CHAPTER 2
NEWTONIAN MECHANICS:
RECTILINEAR MOTION OF A PARTICLE

2.1 (a) $\ddot{x} = \frac{1}{m}\left(F_\circ + ct\right)$

$$\dot{x} = \int_0^t \frac{1}{m}\left(F_\circ + ct\right)dt = \frac{F_\circ}{m}t + \frac{c}{2m}t^2$$

$$x = \int_0^t \left(\frac{F_\circ}{m}t + \frac{c}{2m}t^2\right)dt = \frac{F_\circ}{m}t^2 + \frac{c}{6m}t^3$$

(b) $\ddot{x} = \frac{F_\circ}{m}\sin ct$

$$\dot{x} = \int_0^t \frac{F_\circ}{m}\sin ct\, dt = -\frac{F_\circ}{cm}\cos ct\Big|_0^t = \frac{F_\circ}{cm}\left(1 - \cos ct\right)$$

$$x = \int_0^t \frac{F_\circ}{cm}\left(1 - \cos ct\right)dt = \frac{F_\circ}{cm}\left(1 - \frac{1}{c}\sin ct\right)$$

(c) $\ddot{x} = \frac{F_\circ}{m}e^{ct}$

$$\dot{x} = \frac{F_\circ}{cm}e^{ct}\Big|_0^t = \frac{F_\circ}{cm}\left(e^{ct} - 1\right)$$

$$x = \frac{F_\circ}{cm}\left(\frac{1}{c}e^{ct} - \frac{1}{c} - t\right) = \frac{F_\circ}{c^2 m}\left(e^{ct} - 1 - ct\right)$$

2.2 (a) $\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx}\cdot\frac{dx}{dt} = \dot{x}\frac{d\dot{x}}{dx}$

$$\dot{x}\frac{d\dot{x}}{dx} = \frac{1}{m}\left(F_\circ + cx\right)$$

$$\dot{x}d\dot{x} = \frac{1}{m}\left(F_\circ + cx\right)dx$$

$$\frac{1}{2}\dot{x}^2 = \frac{1}{m}\left(F_\circ x + \frac{cx^2}{2}\right)$$

$$\dot{x} = \left[\frac{x}{m}\left(2F_\circ + cx\right)\right]^{\frac{1}{2}}$$

(b) $\ddot{x} = \dot{x}\frac{d\dot{x}}{dx} = \frac{1}{m}F_\circ e^{-cx}$

$$\dot{x}d\dot{x} = \frac{1}{m} F_\circ e^{-cx} dx$$

$$\frac{1}{2}\dot{x}^2 = -\frac{F_\circ}{cm}\left(e^{-cx} - 1\right) = \frac{F_\circ}{cm}\left(1 - e^{-cx}\right)$$

$$\dot{x} = \left[\frac{2F_\circ}{cm}\left(1 - e^{-cx}\right)\right]^{\frac{1}{2}}$$

(c) $\ddot{x} = \dot{x}\frac{d\dot{x}}{dx} = \frac{1}{m}\left(F_\circ \cos cx\right)$

$$\dot{x}d\dot{x} = \frac{F_\circ}{m}\cos cx\, dx$$

$$\frac{1}{2}\dot{x}^2 = \frac{F_\circ}{cm}\sin cx$$

$$\dot{x} = \left(\frac{2F_\circ}{cm}\sin cx\right)^{\frac{1}{2}}$$

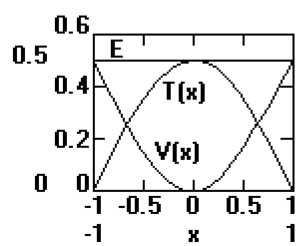
2.3 (a) $V(x) = -\int_{x_\circ}^{x}\left(F_\circ + cx\right)dx = -F_\circ x - \frac{cx^2}{2} + C$

(b) $V(x) = -\int_{x_\circ}^{x} F_\circ e^{-cx} dx = \frac{F_\circ}{c} e^{-cx} + C$

(c) $V(x) = -\int_{x_\circ}^{x} F_\circ \cos cx\, dx = -\frac{F_\circ}{c}\sin cx + C$

2.4 (a) $F(x) = -\frac{dV(x)}{dx} = -kx$

$$V(x) = \int_0^x kx\, dx = \frac{1}{2}kx^2$$



(b) $T_\circ = T(x) + V(x)$

$$T(x) = T_\circ - V(x) = \frac{1}{2}k\left(A - x^2\right)$$

(c) $E = T_\circ = \frac{1}{2}kA^2$

(d) *turning points @* $T(x_1) \to 0 \quad \therefore x_1 = \pm A$

2.5 (a) $F(x) - kx + \frac{kx^3}{A^2}$ so $V(x) = \int_0^x \left(kx - \frac{kx^3}{A^2}\right)dx = \frac{1}{2}kx^2 - \frac{1}{4}\frac{kx^4}{A^2}$

(b) $T(x) = T_\circ - V(x) = T_\circ - \frac{1}{2}kx^2 + \frac{1}{4}\frac{kx^4}{A^2}$

(c) $E = T_\circ$

(d) $V(x)$ has maximum at $|F(x_m)| \to 0$

$$kx_m - \frac{kx_m^3}{A^2} = 0 \qquad x_m = \pm A$$

$$V(x_m) = \frac{1}{2}kA^2 - \frac{1}{4}\frac{kA^4}{A^2} = \frac{1}{4}kA^2$$

If $E < V(x_m)$ turning points exist.

Turning points @ $T(x_1) \to 0$ let $u = x_1^2$

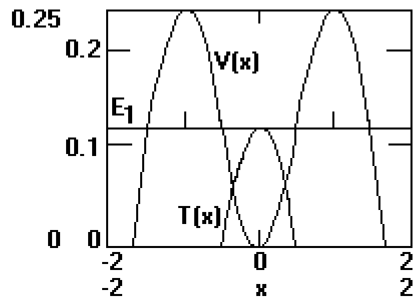
$$E - \frac{1}{2}ku + \frac{1}{4}\frac{ku^2}{A^2} = 0$$

solving for u, we obtain

$$u = A^2\left[1 \pm \left(1 - \frac{4E}{kA^2}\right)^{\frac{1}{2}}\right]$$

or

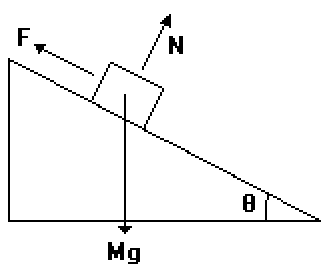
$$x_1 = \pm A\left[1 - \sqrt{\left(1 - \frac{4E}{kA^2}\right)}\right]^{\frac{1}{2}}$$



2.6 $\dot{x} = v(x) = \frac{\alpha}{x} \qquad \ddot{x} = -\frac{\alpha}{x^2}\dot{x} = -\frac{\alpha^2}{x^3}$

$$F(x) = m\ddot{x} = -\frac{m\alpha^2}{x^3}$$

2.7



$$F \geq Mg\sin\theta$$

2.8 $F = m\ddot{x} = m\dot{x}\frac{d\dot{x}}{dx}$

$$\dot{x} = bx^{-3}$$

$$\frac{d\dot{x}}{dx} = -3bx^{-4}$$

$$F = m\left(bx^{-3}\right)\left(-3bx^{-4}\right)$$

$$F = -3mb^2x^{-7}$$

2.9 (a) $V = mgx = (.145kg)\left(9.8\frac{m}{s^2}\right)(1250\,ft)\left(.3048\frac{m}{ft}\right) = 541J$

(b) $T = \frac{1}{2}mv^2 = \frac{1}{2}mv_t^{\,2} = \frac{1}{2}m\left(\frac{mg}{c_2}\right) = \frac{1}{2}\frac{m^2 g}{.22D^2}$

$$T = \frac{(.145kg)^2\left(9.8\frac{m}{s^2}\right)}{(2)(.22)\left[(2)(.0366)\right]^2 \frac{kg}{m}} = 87J$$

$$\int F dx = \int -cv^2 dx = -c\int v^3 dt = -c\int\left(-v_t \tanh\left(\frac{t}{\tau}\right)\right)^3 dt$$

$$= cv_t^{\,3}\tau\left[-\frac{1}{2}\tanh^2\left(\frac{t}{\tau}\right) + \int \tanh\left(\frac{t}{\tau}\right) d\left(\frac{t}{\tau}\right)\right]$$

$$= cv_t^{\,3}\tau\left[-\frac{1}{2}\tanh^2\left(\frac{t}{\tau}\right) + \ln\cosh\left(\frac{t}{\tau}\right)\right]$$

Now $\tanh^2\left(\frac{t}{\tau}\right) \cong 1$ for $t \ll \tau$

Meanwhile $x = \int v dt = \int\left(-v_t \tanh\left(\frac{t}{\tau}\right)\right) dt = v_t \tau \ln\cosh\left(\frac{t}{\tau}\right)$

$$\ln\cosh\left(\frac{t}{\tau}\right) = \frac{x}{v_t \tau}$$

$$x = (1250\,ft)\left(.3048\frac{m}{ft}\right) = 381m$$

$$v_t = \left(\frac{mg}{c_2}\right)^{\frac{1}{2}} = \left[\frac{(.145kg)\left(9.8\frac{m}{s^2}\right)}{(.22)(.0732)^2 \frac{kg}{m}}\right]^{\frac{1}{2}} = 34.72\frac{m}{s}$$

$$\tau = \left(\frac{m}{c_2 g}\right)^{\frac{1}{2}} = \left[\frac{(.145kg)}{(.22)(.0732)^2 \frac{kg}{m}\left(9.8\frac{m}{s^2}\right)}\right]^{\frac{1}{2}} = 3.543s$$

$$\int F dx = (.22)(.0732)^2(34.72)^3(3.543)\left[-.5 + \frac{3.81}{(34.72)(3.54)}\right] = 454J$$

$$V - T = 541J - 87J = 454J$$

2.10 For $0 \le t \le t_1$: $v = \frac{F_\circ}{m}t$, $x = \frac{1}{2}\frac{F_\circ}{m}t^2$

For $t_1 \le t \le 2t_1$: $v_\circ = \frac{F_\circ}{m}t_1$, $x_\circ = \frac{F_\circ}{2m}t_1^{\,2}$, $t_\circ = t_1$

$$x=\frac{F_\circ}{2m}t_1^{\,2}+\frac{F_\circ}{m}t_1\left(t-t_1\right)+\frac{1}{2}\frac{2F_\circ}{m}\left(t-t_1\right)^2$$

At $t=2t_1$: $x=\frac{F_\circ}{2m}t_1^{\,2}+\frac{F_\circ}{m}t_1^{\,2}+\frac{F_\circ}{m}t_1^{\,2}=\frac{5F_\circ}{2m}t_1^{\,2}$

2.11 $a=\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}=v\cdot\frac{dv}{dx}=-\frac{c}{m}v^{\frac{3}{2}}$

$$v^{-\frac{1}{2}}dv=-\frac{c}{m}dx$$

$$\int_{v_\circ}^{v}v^{-\frac{1}{2}}dv=\int_0^{x_{\max}}-\frac{c}{m}dx$$

$$-2v_\circ^{\frac{1}{2}}=-\frac{c}{m}x_{\max}$$

$$x_{\max}=\frac{2mv_\circ^{\frac{1}{2}}}{c}$$

2.12 Going up: $F_x=-mg\sin 30^\circ-\mu mg\cos 30^\circ$

$$\ddot{x}=-g\left(\sin 30^\circ+0.1\cos 30^\circ\right)=-5.749\frac{m}{s^2}$$

$$v=v_\circ+at$$

at the highest point $v=0$ so $t_{up}=-\frac{v_\circ}{a}=0.174v_\circ s$

$$x_{up}=v_\circ t_{up}+\frac{1}{2}at_{up}^2=0.174v_\circ^{\,2}-.087v_\circ^{\,2}=0.087v_\circ^{\,2}m$$

Going down: $x_\circ{}'=0.087v_\circ^{\,2}$, $v_\circ{}'=0$, $a'=-9.8\left(0.5-0.0866\right)$

$$x_{down}=0=0.087v_\circ^{\,2}-\frac{1}{2}4.0513t_{down}^2$$

$$t_{down}=0.207v_\circ s$$

$$t_{total}=t_{up}+t_{down}=0.381v_\circ s$$

2.13 At the top $v=0$ so $e^{-2kx_{\max}}=\dfrac{\frac{g}{k}}{\frac{g}{k}+v_\circ^{\,2}}$

Coming down $x_\circ=x_{\max}$ and at the bottom $x=0$

$$v^2=\frac{g}{k}-\left(\frac{g}{k}\right)^2\frac{1}{\left(\frac{g}{k}+v_\circ^{\,2}\right)}(1)=\frac{\left(\frac{g}{k}\right)v_\circ^{\,2}}{\frac{g}{k}+v_\circ^{\,2}}$$

$$v=\frac{v_t v_\circ}{\left(v_t^2+v_\circ^2\right)^{\frac{1}{2}}},\quad v_t=\sqrt{\frac{g}{k}}=\sqrt{\frac{mg}{c_2}}$$

2.14 Going up: $F_x=-mg-c_2v^2$

$$a=v\frac{dv}{dx}=-g-kv^2,\quad k=\frac{c^2}{m}$$

$$\int_{v_\circ}^{v}\frac{vdv}{-g-kv^2}=\int_0^x dx$$

$$-\frac{1}{2k}\ln\left(-g-kv^2\right)\Big|_{v_\circ}^{v}=x$$

$$\frac{g+kv^2}{g+kv_\circ^2}=e^{-2kx}$$

$$v^2=\left(\frac{g}{k}+v_\circ^2\right)e^{-2kx}-\frac{g}{k}$$

Going down: $F_x=-mg+c_2v^2$

$$v\frac{dv}{dx}=-g+kv^2$$

$$\int_0^{v}\frac{vdv}{-g+kv^2}=\int_0^x dx$$

$$\frac{1}{2k}\ln\left(-g+kv^2\right)\Big|_0^{v}=x-x_\circ$$

$$1-\frac{k}{g}v^2=e^{2kx}e^{-2kx_\circ}$$

$$v^2=\frac{g}{k}-\left(\frac{g}{k}e^{-2kx_\circ}\right)e^{2kx}$$

2.15 $m\frac{dv}{dt}=mg-c_1v-c_2v^2$

$$\int_0^t\frac{dt}{m}=\int_0^v\frac{dv}{mg-c_1v-c_2v^2}$$

Using $\int\frac{dx}{a+bx+cx^2}=\frac{1}{\sqrt{b^2-4ac}}\ln\frac{2cx+b-\sqrt{b^2-4ac}}{2cx+b+\sqrt{b^2-4ac}}$,

$$\frac{t}{m}=\frac{1}{\sqrt{c_1^{\ 2}+4mgc_2}}\ln\frac{-2c_2v-c_1-\sqrt{c_1^{\ 2}+4mgc_2}}{-2c_2v-c_1+\sqrt{c_1^{\ 2}+4mgc_2}}\Bigg|_0^v$$

$$\frac{t}{m}\left(c_1^{\ 2}+4mgc_2\right)^{\frac{1}{2}}=\ln\frac{\left(2c_2v+c_1+\sqrt{c_1^{\ 2}+4mgc_2}\right)\left(c_1-\sqrt{c_1^{\ 2}+4mgc_2}\right)}{\left(2c_2v+c_1-\sqrt{c_1^{\ 2}+4mgc_2}\right)\left(c_1+\sqrt{c_1^{\ 2}+4mgc_2}\right)}$$

as $t\to\infty$, $2c_2v_t+c_1-\sqrt{c_1^{\ 2}+4mgc_2}=0$

$$v_t=-\frac{c_1}{2c_2}+\left[\left(\frac{c_1}{2c_2}\right)^2+\frac{mg}{c_2}\right]^{\frac{1}{2}}$$

Alternatively, when $v=v_t$,

$$m\frac{dv}{dt}=0=mg-c_1v_t-c_2v_t^{\ 2}$$

$$v_t=-\frac{c_1}{2c_2}+\left[\left(\frac{c_1}{2c_2}\right)^2+\frac{mg}{c_2}\right]^{\frac{1}{2}}$$

2.16 $a=v\dfrac{dv}{dx}=-\dfrac{k}{m}x^{-2}$

$$\int_0^v vdv=\int_b^x-\frac{kdx}{mx^2}$$

$$\frac{1}{2}v^2=\frac{k}{m}\left(\frac{1}{x}-\frac{1}{b}\right)$$

$$v=\frac{dx}{dt}=\left[\frac{2k}{m}\left(\frac{1}{x}-\frac{1}{b}\right)\right]^{\frac{1}{2}}=\left[\frac{2k}{mb}\left(\frac{b-x}{x}\right)\right]^{\frac{1}{2}}$$

$$\int_0^t dt=\int_b^0\left[\frac{mb}{2k}\left(\frac{x}{b-x}\right)\right]^{\frac{1}{2}}dx=\left(\frac{mb^3}{2k}\right)^{\frac{1}{2}}\int_1^0\left(\frac{\frac{x}{b}}{1-\frac{x}{b}}\right)^{\frac{1}{2}}d\left(\frac{x}{b}\right)$$

Since $x\le b$, say $\dfrac{x}{b}=\sin^2\theta$

$$t=\left(\frac{mb^3}{2k}\right)^{\frac{1}{2}}\int_{-\frac{\pi}{2}}^0\frac{\sin\theta\left(2\sin\theta\cos\theta d\theta\right)}{\cos\theta}=\left(\frac{2mb^3}{k}\right)^{\frac{1}{2}}\int_{-\frac{\pi}{2}}^0\sin^2\theta d\theta$$

$$t=\left(\frac{mb^3}{8k}\right)^{\frac{1}{2}}\pi$$

2.17 $m\frac{dv}{dt} = mv\frac{dv}{dx} = f(x) \bullet g(v)$

$$\frac{mvdv}{g(v)} = f(x)\,dx$$

By integration, get $v = v(x) = \frac{dx}{dt}$

If $F(x,t) = f(x) \bullet g(t)$:

$$m\frac{d^2x}{dt^2} = m\frac{d}{dt}\left(\frac{dx}{dt}\right) = f(x) \bullet g(t)$$

This cannot, in general, be solved by integration.

If $F(v,t) = f(v) \bullet g(t)$:

$$m\frac{dv}{dt} = f(v) \bullet g(t)$$

$$\frac{mdv}{f(v)} = g(t)\,dt$$

Integration gives $v = v(t)$

$$\frac{dx}{dt} = v(t)$$

$$dx = v(t)\,dt$$

A second integration gives $x = x(t)$

2.18

$$c_1 = \left(1.55\times10^{-4}\right)\left(10^{-2}\right) = 1.55\times10^{-6}\,\frac{kg}{s}$$

$$c_2 = (0.22)\left(10^{-2}\right)^2 = 2.2\times10^{-5}\,\frac{kg}{s}$$

$$v_t = -\frac{1.55\times10^{-6}}{2\times2.2\times10^{-5}} + \left[\left(\frac{1.55\times10^{-6}}{2\times2.2\times10^{-5}}\right)^2 + \frac{\left(10^{-7}\right)(9.8)}{2.2\times10^{-5}}\right]^{\frac{1}{2}}$$

$$v_t = 0.179\,\frac{m}{s}$$

Using equation 2.29, $v_t = \sqrt{\frac{\left(10^{-7}\right)(9.8)}{2.2\times10^{-5}}} = 0.211\,\frac{m}{s}$

2.19

$F(x) = -Ae^{\alpha\dot{x}} = m\ddot{x}$ or $F(v) = -Ae^{\alpha v} = m\dot{v}$ $\quad \frac{dv}{e^{\alpha v}} = -\frac{A}{m}dt$

Let $u = e^{\alpha v}$ $\quad du = \alpha e^{\alpha v}dv$ $\quad dv = \frac{du}{\alpha e^{\alpha v}} = \frac{du}{\alpha u}$ $\quad \therefore \frac{du}{u^2} = -\frac{\alpha A}{m}dt$

Integrating

$$\frac{1}{u}-\frac{1}{u_\circ}=\frac{A}{m}\alpha t \qquad \text{and substituting } e^{\alpha v}=u$$

(a) $v=v_\circ-\frac{1}{\alpha}\ln\left[1+\frac{A}{m}e^{\alpha v_\circ}\alpha t\right]$

(b) $t=T\,@\,v=0$

$$\alpha v_\circ=\ln\left[1+\frac{A}{m}e^{\alpha v_\circ}\alpha T\right]$$

$$e^{\alpha v_\circ}=1+\frac{A}{m}e^{\alpha v_\circ}\alpha T \qquad T=\frac{m}{\alpha A}\left[1-e^{-\alpha v_\circ}\right]$$

(c) $v\frac{dv}{dx}=v=-\frac{A}{m}e^{\alpha v} \qquad \frac{vdv}{e^{\alpha v}}=-\frac{A}{m}dx$

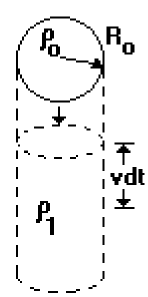
again, let $u=e^{\alpha v}$ $\quad du=\alpha u dv$ $\quad$ or $\quad dv=\frac{du}{\alpha u}$ $\quad v=\frac{1}{\alpha}\ln u$

$$\frac{\left[\frac{1}{\alpha}\ln u\right]\frac{du}{\alpha u}}{u}=-\frac{A}{m}dx \qquad \text{Integrating and solving}$$

$$x=\frac{m}{\alpha^2 A}\left[1-\left(1+\alpha v_\circ\right)e^{-\alpha v_\circ}\right]$$

2.20

$$F=\frac{d\left(mv\right)}{dt}=mv+vm=mg$$



but $\quad m=\rho_\circ\frac{4}{3}\pi r^3 \qquad m=\rho_1\pi r^2 v$

so (1) $\frac{4}{3}\pi\rho_\circ r^3 v+\pi\rho_1 r^2 v^2=\frac{4}{3}\pi^2=\frac{4}{3}\pi\rho_\circ r^3 g$

Now $\frac{\rho_1}{\rho_\circ}\approx 10^{-3}$ $\qquad$ so, second term is negligible-small

hence $v\approx g$ and $\boxed{v\approx gt}$ speed $\propto t$ but

$\dot{m}=\rho_\circ 4\pi r^2\dot{r}=\rho_1\pi r^2 v$ $\quad$ or $\quad \dot{r}\cong\frac{1}{4}\frac{\rho_1}{\rho_\circ}v$ $\qquad$ Hence $\boxed{r\approx\frac{1}{4}\frac{\rho_1}{\rho_\circ}gt}$ and rate of growth $\propto t$

The exact differential equation from (1) above is:

$$\frac{4}{3}\pi\rho_\circ r\left|\frac{4\rho_\circ}{\rho_1}\ddot{r}\right|+\pi\rho_1\left|\frac{4\rho_\circ\dot{r}}{\rho_1}\right|^2=\frac{4}{3}\pi\rho_\circ rg$$

which reduces to: $\ddot{r} + \dfrac{3\dot{r}^2}{r} = \dfrac{\rho_1}{4\rho_\circ} g$

Using Mathcad, solve the above non-linear d.e. letting $\dfrac{\rho_1}{\rho_\circ} \approx 10^{-3}$ and $R_\circ \approx 0.01mm$ (small raindrop). Graphs show that

$$v \propto \dot{r} \propto t \quad \text{and} \quad r \propto t^2$$