_e^2}{(r_e + z)}$

$$V(z) = -mgr_e\left(1 + \frac{z}{r_e}\right)^{-1}$$

From Appendix D, $(1+x)^{-1} = 1 - x + x^2 + \ldots$

$$V(z) = -mgr_e\left(1 - \frac{z}{r_e} + \frac{z^2}{r_e^2} + \ldots\right)$$

$$V(z) = -mgr_e + mgz - \frac{mgz^2}{r_e} + \ldots$$

With $-mgr_e$ an additive constant,

$$V(z) \approx mgz\left(1 - \frac{z}{r_e}\right)$$

$$\vec{F} = -\vec{\nabla} V = -\hat{k}\frac{\partial}{\partial z}V(z)$$

$$= -\hat{k}mg\left[1 - \frac{z}{r_e} + z\left(-\frac{1}{r_e}\right)\right]$$

$$\vec{F} = -\hat{k}mg\left(1 - \frac{2z}{r_e}\right)$$

$$m\ddot{x} = F_x = 0, \qquad m\ddot{y} = F_y = 0 \qquad m\ddot{z} = -mg\left(1 - \frac{2z}{r_e}\right)$$

$$m\dot{z}\frac{d\dot{z}}{dz} = -mg\left(1 - \frac{2z}{r_e}\right)$$

$$\int_{v_{\circ z}}^{0} \dot{z}dz = -g\int_0^h \left(1 - \frac{2z}{r_e}\right)dz$$

$$-\frac{1}{2}v_{\circ z}^2 = -g\left(h - \frac{h^2}{r_e}\right)$$

$$h^2 - r_e h + \frac{r_e v_{\circ z}^2}{2g} = 0$$

$$h = \frac{r_e}{2} - \frac{1}{2}\sqrt{r_e^2 - \frac{2r_e v_{\circ z}^2}{g}} \qquad (h, z \ll r_e)$$

$$h = \frac{r_e}{2} - \frac{r_e}{2}\sqrt{1 - \frac{2v_{\circ z}^2}{gr_e}}$$

From Appendix D, $(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \ldots$

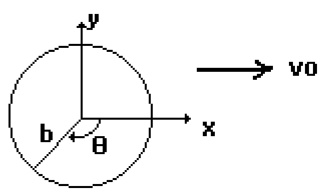
$$h = \frac{r_e}{2} - \frac{r_e}{2} + \frac{v_{\circ z}^2}{2g} + \frac{v_{\circ z}^4}{4gr_e} + \ldots$$

$$h \approx \frac{v_{\circ z}^2}{2g}\left(1 + \frac{v_{\circ z}^2}{2gr_e}\right)$$

From Example 2.3.2, $h = \frac{v_\circ^2}{2g}\left(1 - \frac{v_\circ^2}{2gr_e}\right)^{-1}$

And with $(1-x)^{-1} \approx 1 + x$, $\qquad h \approx \frac{v_\circ^2}{2g}\left(1 + \frac{v_\circ^2}{2gr_e}\right)$

4.7



For a point on the rim measured from the center of the wheel:

$$\vec{r} = \hat{i} b\cos\theta - \hat{j} b\sin\theta$$

$$\theta = \omega t = \frac{v_\circ t}{b}, \qquad \text{so} \quad \dot{\vec{r}} = -\hat{i} v_\circ \sin\theta - \hat{j} v_\circ \cos\theta$$

Relative to the ground, $\vec{v} = \hat{i} v_\circ (1 - \sin\theta) - \hat{j} v_\circ \cos\theta$

For a particle of mud leaving the rim:

$$y_\circ = -b\sin\theta \quad \text{and} \quad v_{\circ y} = -v_\circ \cos\theta$$

So $\qquad v_y = v_{\circ y} - gt = -v_\circ \cos\theta - gt$

and $\qquad y = -b\sin\theta - v_\circ t\cos\theta - \frac{1}{2}gt^2$

At maximum height, $v_y = 0$:

$$t = -\frac{v_\circ \cos\theta}{g}$$

$$h = -b\sin\theta - v_\circ\left(-\frac{v_\circ\cos\theta}{g}\right)\cos\theta - \frac{1}{2}g\left(-\frac{v_\circ\cos\theta}{g}\right)^2$$

$$h = -b\sin\theta + \frac{v_\circ^{\,2}\cos^2\theta}{2g}$$

Maximum h occurs for $\qquad \frac{dh}{d\theta} = 0 = -b\cos\theta - \frac{2v_\circ^2\cos\theta\sin\theta}{2g}$

$$\sin\theta = -\frac{gb}{v_\circ^2}$$

$$\cos^2\theta = 1-\sin^2\theta = \frac{v_\circ^4 - g^2b^2}{v_\circ^4}$$

$$h_{max} = \frac{gb^2}{v_\circ^2} + \frac{v_\circ^4 - g^2b^2}{2gv_\circ^2} = \frac{gb^2}{2v_\circ^2} + \frac{v_\circ^2}{2g}$$

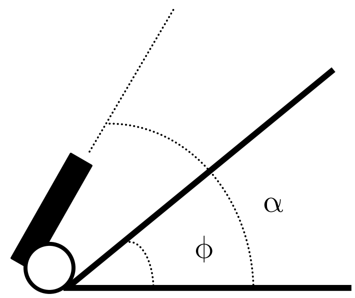
Measured from the ground,

$$h'_{max} = b + \frac{gb^2}{2v_\circ^2} + \frac{v_\circ^2}{2g}$$

The mud leaves the wheel at $\theta = \sin^{-1}\left(-\frac{gb}{v_\circ^2}\right)$

4.8 $x = R\cos\phi$ and $x = v_{\circ x}t = (v_\circ\cos\alpha)t$

so $t = \frac{R\cos\phi}{v_\circ\cos\alpha}$



$y = R\sin\phi$ and $y = v_{\circ y}t - \frac{1}{2}gt^2 = (v_\circ\sin\alpha)t - \frac{1}{2}gt^2$

$$R\sin\phi = (v_\circ\sin\alpha)\frac{R\cos\phi}{v_\circ\cos\alpha} - \frac{1}{2}g\left(\frac{R\cos\phi}{v_\circ\cos\alpha}\right)^2$$

$$\sin\phi = \tan\alpha\cos\phi - \frac{gR\cos^2\phi}{2v_\circ^2\cos^2\alpha}$$

$$R = \frac{2v_\circ^2\cos^2\alpha}{g\cos^2\phi}(\tan\alpha\cos\phi - \sin\phi) = \frac{2v_\circ^2\cos\alpha}{g\cos^2\phi}(\sin\alpha\cos\phi - \cos\alpha\sin\phi)$$

From Appendix B, $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$$R = \frac{2v_\circ^2\cos\alpha}{g\cos^2\phi}\sin(\alpha-\phi)$$

R is a maximum for $\frac{dR}{d\alpha} = 0 = \frac{2v_\circ^2}{g\cos^2\phi}\left[-\sin\alpha\sin(\alpha-\phi) + \cos\alpha\cos(\alpha-\phi)\right]$

Implies that $\cos\alpha\cos(\alpha-\phi) - \sin\alpha\sin(\alpha-\phi) = 0$

From appendix B, $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

so $\cos(2\alpha-\phi) = 0$

$$2\alpha - \phi = \frac{\pi}{2} \qquad \alpha = \frac{\pi}{4} + \frac{\phi}{2}$$

$$R_{max} = \frac{2v_\circ^2}{g\cos^2\phi}\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)$$

Now $\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)=\cos\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\right]=\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$

$$R_{\max}=\frac{2v_\circ^2}{g\cos^2\phi}\cos^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$$

Again using Appendix B, $\cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1$

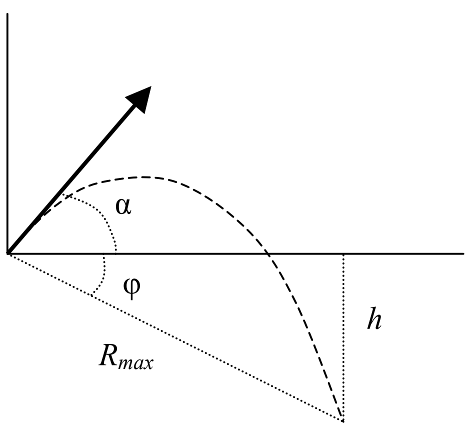
$$R_{\max}=\frac{2v_\circ^2}{g\cos^2\phi}\left[\frac{1}{2}\cos\left(\frac{\pi}{2}+\phi\right)+\frac{1}{2}\right]=\frac{v_\circ^2}{g\cos^2\phi}\left[\cos\left(\frac{\pi}{2}+\phi\right)+1\right]$$

Using $\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$,

$$R_{\max}=\frac{v_\circ^2}{g\left(1-\sin^2\phi\right)}\left(1-\sin\phi\right)$$

$$R_{\max}=\frac{v_\circ{}^2}{g\left(1+\sin\phi\right)}$$

4.9



(a) Here we note that the projectile is launched "downhill" towards the target, which is located a distance h below the cannon along a line at an angle φ below the horizon. α is the angle of projection that yields maximum range, R_{max}. We can use the results from problem 4.8 for this problem. We simply have to replace the angle φ in the above result with the angle -φ, to account for the downhill slope. Thus, we get for the downhill range …

$$R=\frac{2v_0{}^2}{g}\frac{\cos\alpha\sin\left(\alpha+\varphi\right)}{\cos^2\varphi}$$

The maximum range and the angle is α are obtained from the problem above again by replacing φ with the angle -φ … $R_{\max}=\frac{v_0{}^2}{g}\frac{\left(1+\sin\varphi\right)}{\cos^2\varphi}$ and … $2\alpha=\frac{\pi}{2}-\varphi$.

We can now calculate α … $R_{\max}=\frac{h}{\sin\varphi}=\frac{v_0{}^2}{g}\frac{\left(1+\sin\varphi\right)}{\cos^2\varphi}=\frac{v_0{}^2}{g\left(1-\sin\varphi\right)}$

Solving for $\sin\varphi$ … $\sin\varphi=\frac{gh}{v_0{}^2}\bigg/\left(1+\frac{gh}{v_0{}^2}\right)$

But, from the above … $\sin\varphi=\sin\left(\frac{\pi}{2}-2\alpha\right)=\cos 2\alpha=1-2\sin^2\alpha$

Thus … $1-2\sin^2\alpha=\frac{gh}{v_0{}^2}\bigg/\left(1+\frac{gh}{v_0{}^2}\right)$

$$2\sin^2\alpha = \frac{2}{\csc^2\alpha} = 1 - \frac{gh}{v_0^{\ 2}} \Big/ \left(1 + \frac{gh}{v_0^{\ 2}}\right) = \frac{1}{1 + \frac{gh}{v_0^{\ 2}}}$$

Finally … $\csc^2\alpha = 2\left(1 + \frac{gh}{v_0^{\ 2}}\right)$

(b)

Solving for R_{max} … $R_{\max} = \frac{h}{\sin\varphi} = \frac{h}{1 - 2\sin^2\alpha} = \frac{h}{1 - 2/\csc^2\alpha}$

Substituting for $\csc^2\alpha$ and solving …

$$R_{\max} = \frac{v_0^{\ 2}}{g}\left(1 + \frac{gh}{v_0^{\ 2}}\right)$$

4.10 We can again use the results of problem 4.8. The maximum slope range from problem 4.8 is given by …

$$R_{\max} = \frac{v_0^{\ 2}}{g(1 + \sin\varphi)} = \frac{h}{\sin\varphi}$$

Solving for $\sin\varphi$ …

$$\sin\varphi = \frac{gh}{v_0^{\ 2}} \Big/ \left(1 - \frac{gh}{v_0^{\ 2}}\right)$$

Thus …

$$x_{\max} = R_{\max}\cos\varphi = h\frac{\cos\varphi}{\sin\varphi}$$

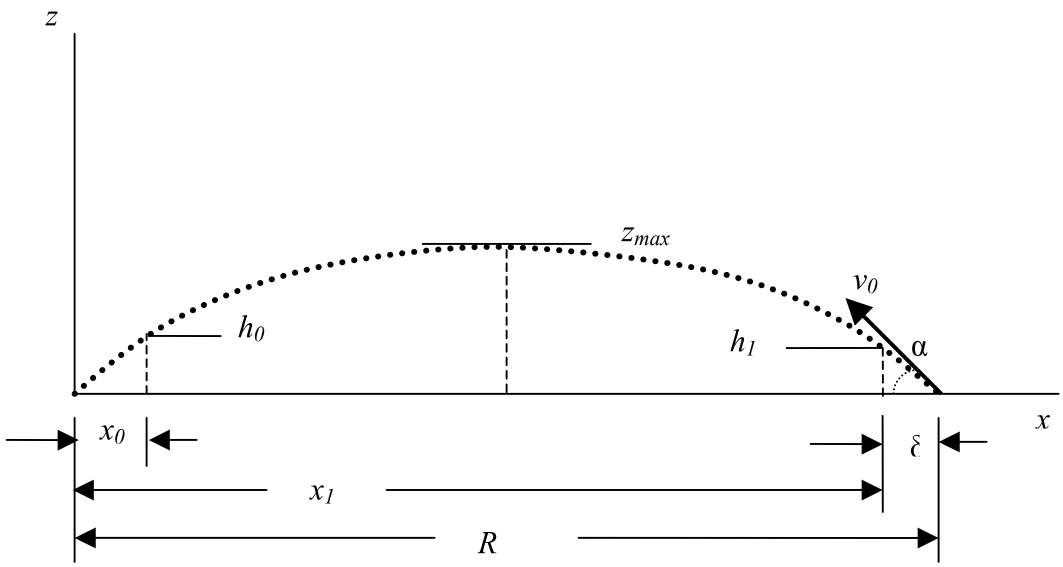
We can calculate $\cos\varphi$ from the above relation for $\sin\varphi$ …

$$\cos\varphi = \left(1 - \sin^2\varphi\right)^{\frac{1}{2}} = \left(1 - 2\frac{gh}{v_0^{\ 2}}\right)^{\frac{1}{2}} \Big/ \left(1 - \frac{gh}{v_0^{\ 2}}\right)$$

Inserting the results for $\sin\varphi$ and $\cos\varphi$ into the above …

$$x_{\max} = h\frac{\cos\varphi}{\sin\varphi} = \frac{v_0^{\ 2}}{g}\left(1 - 2\frac{gh}{v_0^{\ 2}}\right)^{\frac{1}{2}}$$

4.11 We can simplify this problems somewhat by noting that the trajectory is symmetric about a vertical line that passes through the highest point of the trajectory. Thus we have the following picture …



We have "reversed the trajectory so that h_0 (= 9.8 ft), and x_0 , the height and range within which Mickey can catch the ball represent the starting point of the trajectory. h_1 (=3.28 ft) is the height of the ball when Mickey strikes it at home plate. δ is the distance behind home plate where the ball would be hypothetically launched at some angle α to achieve the total range *R*. x_1 (=328 ft) is the distance the ball actually would travel from home plate if not caught by Mickey. (Note, because of the symmetry, v_0 is the speed of the ball when it strikes the ground … also at the same angle α at which was launched. We will calculate the value of x_0 assuming a time-reversed trajectory!)

(1) The range of the ball … $R = \dfrac{v_0^2 \sin 2\alpha}{g} = \dfrac{2v_0^2 \sin\alpha \cos\alpha}{g}$

(2) The maximum height … $z_{\max} = \dfrac{R}{2}\tan\alpha - \dfrac{g}{2v_0^2\cos^2\alpha}\left(\dfrac{R}{2}\right)^2$

(3) The height at x_1 … $h_1 = x_1 \tan\alpha - \dfrac{g}{2v_0^2\cos^2\alpha}(x_1)^2$

From (1) … $\dfrac{g}{2v_0^2\cos^2\alpha} = \dfrac{\tan\alpha}{R}$ and inserting this into (2) gives …

$$z_{\max} = \frac{R}{2}\tan\alpha - \frac{R}{4}\tan\alpha = \frac{R}{4}\tan\alpha$$

Thus, $R = \dfrac{4z_{\max}}{\tan\alpha}$ and inserting this expression and the first previously derived into (3)

(4) $h_1 = x_1\tan\alpha - \dfrac{(x_1\tan\alpha)^2}{4z_{\max}}$

Let $u = x_1\tan\alpha$ and we obtain the following quadratic …

$u^2 - 4z_{\max}u + 4z_{\max}h_1 = 0$ and solving for u …

$u = 2z_{\max}\left[1 \pm (1 - h_1/z_{\max})^{\frac{1}{2}}\right]$ and letting $\varepsilon = \dfrac{h_1}{z_{\max}}$, we get …

$$u \approx z_{max}\varepsilon = h_1$$

or $\quad u \approx 2z_{max}\left(2-\varepsilon\right) = 2z_{max}\left(2-.0475\right) = 3.9z_{max}$. This result is the correct one …

Thus, $\quad \tan\alpha = \dfrac{3.9z_{max}}{x_1} = 0.821 \quad \therefore\ \alpha = 39.4^0$

Now solve for x_0 using a relation identical to (4) …

$$h_0 = x_0 \tan\alpha - \frac{\left(x_0 \tan\alpha\right)^2}{4z_{max}}$$

Again we obtain a quadratic expression for $u = x_0 \tan\alpha$ which we solve as before. This time, though, the first result for *u* is the correct one to use …

$u = z_{max}\varepsilon \approx h_0$ and we obtain …

$$x_0 = \frac{h_0}{\tan\alpha} = 11.9\,ft$$

4.12 The x and z positions of the ball vs. time are

$$x = v_\circ t\cos\frac{1}{2}\theta_\circ \qquad z = v_\circ t\cos\frac{1}{2}\theta_\circ \sin\theta_\circ - \frac{1}{2}gt^2$$

Since $v_x = v_\circ \cos\frac{1}{2}\theta_\circ$

The horizontal range is $\quad R = \dfrac{v_\circ^{\,2}}{g}\cos^2\dfrac{1}{2}\theta_\circ \sin 2\theta_\circ$

The maximum range occurs @ $\dfrac{dR}{d\theta_\circ} = 0$

$$\frac{dR}{d\theta_\circ} = \frac{v_\circ^{\,2}}{g}\left(2\cos^2\frac{1}{2}\theta_\circ \cos 2\theta_\circ - \cos\frac{1}{2}\theta_\circ \sin\frac{1}{2}\theta_\circ \sin 2\theta_\circ\right) = 0$$

Thus, $2\cos^2\frac{1}{2}\theta_\circ \cos 2\theta_\circ = \cos\frac{1}{2}\theta_\circ \sin\frac{1}{2}\theta_\circ \sin 2\theta_\circ$

Using the identities: $\quad 2\cos^2\theta_\circ = 1+\cos 2\theta_\circ \quad$ and $\quad \sin 2\theta_\circ = 2\sin\theta_\circ \cos\theta_\circ$

We get:

$$\left(1+\cos\theta_\circ\right)\left(2\cos^2\theta_\circ - 1\right) = \sin\theta_\circ \sin\theta_\circ \cos\theta_\circ = \left(1-\cos^2\theta_\circ\right)\cos\theta_\circ$$

or $\quad \left(1+\cos\theta_\circ\right)\left(3\cos^2\theta_\circ - \cos\theta_\circ - 1\right) = 0$

Thus $\quad \cos\theta_\circ = -1, \quad \cos\theta_\circ = \dfrac{1}{6}\left(1\pm\sqrt{13}\right)$

Only the positive root applies for the $\theta_\circ$-range: $\quad 0 \le \theta_\circ \le \dfrac{\pi}{2}$

$$\cos\theta_\circ = \frac{1}{6}\left(1+\sqrt{13}\right) = 0.7676 \qquad \underline{\theta_\circ = 39^\circ 51'}$$

Thus (b) for $v_\circ = 25\,m\,s^{-1}$ $\quad \underline{R_{max} = 55.4\,m \quad @\theta_\circ = 39^\circ 51'}$

(a) The maximum height occurs at $\frac{dz}{dt} = 0$

$$v_\circ \cos\frac{1}{2}\theta_\circ \sin\theta_\circ = gT \qquad \text{or at} \qquad T = \frac{v_\circ \cos\frac{1}{2}\theta_\circ \sin\theta_\circ}{g}$$

or $$H = \frac{v_\circ^2}{2g}\cos^2\frac{1}{2}\theta_\circ \sin^2\theta_\circ \qquad \text{maximum at fixed } \theta_\circ$$

The maximum <u>possible</u> height occurs @ $\frac{dH}{d\alpha} = 0$

$$\frac{dH}{d\alpha} = \frac{v_\circ^2}{2g}\left(2\cos^2\frac{1}{2}\theta_\circ \sin\theta_\circ \cos\theta_\circ - \cos\frac{1}{2}\theta_\circ \sin\frac{1}{2}\theta_\circ \sin^2\theta_\circ\right) = 0$$

Using the above trigonometric identities, we get

$$\left(1+\cos\theta_\circ\right)\sin\theta_\circ \cos\theta_\circ = \frac{1}{2}\sin\theta_\circ \sin^2\theta_\circ = \frac{1}{2}\sin\theta_\circ\left(1-\cos^2\theta_\circ\right)$$

or $$\sin\theta_\circ\left(1+\cos\theta_\circ\right)\left(3\cos\theta_\circ - 1\right) = 0$$

There are 3-roots: $\sin\theta_\circ = 0$, $\cos\theta_\circ = -1$, $\cos\theta_\circ = \frac{1}{3}$

The first two roots give minimum heights; the last gives the maximum

Thus, $H_{max} = 18.9m \quad @\theta_\circ = \cos^{-1}\frac{1}{3} = 70^\circ 32'$

4.13 The trajectory of the shell is given by Eq. 4.3.11 with r replacing x

$$z = \frac{\dot{z}_\circ}{\dot{r}_\circ}r - \frac{g}{2\dot{r}_\circ^2}r^2 \qquad \text{where} \quad \dot{r}_\circ = v_\circ\cos\theta_\circ \quad \dot{z}_\circ = v_\circ\sin\theta_\circ$$

Thus, $z = r\tan\theta_\circ - \frac{g r^2}{2v_\circ^2}\sec^2\theta_\circ$

Since $\sec^2\theta_\circ = 1+\tan^2\theta_\circ$

We have:

$$\frac{g r^2}{2v_\circ^2}\tan^2\theta_\circ - r\tan\theta_\circ + z + \frac{g r^2}{2v_\circ^2} = 0$$

(r,z) are target coordinates.

The above equation yields two possible roots:

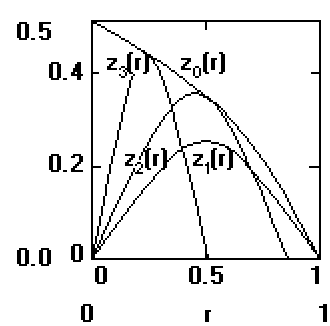
$$\tan\theta_\circ = \frac{1}{gr}\left[v_\circ^2 \pm \left(v_\circ^4 - 2gzv_\circ^2 - g^2r^2\right)^{\frac{1}{2}}\right]$$

The roots are only real if

$v_\circ^4 - 2gzv_\circ^2 - g^2r^2 \geq 0$

The <u>critical</u> surface is therefore:

<u>$v_\circ^4 - 2gzv_\circ^2 - g^2r^2 = 0$</u>



4.14 If the velocity vector, of magnitude $\dot{s}$, makes an angle θ with the z-axis, and its projection on the xy-plane make an angle ϕ with the x-axis:

$$\dot{x} = \dot{s}\sin\theta\cos\phi \ , \qquad \text{and} \quad F_x = F_r \sin\theta\cos\phi = m\ddot{x}$$

$$\dot{y} = \dot{s}\sin\theta\sin\phi \ , \qquad \text{and} \quad F_y = F_r \sin\theta\sin\phi = m\ddot{y}$$

$$\dot{z} = \dot{s}\cos\theta \ , \qquad \text{and} \quad F_z = -mg + F_r\cos\theta = m\ddot{z}$$

Since $F_r = -c_2\dot{s}^2 = -c_2\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$, the differential equations of motion are not separable.

$$m\ddot{x} = -c_2\dot{s}^2\sin\theta\cos\phi = -c_2\dot{s}\dot{x}$$

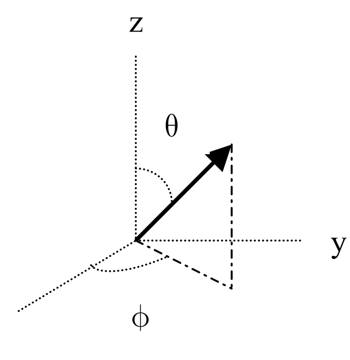
$$m\frac{d\dot{x}}{dt} = m\frac{d\dot{x}}{ds}\cdot\frac{ds}{dt} = m\dot{s}\frac{d\dot{x}}{ds} = -c_2\dot{s}\dot{x}$$

$$\frac{d\dot{x}}{\dot{x}} = -\frac{c_2}{m}ds = -\gamma ds \text{, where } \gamma = \frac{c_2}{m}$$

$$\ln\dot{x} - \ln\dot{x}_\circ = \ln\frac{\dot{x}}{\dot{x}_\circ} = -\gamma s$$

$$\dot{x} = \dot{x}_\circ e^{-\gamma s}$$

Similarly $\qquad \dot{y} = \dot{y}_\circ e^{-\gamma s}$



4.15 From eqn 4.3.16, $\left(\frac{\dot{z}_\circ}{\gamma} + \frac{g}{\gamma^2}\right)\frac{\gamma x_{max}}{\dot{x}_\circ} + \frac{g}{\gamma^2}\ln\left(1 - \frac{\gamma x_{max}}{\dot{x}_\circ}\right) = 0$

From Appendix D: $\ln(1-u) = -u - \frac{u^2}{2} - \frac{u^3}{3} - \ldots \qquad \text{for} \quad |u| < 1$

$$\ln\left(1 - \frac{\gamma x_{max}}{\dot{x}_\circ}\right) = -\frac{\gamma x_{max}}{\dot{x}_\circ} - \frac{\gamma^2 x_{max}{}^2}{2\dot{x}_\circ^2} - \frac{\gamma^3 x_{max}^3}{3\dot{x}_\circ^3} + \text{ terms in } \gamma^4$$

$$\frac{\dot{z}_\circ x_{max}}{\dot{x}_\circ} + \frac{g x_{max}}{\gamma\dot{x}_\circ} - \frac{g x_{max}}{\gamma\dot{x}_\circ} - \frac{g x_{max}^2}{2\dot{x}_\circ^2} - \frac{g\gamma x_{max}^3}{3\dot{x}_\circ^3} + \text{ terms in } \gamma^2 = 0$$

$$x_{max}^2 + \frac{3\dot{x}_\circ}{2\gamma}x_{max} - \frac{3\dot{x}_\circ^2\dot{z}_\circ}{g\gamma} \approx 0$$

$$x_{max} \approx -\frac{3\dot{x}_\circ}{4\gamma} \pm \left(\frac{9\dot{x}_\circ^2}{16\gamma^2} + \frac{3\dot{x}_\circ^2\dot{z}_\circ}{g\gamma}\right)^{\frac{1}{2}}$$

$$x_{max} \approx -\frac{3\dot{x}_\circ}{4\gamma} \pm \frac{3\dot{x}_\circ}{4\gamma}\left(1 + \frac{16\gamma\dot{z}_\circ}{3g}\right)^{\frac{1}{2}}$$

Since $x_{max} > 0$, the + sign is used.

From Appendix D:

$$\left(1 + \frac{16\gamma\dot{z}_\circ}{3g}\right)^{\frac{1}{2}} = 1 + \frac{8\gamma\dot{z}_\circ}{3g} - \frac{1}{8}\left(\frac{16\gamma\dot{z}_\circ}{3g}\right)^2 + \text{ terms in } \gamma^3$$

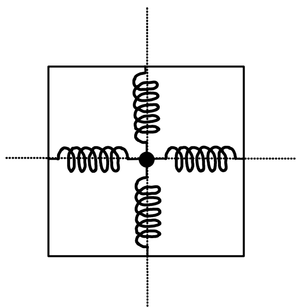
$$x_{max} = -\frac{3\dot{x}_\circ}{4\gamma} + \frac{3\dot{x}_\circ}{4\gamma} + \frac{2\dot{x}_\circ\dot{z}_\circ}{g} - \frac{8\dot{x}_\circ\gamma\dot{z}_\circ^2}{3g^2} + \text{ terms in } \gamma^2$$

$$x_{max} = \frac{2\dot{x}_\circ \dot{z}_\circ}{g} - \frac{8\dot{x}_\circ \dot{z}_\circ^2}{3g^2}\gamma + \ldots$$

For $\quad \dot{z}_\circ = v_\circ \sin\alpha$ and $2\dot{x}_\circ \dot{z}_\circ = v_\circ^2 \sin 2\alpha$:

$$x_{max} = \frac{v_\circ^2 \sin 2\alpha}{g} - \frac{4v_\circ^3 \sin 2\alpha \sin\alpha}{3g^2}\gamma + \ldots$$

4.16



$$x = A\cos(\omega t + \alpha), \qquad \dot{x} = -A\omega \sin(\omega t + \alpha)$$

from $\quad \dot{x}_\circ = 0, \qquad \alpha = 0$

from $\quad x_\circ = A, \qquad x = A\cos\omega t$

$$y = B\cos(\omega t + \beta), \qquad \dot{y} = -\omega B \sin(\omega t + \beta)$$

$$\frac{1}{2}kB^2 = \frac{1}{2}ky_\circ^2 + \frac{1}{2}m\dot{y}_\circ^2$$

with $\quad y_\circ = 4A,\ \dot{y}_\circ = 3\omega A \quad$ and $\omega = \sqrt{\frac{k}{m}}$:

$$B^2 = 16A^2 + \frac{1}{\omega^2}\left(9\omega^2 A^2\right) = 25A^2$$

$$B = 5A$$

Then $\quad 4A = 5A\cos\beta \quad$ and $\quad 3\omega A = -5\omega A \sin\beta$

$$\beta = \cos^{-1}\left(\frac{4}{5}\right) = \sin^{-1}\left(-\frac{3}{5}\right) = -36.9^\circ$$

$$y = 5A\cos\left(\omega t - 36.9^\circ\right)$$

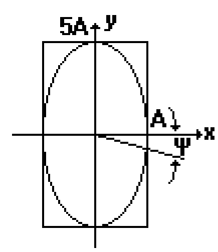
Since maximum x and y displacements are $\pm A$ and $\pm 5A$, respectively, the motion takes place entirely within a rectangle of dimension $2A$ and $10A$.

$$\Delta = \beta - \alpha = -36.9^\circ - 0 = -36.9^\circ$$

From eqn 4.4.15, $\tan 2\psi = \frac{2AB\cos\Delta}{A^2 - B^2}$

$$\tan 2\psi = \frac{(2A)(5A)\cos\left(-36.9^\circ\right)}{A^2 - (5A)^2} = \frac{10\left(\frac{4}{5}\right)}{-24} = -\frac{1}{3}$$

$$\psi = \frac{1}{2}\tan^{-1}\left(-\frac{1}{3}\right) = -9.2^\circ$$



4.17 $\quad m\ddot{x} = F_x = -\frac{\partial V}{\partial x} = -kx = -\pi^2 mx$

$$\dot{x} = A\cos\left(\sqrt{\frac{k}{m}}t + \alpha\right) = A\cos(\pi t + \alpha)$$

$$m\ddot{y} = -\frac{\partial V}{\partial y} = -4\pi^2 mx$$

$$y = B\cos(2\pi t + \beta)$$

$$m\ddot{z} = -\frac{\partial V}{\partial z} = -9\pi^2 mz$$

$$z = C\cos(3\pi t + \gamma)$$

Since $x = y = z = 0$ at $t = 0$, $\alpha = \beta = \gamma = -\frac{\pi}{2}$

$$x = A\cos\left(\pi t - \frac{\pi}{2}\right) = A\sin \pi t$$

$$\dot{x} = A\pi \cos \pi t$$

Since $v_\circ^2 = \dot{x}_\circ^2 + \dot{y}_\circ^2 + \dot{z}_\circ^2$ and $\dot{x}_\circ = \dot{y}_\circ = \dot{z}_\circ$,

$$\dot{x}_\circ = \frac{v_\circ}{\sqrt{3}} = A\pi$$

$$A = \frac{v_\circ}{\pi\sqrt{3}}$$

$$x = \frac{v_\circ}{\pi\sqrt{3}}\sin \pi t$$

$$y = B\sin 2\pi t, \qquad \dot{y} = 2B\pi\cos 2\pi t$$

$$\dot{y}_\circ = \frac{v_\circ}{\sqrt{3}} = 2\pi B$$

$$B = \frac{v_\circ}{2\pi\sqrt{3}}$$

$$y = \frac{v_\circ}{2\pi\sqrt{3}}\sin 2\pi t$$

$$z = C\sin 3\pi t, \qquad \dot{z} = 3C\pi\cos 3\pi t$$

$$\dot{z}_\circ = \frac{v_\circ}{\sqrt{3}} = 3C\pi$$

$$C = \frac{v_\circ}{3\pi\sqrt{3}}$$

$$z = \frac{v_\circ}{3\pi\sqrt{3}}\sin 3\pi t$$

Since $\omega_x = \pi$, $\omega_y = 2\pi$, and $\omega_z = 3\pi$ the ball does retrace its path.

$$t_{\min} = \frac{2\pi n_1}{\omega_x} = \frac{2\pi n_2}{\omega_y} = \frac{2\pi n_3}{\omega_z}$$

The minimum time occurs at $n_1 = 1$, $n_2 = 2$, $n_3 = 3$.

$$t_{\min} = \frac{2\pi}{\pi} = 2$$

4.18 Equation 4.4.15 is $\tan 2\psi = \dfrac{2AB\cos\Delta}{A^2 - B^2}$

Transforming the coordinate axes xyz to the new axes $x'y'z'$ by a rotation about the z-axis through an angle ψ given, from Section 1.8:

$$x' = x\cos\psi + y\sin\psi, \qquad y' = -x\sin\psi + y\cos\psi$$

or, $x = x'\cos\psi - y'\sin\psi$, and $y = x'\sin\psi + y'\cos\psi$

From eqn. 4.4.10: $\dfrac{x^2}{A^2} - xy\dfrac{2\cos\Delta}{AB} + \dfrac{y^2}{B^2} = \sin^2\Delta$

Substituting:

$$\frac{1}{A^2}\left(x'^2\cos^2\psi - 2x'y'\cos\psi\sin\psi + y'^2\sin^2\psi\right)$$
$$-\frac{2\cos\Delta}{AB}\left[x'^2\cos\psi\sin\psi + x'y'\left(\cos^2\psi - \sin^2\psi\right) - y'^2\cos\psi\sin\psi\right]$$
$$+\frac{1}{B^2}\left(x'^2\sin^2\psi + 2x'y'\cos\psi\sin\psi + y'^2\cos^2\psi\right) = \sin^2\Delta$$

For x' to be a major or minor axis of the ellipse, the coefficient of $x'y'$ must vanish.

$$-\frac{2\cos\psi\sin\psi}{A^2} - \frac{2\cos\Delta}{AB}\left(\cos^2\psi - \sin^2\psi\right) + \frac{2\cos\psi\sin\psi}{B^2} = 0$$

From Appendix B, $2\cos\psi\sin\psi = \sin 2\psi$ and $\cos^2\psi - \sin^2\psi = \cos 2\psi$

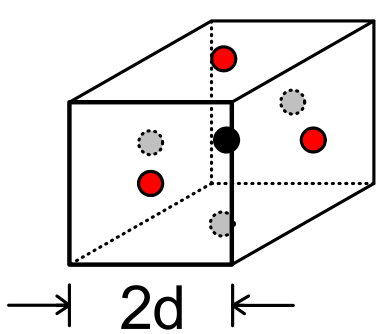
$$-\frac{\sin 2\psi}{A^2} - \frac{2\cos\Delta\cos 2\psi}{AB} + \frac{\sin 2\psi}{B^2} = 0$$

$$\tan 2\psi\left(\frac{1}{B^2} - \frac{1}{A^2}\right) = \frac{2\cos\Delta}{AB}$$

$$\tan 2\psi = \frac{2AB\cos\Delta}{A^2 - B^2}$$

4.19 Shown below is a face-centered cubic lattice. Each atom in the lattice is centered within a cube on whose 6 faces lies another adjacent atom. Thus each atom is surrounded by 6 nearest neighbors at a distance d. We neglect the influence of atoms that lie at further distances. Thus, the potential energy of the central atom can be approximated as

$$V = \sum_{i=1}^{6} c r_i^{-\alpha}$$



$$r_1 = \left[(d-x)^2 + y^2 + z^2\right]^{\frac{1}{2}}$$

$$r_1^{-\alpha} = \left(d^2 - 2dx + x^2 + y^2 + z^2\right)^{-\frac{\alpha}{2}} = d^{-\alpha}\left(1 - \frac{2x}{d} + \frac{x^2+y^2+z^2}{d^2}\right)^{-\frac{\alpha}{2}}$$

From Appendix D, $(1+x)^n = 1 + nx + \frac{1}{2}n(n-1)x^2 + \dots$

$$r_1^{-\alpha} = d^{-\alpha}\left\{1 - \frac{\alpha}{2}\left(-\frac{2x}{d} + \frac{x^2+y^2+z^2}{d^2}\right) + \frac{1}{2}\left(-\frac{\alpha}{2}\right)\left(-\frac{\alpha}{2} - 1\right)\right.$$

$$\left.\left[\left(-\frac{2x}{d}\right)^2 - 2\left(\frac{2x}{d}\right)\left(\frac{x^2+y^2+z^2}{d^2}\right) + \left(\frac{x^2+y^2+z^2}{d^2}\right)^2\right] + \text{terms in } \frac{x^3}{d^3}\right\}$$

$$r_1^{-\alpha} = d^{-\alpha}\left\{1 + \frac{\alpha x}{d} - \frac{\alpha}{2d^2}\left(x^2+y^2+z^2\right) + \frac{\alpha}{4}\left(\frac{\alpha}{2}+1\right)\left[\frac{4x^2}{d^2} + \text{terms in } \frac{x^3}{d^3}\right]\right\}$$

$$r_1^{-\alpha} \approx d^{-\alpha}\left[1 + \frac{\alpha x}{d} - \frac{\alpha}{2d^2}\left(x^2+y^2+z^2\right) + \frac{\alpha}{d^2}\left(\frac{\alpha}{2}+1\right)x^2\right]$$

$$r_2 = \left[(-d-x)^2 + y^2 + z^2\right]^{\frac{1}{2}} = \left[d^2 + 2dx + x^2 + y^2 + z^2\right]^{\frac{1}{2}}$$

$$r_2^{-\alpha} = d^{-\alpha}\left(1 + \frac{2x}{d} + \frac{x^2+y^2+z^2}{d^2}\right)^{-\frac{\alpha}{2}}$$

$$r_2^{-\alpha} \approx d^{-\alpha}\left[1 - \frac{\alpha x}{d} - \frac{\alpha}{2d^2}\left(x^2+y^2+z^2\right) + \frac{\alpha}{d^2}\left(\frac{\alpha}{2}+1\right)x^2\right]$$

$$r_1^{-\alpha} + r_2^{-\alpha} \approx d^{-\alpha}\left[2 - \frac{\alpha}{d^2}\left(x^2+y^2+z^2\right) + \frac{\alpha}{d^2}(\alpha+2)x^2\right]$$

Similarly:

$$r_3^{-\alpha} + r_4^{-\alpha} \approx d^{-\alpha}\left[2 - \frac{\alpha}{d^2}\left(x^2+y^2+z^2\right) + \frac{\alpha}{d^2}(\alpha+2)y^2\right]$$

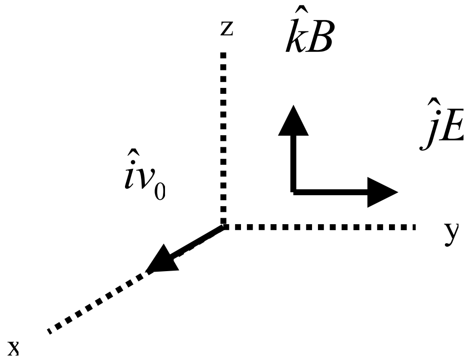
$$r_5^{-\alpha} + r_6^{-\alpha} \approx d^{-\alpha}\left[2 - \frac{\alpha}{d^2}\left(x^2+y^2+z^2\right) + \frac{\alpha}{d^2}(\alpha+2)z^2\right]$$

$$V \approx cd^{-\alpha}\left[6 - \frac{3\alpha}{d^2}\left(x^2+y^2+z^2\right) + \left(\frac{\alpha^2}{d^2} + \frac{2\alpha}{d^2}\right)\left(x^2+y^2+z^2\right)\right]$$

$$\approx 6cd^{-\alpha} + cd^{-\alpha-2}\left(\alpha^2 - \alpha\right)\left(x^2+y^2+z^2\right)$$

$$V \approx A + B\left(x^2+y^2+z^2\right)$$

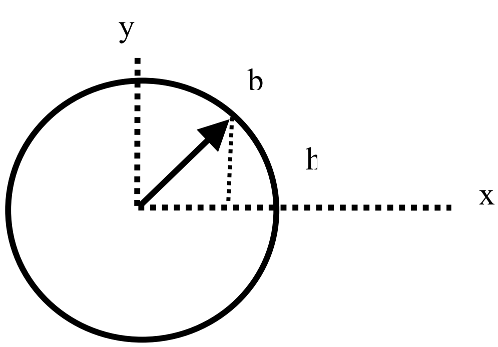
4.20



$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$
$$\vec{v}\times\vec{B} = \left(\hat{i}\dot{x} + \hat{j}\dot{y} + \hat{k}\dot{z}\right)\times\hat{k}B = \hat{i}\dot{y}B - \hat{j}\dot{x}B$$
$$\vec{F} = \hat{i}q\dot{y}B + \hat{j}q\left(E - \dot{x}B\right)$$
$$m\ddot{x} = F_x = q\dot{y}B$$
$$\dot{x} - \dot{x}_\circ = \frac{qB}{m}y$$

$$m\ddot{y} = F_y = qE - q\dot{x}B = qE - qB\left(\dot{x}_\circ + \frac{qB}{m}y\right)$$
$$\ddot{y} = \frac{qE}{m} - \frac{qB\dot{x}_\circ}{m} - \left(\frac{qB}{m}\right)^2 y = -\frac{eE}{m} + \frac{eB\dot{x}_\circ}{m} - \left(\frac{eB}{m}\right)^2 y$$
$$\ddot{y} + \omega^2 y = -\frac{eE}{m} + \omega\dot{x}_\circ, \qquad \omega = \frac{eB}{m}$$
$$y = \frac{1}{\omega^2}\left(-\frac{eE}{m} + \omega x_\circ\right) + A\cos\left(\omega t + \theta_\circ\right)$$
$$\dot{y} = -A\omega\sin\left(\omega t + \theta_\circ\right)$$
$$\dot{y}_\circ = 0, \text{ so } \quad \theta_\circ = 0$$
$$y_\circ = 0, \text{ so } \quad A = -\frac{1}{\omega^2}\left(-\frac{eE}{m} + \omega\dot{x}_\circ\right)$$
$$y = a\left(1 - \cos\omega t\right), \qquad a = \frac{1}{\omega^2}\left(-\frac{eE}{m} + \omega\dot{x}_\circ\right)$$
$$\dot{x} = \dot{x}_\circ + \frac{qB}{m}y = \dot{x}_\circ - \omega y = \dot{x}_\circ - \omega a\left(1 - \cos\omega t\right)$$
$$\dot{x} = \left(\dot{x}_\circ - \omega a\right) + \omega a\cos\omega t$$
$$x = \left(\dot{x}_\circ - \omega a\right)t + a\sin\omega t$$
$$x = a\sin\omega t + bt, \qquad b = \dot{x}_\circ - \omega a$$
$$m\ddot{z} = F_z = 0$$
$$z = \dot{z}_\circ t + z_\circ = 0$$

4.21



$$\frac{1}{2}mv^2 + mqh = mg\frac{b}{2}$$
$$v^2 = g\left(b - 2h\right)$$
$$F_r = -\frac{mv^2}{b} = -mg\cos\theta + R$$

$$\cos\theta = \frac{h}{b}$$

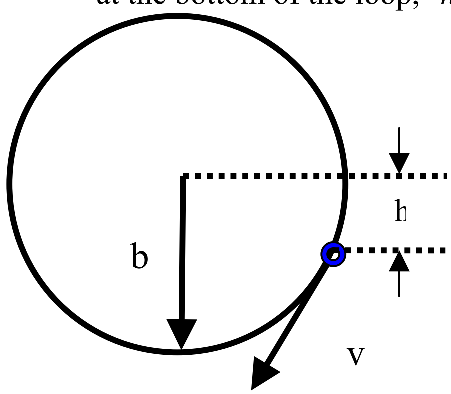
$$R = mg\frac{h}{b} - \frac{mv^2}{b} = \frac{mg}{b}\left[h - (b - 2h)\right] = \frac{mg}{b}(3h - b)$$

the particle leaves the side of the sphere when $R = 0$

$h = \frac{b}{3}$, i.e., $\frac{b}{3}$ above the central plane

4.22 $\frac{1}{2}mv^2 + mgh = 0$

at the bottom of the loop, $h = -b$



so $\frac{1}{2}mv^2 = mgb$,

$$v = \sqrt{2gb}$$

$$F_r = -mg + R = \frac{mv^2}{b}$$

$$R = mg + \frac{mv^2}{b} = mg + 2mg = 3mg$$

4.23 From the equation for the energy as a function of s in Example 4.6.2,

$$E = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}\left(\frac{mg}{4A}\right)s^2,$$

s is undergoing harmonic motion with:

$$\omega = \sqrt{\frac{"k"}{m}} = \sqrt{\frac{g}{4A}} = \frac{1}{2}\sqrt{\frac{g}{A}}$$

Since $s = 4A\sin\phi$, ϕ increases by 2π radians during the time interval:

$$T' = \frac{2\pi}{\omega} = 2\pi\left(2\sqrt{\frac{A}{g}}\right)$$

For cycloidal motion, x and z are functions of 2ϕ so they undergo a complete cycle every time ϕ changes by π. Therefore, the period for the cycloidal motion is one-half the period for s.

$$T = \frac{1}{2}T' = 2\pi\sqrt{\frac{A}{g}}$$
